CHAPTER 13 INTEGRATION IN VECTOR FIELDS

13.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t$ and $y = 1-t \Rightarrow y = 1-x \Rightarrow$ (c)

2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1$, $y = 1$, and $z = t \Rightarrow$ (e)

3. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow x = 2\cos t$ and $y = 2\sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow$ (g)

4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t$, $y = 0$, and $z = 0 \Rightarrow$ (a)

5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t$, $y = t$, and $z = t \Rightarrow$ (d)

6. $\mathbf{r} = t\mathbf{j} + (2-2t)\mathbf{k} \Rightarrow y = t$ and $z = 2-2t \Rightarrow z = 2-2y \Rightarrow$ (b)

7. $\mathbf{r} = (t^2-1)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1$ and $z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow$ (f)

8. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k} \Rightarrow x = 2\cos t$ and $z = 2\sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow$ (h)

9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}\,\mathbf{j}$; $x = t$ and $y = 1-t \Rightarrow x + y = t + (1-t) = 1$

$$\Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 f(t,1-t,0)\,\frac{d\mathbf{r}}{dt}\,dt = \int_0^1 (1)(\sqrt{2})\,dt = \left[\sqrt{2}\,t\right]_0^1 = \sqrt{2}$$

10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}$; $x = t$, $y = 1-t$, and $z = 1 \Rightarrow x - y + z - 2$

$$= t - (1-t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t-2)\sqrt{2}\,dt = \sqrt{2}\left[t^2 - 2t\right]_0^1 = -\sqrt{2}$$

11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4+1+4} = 3$; $xy + y + z$

$$= (2t)t + t + (2-2t) \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t^2 - t + 2)\,3\,dt = 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}$$

12. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}$, $-2\pi \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k}$

$$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5;\ \sqrt{x^2+y^2} = \sqrt{16\cos^2 t + 16\sin^2 t} = 4 \Rightarrow \int_C f(x,y,z)\,ds = \int_{-2\pi}^{2\pi} (4)(5)\,dt$$

$$= [20t]_{-2\pi}^{2\pi} = 80\pi$$

13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+9+4} = \sqrt{14}$; $x + y + z = (1-t) + (2-3t) + (3-2t) = 6 - 6t \Rightarrow \int_C f(x,y,z)\,ds$

$= \int_0^1 (6-6t)\sqrt{14}\,dt = 6\sqrt{14}\left[1 - \frac{t^2}{2}\right]_0^1 = \left(6\sqrt{14}\right)\left(\frac{1}{2}\right) = 3\sqrt{14}$

14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 1 \le t \le \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}$; $\frac{\sqrt{3}}{x^2+y^2+z^2} = \frac{\sqrt{3}}{t^2+t^2+t^2} = \frac{\sqrt{3}}{3t^2}$

$\Rightarrow \int_C f(x,y,z)\,ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2}\right)\sqrt{3}\,dt = \left[-\frac{1}{t}\right]_1^\infty = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$

15. C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2}$; $x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$

$\Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt = \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}\left(5\sqrt{5}-1\right)$;

C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$

$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(2-t^2\right)(1)\,dt = \left[2t - \frac{1}{3}t^3\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$; therefore $\int_C f(x,y,z)\,ds$

$= \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}$

16. C_1: $\mathbf{r}(t) = t\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$

$\Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 \left(-t^2\right)(1)\,dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$;

C_2: $\mathbf{r}(t) = t\mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$

$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(\sqrt{t}-1\right)(1)\,dt = \left[\frac{2}{3}t^{3/2} - t\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$;

C_3: $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t$

$\Rightarrow \int_{C_3} f(x,y,z)\,ds = \int_0^1 (t)(1)\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} \Rightarrow \int_C f(x,y,z)\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \frac{1}{2}$

$= -\frac{1}{6}$

17. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 < a \le t \le b \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3};\ \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$

$\Rightarrow \int_C f(x,y,z)\ ds = \int_a^b \left(\frac{1}{t}\right)\sqrt{3}\ dt = \left[\sqrt{3}\ \ln|t|\right]_a^b = \sqrt{3}\ \ln\left(\frac{b}{a}\right)$, since $0 < a \le b$

18. $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a|;$

$-\sqrt{x^2+z^2} = -\sqrt{0+a^2\sin^2 t} = \begin{cases} -|a|\sin t, & 0 \le t \le \pi \\ |a|\sin t, & \pi \le t \le 2\pi \end{cases} \Rightarrow \int_C f(x,y,z)\ ds = \int_0^\pi -|a|^2 \sin t\ dt + \int_\pi^{2\pi} |a|^2 \sin t\ dt$

$= \left[a^2\cos t\right]_0^\pi - \left[a^2\cos t\right]_\pi^{2\pi} = \left[a^2(-1) - a^2\right] - \left[a^2 - a^2(-1)\right] = -4a^2$

19. $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j},\ 0 \le x \le 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1+x^2};\ f(x,y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x \Rightarrow \int_C f\ ds$

$= \int_0^2 (2x)\sqrt{1+x^2}\ dx = \left[\frac{2}{3}\left(1+x^2\right)^{3/2}\right]_0^2 = \frac{2}{3}\left(5^{3/2} - 1\right) = \frac{10\sqrt{5} - 2}{3}$

20. $\mathbf{r}(x) = x\mathbf{i} + \frac{x^2}{2}\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j},\ 0 \le x \le 1 \Rightarrow \left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1+x^2};\ f(x,y) = f\left(x, \frac{x^2}{2}\right) = \frac{x + \left(\frac{x^4}{4}\right)}{\sqrt{1+x^2}} = \frac{4x + x^4}{4\sqrt{1+x^2}}$

$\Rightarrow \int_C f\ ds = \int_0^1 \left(\frac{4x+x^4}{4\sqrt{1+x^2}}\right)\sqrt{1+x^2}\ dx = \int_0^1 \left(x + \frac{x^4}{4}\right) dx = \left[\frac{x^2}{2} + \frac{x^5}{20}\right]_0^1 = \frac{1}{2} + \frac{1}{20} = \frac{11}{20}$

21. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ 0 \le t \le \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x,y) = f(2\cos t, 2\sin t)$

$= 2\cos t + 2\sin t \Rightarrow \int_C f\ ds = \int_0^{\pi/2} (2\cos t + 2\sin t)(2)\ dt = [4\sin t - 4\cos t]_0^{\pi/2} = 4 - (-4) = 8$

22. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ \frac{\pi}{2} \ge t \ge \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x,y) = f(2\cos t, 2\sin t)$

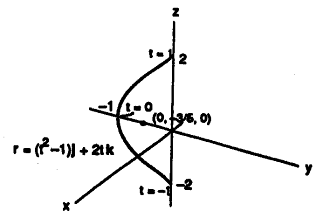
$= 4\cos^2 t - 2\sin t \Rightarrow \int_C f\ ds = \int_{\pi/2}^{\pi/4} \left(4\cos^2 t - 2\sin t\right)(2)\ dt = \left[4t + 2\sin 2t + 4\cos t\right]_{\pi/2}^{\pi/4}$

$= \left(\pi + 2 + 2\sqrt{2}\right) - (2\pi + 0 + 0) = 2\left(1 + \sqrt{2}\right) - \pi$

23. $\mathbf{r}(t) = \left(t^2 - 1\right)\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x,y,z)\ ds = \int_0^1 \delta(t)\left(2\sqrt{t^2+1}\right) dt$

$= \int_0^1 \left(\frac{3}{2}t\right)\left(2\sqrt{t^2+1}\right) dt = \left[\left(t^2+1\right)\right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$

24. $\mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k},\ -1 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$



$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x,y,z)\ ds$

$= \int_{-1}^{1} \left(15\sqrt{(t^2-1)+2}\right)\left(2\sqrt{t^2+1}\right) dt$

$= \int_{-1}^{1} 30\left(t^2+1\right) dt = \left[30\left(\frac{t^3}{3}+t\right)\right]_{-1}^{1} = 60\left(\frac{1}{3}+1\right) = 80;$

$M_{xz} = \int_C y\delta(x,y,z)\ ds = \int_{-1}^{1} (t^2-1)[30(t^2+1)]\ dt = \int_{-1}^{1} 30\left(t^4-1\right) dt = \left[30\left(\frac{t^5}{5}-t\right)\right]_{-1}^{1} = 60\left(\frac{1}{5}-1\right) = -48$

$\Rightarrow \bar{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5};\ M_{yz} = \int_C x\delta(x,y,z)\ ds = \int_C 0\,\delta\ ds = 0 \Rightarrow \bar{x} = 0;\ \bar{z} = 0$ by symmetry (since δ is independent of z) $\Rightarrow (\bar{x},\bar{y},\bar{z}) = \left(0,-\frac{3}{5},0\right)$

25. $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + (4-t^2)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2+2+4t^2} = 2\sqrt{1+t^2};$

(a) $M = \int_C \delta\ ds = \int_0^1 (3t)\left(2\sqrt{1+t^2}\right) dt = \left[2\left(1+t^2\right)^{3/2}\right]_0^1 = 2\left(2^{3/2}-1\right) = 4\sqrt{2}-2$

(b) $M = \int_C \delta\ ds = \int_0^1 (1)\left(2\sqrt{1+t^2}\right) dt = \left[t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\right]_0^1 = \left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right] - (0+\ln 1)$
$= \sqrt{2}+\ln\left(1+\sqrt{2}\right)$

26. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4+t} = \sqrt{5+t};$

$M = \int_C \delta\ ds = \int_0^2 \left(3\sqrt{5+t}\right)\left(\sqrt{5+t}\right) dt = \int_0^2 3(5+t)\ dt = \left[\frac{3}{2}(5+t)^2\right]_0^2 = \frac{3}{2}\left(7^2-5^2\right) = \frac{3}{2}(24) = 36;$

$M_{yz} = \int_C x\delta\ ds = \int_0^2 t[3(5+t)]\ dt = \int_0^2 \left(15t+3t^2\right) dt = \left[\frac{15}{2}t^2+t^3\right]_0^2 = 30+8 = 38;$

$M_{xz} = \int_C y\delta\ ds = \int_0^2 2t[3(5+t)]\ dt = 2\int_0^2 \left(15t+3t^2\right) dt = 76;\ M_{xy} = \int_C z\delta\ ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5+t)]\ dt$

$= \int_0^2 \left(10t^{3/2}+2t^{5/2}\right) dt = \left[4t^{5/2}+\frac{4}{7}t^{7/2}\right]_0^2 = 4(2)^{5/2}+\frac{4}{7}(2)^{7/2} = 16\sqrt{2}+\frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M}$

$= \frac{38}{36} = \frac{19}{18},\ \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7\cdot 36} = \frac{4}{7}\sqrt{2}$

27. Let $x = a\cos t$ and $y = a\sin t$, $0 \le t \le 2\pi$. Then $\frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = a\cos t$, $\frac{dz}{dt} = 0$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = a\,dt;\ I_z = \int_C (x^2+y^2)\delta\,ds = \int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)a\delta\,dt$$

$$= \int_0^{2\pi} a^3\delta\,dt = 2\pi\delta a^3;\ M = \int_C \delta(x,y,z)\,ds = \int_0^{2\pi} \delta a\,dt = 2\pi\delta a \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{2\pi a^3\delta}{2\pi a\delta}} = a.$$

28. $\mathbf{r}(t) = t\mathbf{j} + (2-2t)\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{5}$; $M = \int_C \delta\,ds = \int_0^1 \delta\sqrt{5}\,dt = \delta\sqrt{5}$;

$$I_x = \int_C (y^2+z^2)\delta\,ds = \int_0^1 [t^2 + (2-2t)^2]\delta\sqrt{5}\,dt = \int_0^1 (5t^2 - 8t + 4)\delta\sqrt{5}\,dt = \delta\sqrt{5}\left[\tfrac{5}{3}t^3 - 4t^2 + 4t\right]_0^1 = \tfrac{5}{3}\delta\sqrt{5};$$

$$I_y = \int_C (x^2+z^2)\delta\,ds = \int_0^1 [0^2 + (2-2t)^2]\delta\sqrt{5}\,dt = \int_0^1 (4t^2 - 8t + 4)\delta\sqrt{5}\,dt = \delta\sqrt{5}\left[\tfrac{4}{3}t^3 - 4t^2 + 4t\right]_0^1 = \tfrac{4}{3}\delta\sqrt{5};$$

$$I_z = \int_C (x^2+y^2)\delta\,ds = \int_0^1 (0^2 + t^2)\delta\sqrt{5}\,dt = \delta\sqrt{5}\left[\frac{t^3}{3}\right]_0^1 = \tfrac{1}{3}\delta\sqrt{5} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\tfrac{5}{3}},\ R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\tfrac{4}{3}} = \frac{2}{\sqrt{3}},$$

and $R_z = \sqrt{\frac{I_z}{M}} = \frac{1}{\sqrt{3}}$

29. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$;

(a) $M = \int_C \delta\,ds = \int_0^{2\pi} \delta\sqrt{2}\,dt = 2\pi\delta\sqrt{2}$; $I_z = \int_C (x^2+y^2)\delta\,ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t)\delta\sqrt{2}\,dt = 2\pi\delta\sqrt{2}$

$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$

(b) $M = \int_C \delta(x,y,z)\,ds = \int_0^{4\pi} \delta\sqrt{2}\,dt = 4\pi\delta\sqrt{2}$ and $I_z = \int_C (x^2+y^2)\delta\,ds = \int_0^{4\pi} \delta\sqrt{2}\,dt = 4\pi\delta\sqrt{2}$

$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$

30. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \sqrt{2}\mathbf{k}$

$$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(t+1)^2} = t+1 \text{ for } 0 \le t \le 1;\ M = \int_C \delta\,ds = \int_0^1 (t+1)\,dt = \left[\tfrac{1}{2}(t+1)^2\right]_0^1 = \tfrac{1}{2}(2^2 - 1^2) = \tfrac{3}{2};$$

$$M_{xy} = \int_C z\delta\,ds = \int_0^1 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)(t+1)\,dt = \frac{2\sqrt{2}}{3}\int_0^1 (t^{5/2} + t^{3/2})\,dt = \frac{2\sqrt{2}}{3}\left[\tfrac{2}{7}t^{7/2} + \tfrac{2}{5}t^{5/2}\right]_0^1$$

$= \frac{2\sqrt{2}}{3}\left(\frac{2}{7}+\frac{2}{5}\right) = \frac{2\sqrt{2}}{3}\left(\frac{24}{35}\right) = \frac{16\sqrt{2}}{35} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35}\right)\left(\frac{2}{3}\right) = \frac{32\sqrt{2}}{105}$; $I_z = \int_C (x^2+y^2)\delta\, ds$

$= \int_0^1 (t^2\cos^2 t + t^2\sin^2 t)(t+1)\, dt = \int_0^1 (t^3+t^2)\, dt = \left[\frac{t^4}{4}+\frac{t^3}{3}\right]_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{18}}$

31. $\delta(x,y,z) = 2 - z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, $0 \le t \le \pi \Rightarrow M = 2\pi - 2$ as found in Example 4 of the text; also $\left|\frac{d\mathbf{r}}{dt}\right| = 1$; $I_x = \int_C (y^2+z^2)\delta\, ds = \int_0^\pi (\cos^2 t + \sin^2 t)(2-\sin t)\, dt = \int_0^\pi (2-\sin t)\, dt = 2\pi - 2 \Rightarrow R_x = \sqrt{\frac{I_x}{M}}$ $= 1$

32. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}$, $0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2}\,t^{1/2}\mathbf{j} + t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = 1+t$ for $0 \le t \le 2$; $M = \int_C \delta\, ds = \int_0^2 \left(\frac{1}{t+1}\right)(1+t)\, dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta\, ds = \int_0^2 t\left(\frac{1}{t+1}\right)(1+t)\, dt = \left[\frac{t^2}{2}\right]_0^2 = 2$; $M_{xz} = \int_C y\delta\, ds = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\, dt = \left[\frac{4\sqrt{2}}{15}t^{5/2}\right]_0^2 = \frac{32}{15}$; $M_{xy} = \int_C z\delta\, ds = \int_0^2 \frac{t^2}{2}\, dt = \left[\frac{t^3}{6}\right]_0^2 = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = 1$, $\bar{y} = \frac{M_{xz}}{M} = \frac{16}{15}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{4}{3}$; $I_x = \int_C (y^2+z^2)\delta\, ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{1}{4}t^4\right) dt = \left[\frac{2}{9}t^4 + \frac{t^5}{20}\right]_0^2 = \frac{32}{9} + \frac{32}{20} = \frac{232}{45}$; $I_y = \int_C (x^2+z^2)\delta\, ds = \int_0^2 \left(t^2 + \frac{1}{4}t^4\right) dt = \left[\frac{t^3}{3} + \frac{t^5}{20}\right]_0^2 = \frac{8}{3} + \frac{32}{20} = \frac{64}{15}$; $I_z = \int_C (x^2+y^2)\delta\, ds$ $= \int_0^2 \left(t^2 + \frac{8}{9}t^3\right) dt = \left[\frac{t^3}{3} + \frac{2}{9}t^4\right]_0^2 = \frac{8}{3} + \frac{32}{9} = \frac{56}{9} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{2}{3}\sqrt{\frac{29}{5}}$, $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{32}{15}}$, and $R_z = \sqrt{\frac{I_z}{M}} = \frac{2}{3}\sqrt{7}$

33-36. Example CAS commands:

Maple:
```
x:= t -> cos(2*t); y:= t -> sin(2*t);
z:= t -> t^(5/2);
f:= (x,y,z) -> (1 + (9/4)*z^(1/3))^(1/4);
sqrt(D(x)(t)^2 + D(y)(t)^2 + D(z)(t)^2: absvee := unapply(%,t);
a:= 0: b:= 2*Pi:
integrand:= simplify(f(x(t),y(t),z(t))*absvee(t));
int(integrand,t=a..b);
evalf(%);
```

Mathematica:
```
Clear[x,y,z,t]
r[t_] = {x[t],y[t],z[t]}
f[x_,y_,z_] = (1 + 9/4 z^(1/3))^(1/4)
x[t_] = Cos[2 t]
y[t_] = Sin[2 t]
```

```
z[t_] = t^(5/2)
{a,b} = {0,2Pi};
v[t_] = r'[t]
s[t_] = Sqrt[ v[t] . v[t] ]
integrand = f[x[t],y[t],z[t]] s[t]
NIntegrate[ integrand, {t,a,b} ]
```

13.2 VECTOR FIELDS, WORK, CIRCULATION, AND FLUX

1. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$; similarly, $\frac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2} \Rightarrow \nabla f = \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$

2. $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2+z^2}\right)(2x) = \frac{x}{x^2+y^2+z^2}$;

   similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}$ and $\frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2} \Rightarrow \nabla f = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$

3. $g(x,y,z) = e^x - \ln(x^2+y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2+y^2}$, $\frac{\partial g}{\partial y} = -\frac{2y}{x^2+y^2}$ and $\frac{\partial g}{\partial z} = e^z$

   $\Rightarrow \nabla g = \left(\frac{-2x}{x^2+y^2}\right)\mathbf{i} - \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} + e^z\mathbf{k}$

4. $g(x,y,z) = xy+yz+xz \Rightarrow \frac{\partial g}{\partial x} = y+z$, $\frac{\partial g}{\partial y} = x+z$, and $\frac{\partial g}{\partial z} = y+x \Rightarrow \nabla g = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$

5. $|\mathbf{F}|$ inversely proportional to the square of the distance from (x,y) to the origin $\Rightarrow \sqrt{(M(x,y))^2+(N(x,y))^2}$ $= \frac{k}{x^2+y^2}$, $k > 0$; $\mathbf{F}$ points toward the origin $\Rightarrow \mathbf{F}$ is in the direction of $\mathbf{n} = \frac{-x}{\sqrt{x^2+y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$

   $\Rightarrow \mathbf{F} = a\mathbf{n}$, for some constant $a > 0$. Then $M(x,y) = \frac{-ax}{\sqrt{x^2+y^2}}$ and $N(x,y) = \frac{-ay}{\sqrt{x^2+y^2}}$

   $\Rightarrow \sqrt{(M(x,y))^2+(N(x,y))^2} = a \Rightarrow a = \frac{k}{x^2+y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2+y^2)^{3/2}}\mathbf{i} - \frac{ky}{(x^2+y^2)^{3/2}}\mathbf{j}$, for any constant $k > 0$

6. Given $x^2+y^2 = a^2+b^2$, let $x = \sqrt{a^2+b^2}\cos t$ and $y = -\sqrt{a^2+b^2}\sin t$. Then $\mathbf{r} = \left(\sqrt{a^2+b^2}\cos t\right)\mathbf{i} - \left(\sqrt{a^2+b^2}\sin t\right)\mathbf{j}$ traces the circle in a clockwise direction as t goes from 0 to 2π $\Rightarrow \mathbf{v} = \left(-\sqrt{a^2+b^2}\sin t\right)\mathbf{i} - \left(\sqrt{a^2+b^2}\cos t\right)\mathbf{j}$ is tangent to the circle in a clockwise direction. Thus, let $\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and $\mathbf{F}(0,0) = \mathbf{0}$.

7. Substitute the parametric representations for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 9t \Rightarrow W = \int_0^1 9t\, dt = \frac{9}{2}$

(b) $\mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow W = \int_0^1 (7t^2 + 16t^7)\, dt = \left[\frac{7}{3}t^3 + 2t^8\right]_0^1$
$= \frac{7}{3} + 2 = \frac{13}{3}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow W_1 = \int_0^1 5t\, dt = \frac{5}{2}$;

$\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow W_2 = \int_0^1 4t\, dt = 2 \Rightarrow W = W_1 + W_2 = \frac{9}{2}$

8. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2+1} \Rightarrow W = \int_0^1 \frac{1}{t^2+1}\, dt = \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$

(b) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} \Rightarrow W = \int_0^1 \frac{2t}{t^2+1}\, dt = \left[\ln(t^2+1)\right]_0^1 = \ln 2$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2+1}$; $\mathbf{F}_2 = \frac{1}{2}\mathbf{j}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k}$
$\Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow W = \int_0^1 \frac{1}{t^2+1}\, dt = \frac{\pi}{4}$

9. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow W = \int_0^1 (2\sqrt{t} - 2t)dt = \left[\frac{4}{3}t^{3/2} - t^2\right]_0^1 = \frac{1}{3}$

(b) $\mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow W = \int_0^1 (4t^4 - 3t^2)\, dt = \left[\frac{4}{5}t^5 - t^3\right]_0^1 = -\frac{1}{5}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow W_1 = \int_0^1 -2t\,dt$

$= -1$; $\mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1 \Rightarrow W = W_1 + W_2 = 0$

10. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow W = \int_0^1 3t^2\,dt = 1$

(b) $\mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow W = \int_0^1 \left(t^3 + 2t^7 + 4t^8\right) dt$

$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t^2\mathbf{i}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow W_1 = \int_0^1 t^2\,dt = \frac{1}{3}$;

$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow W_2 = \int_0^1 t\,dt = \frac{1}{2} \Rightarrow W = W_1 + W_2 = \frac{5}{6}$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow W = \int_0^1 \left(3t^2 + 1\right) dt = \left[t^3 + t\right]_0^1 = 2$

(b) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$

$\Rightarrow W = \int_0^1 \left(6t^5 + 4t^3 + 3t^2 - 3t\right) dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2\right]_0^1 = \frac{3}{2}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(3t^2 - 3t\right)\mathbf{i} + \mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$

$\Rightarrow W_1 = \int_0^1 \left(3t^2 - 3t\right) dt = \left[t^3 - \frac{3}{2}t^2\right]_0^1 = -\frac{1}{2}$; $\mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1$

$\Rightarrow W = W_1 + W_2 = \frac{1}{2}$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow W = \int_0^1 6t\, dt = \left[3t^2\right]_0^1 = 3$

(b) $\mathbf{F} = (t^2 + t^4)\mathbf{i} + (t^4 + t)\mathbf{j} + (t + t^2)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$

$\Rightarrow W = \int_0^1 (6t^5 + 5t^4 + 3t^2)\, dt = \left[t^6 + t^5 + t^3\right]_0^1 = 3$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t$

$\Rightarrow W_1 = \int_0^1 2t\, dt = 1$; $\mathbf{F}_2 = (1 + t)\mathbf{i} + (t + 1)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow W_2 = \int_0^1 2\, dt = 2$

$\Rightarrow W = W_1 + W_2 = 3$

13. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow$ work $= \int_0^1 2t^3\, dt = \frac{1}{2}$

14. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$

$\Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$

$= 5 \cos^2 t - 2 + \frac{1}{6} \cos t + \frac{1}{6} \sin t \Rightarrow$ work $= \int_0^{2\pi} \left(5 \cos^2 t - 2 + \frac{1}{6} \cos t + \frac{1}{6} \sin t\right) dt$

$= \left[\frac{5}{2}t + \frac{5}{4} \sin 2t - 2t + \frac{1}{6} \sin t - \frac{1}{6} \cos t\right]_0^{2\pi} = 5\pi - 4\pi = \pi$

15. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$ and

$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin^2 t + \cos t \Rightarrow$ work $= \int_0^{2\pi} \left(t \cos t - \sin^2 t + \cos t\right) dt$

$= \left[\cos t + t \sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t\right]_0^{2\pi} = -\pi$

16. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12 \sin t)\mathbf{k}$ and

$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin t \cos^2 t + 2 \sin t$

$$\Rightarrow \text{work} = \int_0^{2\pi} \left(t \cos t - \sin t \cos^2 t + 2 \sin t\right) dt = \left[\cos t + t \sin t + \tfrac{1}{3}\cos^3 t - 2\cos t\right]_0^{2\pi} = 0$$

17. $x = t$ and $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j},\ -1 \le t \le 2$, and $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + \left(t + t^2\right)\mathbf{j}$ and

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t^3 + \left(2t^2 + 2t^3\right) = 3t^3 + 2t^2 \Rightarrow \int_C xy\,dx + (x+y)\,dy = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_{-1}^{2} \left(3t^3 + 2t^2\right) dt$$

$$= \left[\tfrac{3}{4}t^4 + \tfrac{2}{3}t^3\right]_{-1}^{2} = \left(12 + \tfrac{16}{3}\right) - \left(\tfrac{3}{4} - \tfrac{2}{3}\right) = \tfrac{45}{4} + \tfrac{18}{3} = \tfrac{69}{4}$$

18. Along $(0,0)$ to $(1,0)$: $\mathbf{r} = t\mathbf{i},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t$;

Along $(1,0)$ to $(0,1)$: $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$ and

$\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 2t$;

Along $(0,1)$ to $(0,0)$: $\mathbf{r} = (1-t)\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$ and

$$\frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t - 1 \Rightarrow \int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt = \int_0^1 (4t-1)\,dt$$

$$= \left[2t^2 - t\right]_0^1 = 2 - 1 = 1$$

19. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j},\ 2 \ge y \ge -1$, and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F}\cdot\frac{d\mathbf{r}}{dy} = 2y^5 - y$

$$\Rightarrow \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_2^{-1} \mathbf{F}\cdot\frac{d\mathbf{r}}{dy}\,dy = \int_2^{-1} \left(2y^5 - y\right) dy = \left[\tfrac{1}{3}y^6 - \tfrac{1}{2}y^2\right]_2^{-1} = \left(\tfrac{1}{3} - \tfrac{1}{2}\right) - \left(\tfrac{64}{3} - \tfrac{4}{2}\right) = \tfrac{3}{2} - \tfrac{63}{3} = -\tfrac{39}{2}$$

20. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$, and $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-1)\,dt = -\tfrac{\pi}{2}$$

21. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = \left(1 + 3t + 2t^2\right)\mathbf{i} + t\mathbf{j}$ and

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 1 + 5t + 2t^2 \Rightarrow \text{work} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^1 \left(1 + 5t + 2t^2\right) dt = \left[t + \tfrac{5}{2}t^2 + \tfrac{2}{3}t^3\right]_0^1 = \tfrac{25}{6}$$

22. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ 0 \le t \le 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$

$\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}$

$$= -8\left(\sin t\cos t + \sin^2 t\right) + 8\left(\cos^2 t + \cos t\sin t\right) = 8\left(\cos^2 t - \sin^2 t\right) = 8\cos 2t \Rightarrow \text{work} = \int_C \nabla f\cdot d\mathbf{r}$$

$$= \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} 8\cos 2t\,dt = [4\sin 2t]_0^{2\pi} = 0$$

23. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$,

$\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$

$\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0\, dt = 0$ and $\text{Circ}_2 = \int_0^{2\pi} dt = 2\pi$; $\mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1$ and

$\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi$ and $\text{Flux}_2 = \int_0^{2\pi} 0\, dt = 0$

(b) $\mathbf{r} = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4 \cos t)\mathbf{j}$, $\mathbf{F}_1 = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, and

$\mathbf{F}_2 = (-4 \sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15 \sin t \cos t$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \sin t \cos t\, dt$

$= \left[\frac{15}{2} \sin^2 t\right]_0^{2\pi} = 0$ and $\text{Circ}_2 = \int_0^{2\pi} 4\, dt = 8\pi$; $\mathbf{n} = \left(\frac{4}{\sqrt{17}} \cos t\right)\mathbf{i} + \left(\frac{1}{\sqrt{17}} \sin t\right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$

$= \frac{4}{\sqrt{17}} \cos^2 t + \frac{4}{\sqrt{17}} \sin^2 t$ and $\mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}} \sin t \cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n})\, |\mathbf{v}|\, dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}}\right)\sqrt{17}\, dt$

$= 8\pi$ and $\text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n})\, |\mathbf{v}|\, dt = \int_0^{2\pi} \left(-\frac{15}{\sqrt{17}} \sin t \cos t\right) \sqrt{17}\, dt = \left[-\frac{15}{2} \sin^2 t\right]_0^{2\pi} = 0$

24. $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \le t \le 2\pi$, $\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$, and $\mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$,
$\mathbf{F}_1 = (2a \cos t)\mathbf{i} - (3a \sin t)\mathbf{j}$, and $\mathbf{F}_2 = (2a \cos t)\mathbf{i} + (a \cos t - a \sin t)\mathbf{j} \Rightarrow \mathbf{n}\,|\mathbf{v}| = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$,
$\mathbf{F}_1 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2 \cos^2 t - 3a^2 \sin^2 t$, and $\mathbf{F}_2 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t$

$\Rightarrow \text{Flux}_1 = \int_0^{2\pi} \left(2a^2 \cos^2 t + 3a^2 \sin^2 t\right) dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$, and

$\text{Flux}_2 = \int_0^{2\pi} \left(2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t\right) dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} + \frac{a^2}{2}\left[\sin^2 t\right]_0^{2\pi} - a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi a^2$

25. $\mathbf{F}_1 = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0$; $M_1 = a \cos t$,

$N_1 = a \sin t$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} \left(a^2 \cos^2 t + a^2 \sin^2 t\right) dt$

$= \int_0^{\pi} a^2\, dt = a^2\pi$;

$\mathbf{F}_2 = t\mathbf{i}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t\,dt = 0$; $M_2 = t$, $N_2 = 0$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2$

$= \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} 0\,dt = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2\pi$

26. $\mathbf{F}_1 = (a^2 \cos^2 t)\mathbf{i} + (a^2 \sin^2 t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t$

$\Rightarrow \text{Circ}_1 = \int_0^{\pi} (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t)\,dt = -\frac{2a^3}{3}$; $M_1 = a^2 \cos^2 t$, $N_1 = a^2 \sin^2 t$, $dy = a \cos t\,dt$,

$dx = -a \sin t\,dt \Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi} (a^3 \cos^3 t + a^3 \sin^3 t)\,dt = \frac{4}{3}a^3$;

$\mathbf{F}_2 = t^2\mathbf{i}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t^2\,dt = \frac{2a^3}{3}$; $M_2 = t^2$, $N_2 = 0$, $dy = 0$, $dx = dt$

$\Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3}a^3$

27. $\mathbf{F}_1 = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2$

$\Rightarrow \text{Circ}_1 = \int_0^{\pi} a^2\,dt = a^2\pi$; $M_1 = -a \sin t$, $N_1 = a \cos t$, $dx = -a \sin t\,dt$, $dy = a \cos t\,dt$

$\Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi} (-a^2 \sin t \cos t + a^2 \sin t \cos t)\,dt = 0$; $\mathbf{F}_2 = t\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$

$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} -t\,dt = 0$; therefore,

$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2\pi$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

28. $\mathbf{F}_1 = (-a^2 \sin^2 t)\mathbf{i} + (a^2 \cos^2 t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t$

$\Rightarrow \text{Circ}_1 = \int_0^{\pi} (a^2 \sin^3 t + a^3 \cos^3 t)\,dt = \frac{4}{3}a^3$; $M_1 = -a^2 \sin^2 t$, $N_1 = a^2 \cos^2 t$, $dy = a \cos t\,dt$, $dx = -a \sin t\,dt$

$\Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi} (-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t)\,dt = \frac{2}{3}a^3$; $\mathbf{F}_2 = t^2\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$

$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} -t^2\,dt = -\frac{2}{3}a^3$; therefore,

$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3}a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and

$\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds$

$= \int_0^{\pi} (-\sin t \cos t - \sin^2 t - \cos t)\, dt = \left[-\frac{1}{2}\sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t\right]_0^{\pi} = -\frac{\pi}{2}$

(b) $\mathbf{r} = (1-2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\mathbf{j} \Rightarrow$

$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t - 1 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^1 (2t-1)\, dt = \left[t^2 - t\right]_0^1 = 0$

(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2\, dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}$,

$0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - (t^2 + t^2 - 2t + 1)\mathbf{j}$

$= -\mathbf{i} - (2t^2 - 2t + 1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2 - 2t + 1) = 2t - 2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t - 2t^2)\, dt$

$= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

30. From $(1,0)$ to $(0,1)$: $\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,

$\mathbf{F} = \mathbf{i} - (1 - 2t + 2t^2)\mathbf{j}$, and $\mathbf{n}_1 |\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1 |\mathbf{v}_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t - 2t^2)\, dt$

$= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3}$;

From $(0,1)$ to $(-1,0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,

$\mathbf{F} = (1-2t)\mathbf{i} - (1 - 2t + 2t^2)\mathbf{j}$, and $\mathbf{n}_2 |\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2 |\mathbf{v}_2| = (2t-1) + (-1 + 2t - 2t^2) = -2 + 4t - 2t^2$

$\Rightarrow \text{Flux}_2 = \int_0^1 (-2 + 4t - 2t^2)\, dt = \left[-2t + 2t^2 - \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}$;

From $(-1,0)$ to $(1,0)$: $\mathbf{r}_3 = (-1+2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,

$\mathbf{F} = (-1+2t)\mathbf{i} - (1 - 4t + 4t^2)\mathbf{j}$, and $\mathbf{n}_3 |\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3 |\mathbf{v}_3| = 2(1 - 4t + 4t^2)$

$\Rightarrow \text{Flux}_3 = 2\int_0^1 (1 - 4t + 4t^2)\, dt = 2\left[t - 2t^2 + \frac{4}{3}t^3\right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

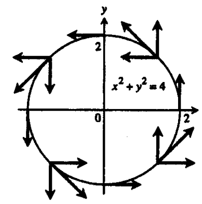
31. $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2+y^2=4$;

at $(2,0)$, $\mathbf{F} = \mathbf{j}$; at $(0,2)$, $\mathbf{F} = -\mathbf{i}$; at $(-2,0)$,

$\mathbf{F} = -\mathbf{j}$; at $(0,-2)$, $\mathbf{F} = \mathbf{i}$; at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$;

at $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$,

$\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



32. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $x^2+y^2=1$; at $(1,0)$, $\mathbf{F} = \mathbf{i}$;

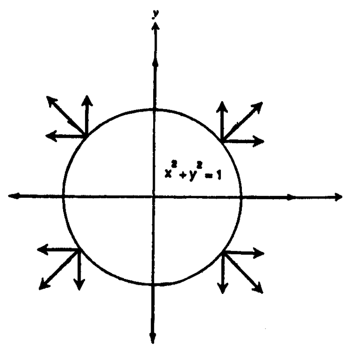
at $(-1,0)$, $\mathbf{F} = -\mathbf{i}$; at $(0,1)$, $\mathbf{F} = \mathbf{j}$; at $(0,-1)$,

$\mathbf{F} = -\mathbf{j}$; at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$;

at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$;

at $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$,

$\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$.



33. (a) $\mathbf{G} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2+b^2}$ and to be tangent to $x^2+y^2=a^2+b^2$ in a counterclockwise direction. Thus $x^2+y^2=a^2+b^2 \Rightarrow 2x+2yy'=0 \Rightarrow y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y' = -\frac{a}{b}$ at (a,b). Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2+b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. Then let $P(x,y) = -y$ and $Q(x,y) = x$ $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$ for (a,b) on $x^2+y^2=a^2+b^2$ we have $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$ and $|\mathbf{G}| = \sqrt{a^2+b^2}$.

(b) $\mathbf{G} = \left(\sqrt{x^2+y^2}\right)\mathbf{F} = \left(\sqrt{a^2+b^2}\right)\mathbf{F}$, since $x^2+y^2=a^2+b^2$

34. (a) From Exercise 33, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G} = -\mathbf{F}$

35. The slope of the line through (x,y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

36. (a) From Exercise 35, $-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x,y) pointing toward the origin and we want $|\mathbf{F}|$ to have magnitude $\sqrt{x^2+y^2} \Rightarrow \mathbf{F} = \sqrt{x^2+y^2}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -x\mathbf{i} - y\mathbf{j}$.

(b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2+y^2}} \Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}\right)$, $C \neq 0$, and constant

37. $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3\,dt = \left[3t^4\right]_0^2 = 48$

38. $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2\,dt = \left[24t^3\right]_0^1 = 24$

39. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin t\cos t + 1$

$\Rightarrow \text{Flow} = \int_0^{\pi} (-\sin t\cos t + 1)\,dt = \left[\frac{1}{2}\cos^2 t + t\right]_0^{\pi} = \left(\frac{1}{2}+\pi\right) - \left(\frac{1}{2}+0\right) = \pi$

40. $\mathbf{F} = (-2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -4\sin^2 t - 4\cos^2 t + 4 = 0$
$\Rightarrow \text{Flow} = 0$

41. C_1: $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \frac{\pi}{2} \Rightarrow \mathbf{F} = (2\cos t)\mathbf{i} + 2t\mathbf{j} + (2\sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -2\cos t\sin t + 2t\cos t + 2\sin t = -\sin 2t + 2t\cos t + 2\sin t$

$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t\cos t + 2\sin t)\,dt = \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2} = -1 + \pi;$

C_2: $\mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}$, $0 \le t \le 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\pi$

$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi\,dt = [-\pi t]_0^1 = -\pi;$

C_3: $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 2t$

$\Rightarrow \text{Flow}_3 = \int_0^1 2t\,dt = \left[t^2\right]_0^1 = 1 \Rightarrow \text{Circulation} = (-1+\pi) - \pi + 1 = 0$

```
a:=0; b:= Pi;
M:= (x,y,z) ->3/(1 + x^2);
N:= (x,y,z) ->2/(1 + y^2);
P:= (x,y,z) -> 0;
F:= t -> vector([M(x(t),y(t),z(t)), N(x(t),y(t),z(t)), P(x(t),y(t),z(t))]);
dr:= t -> vector([D(x)(t), D(y)(t), D(z)(t)]);
integrand:= dotprod(F(t), dr(t), orthogonal);
int(integrand, t=a..b);
evalf(%);
```

Mathematica:

```
Clear[x,y,z,t]
r[t_] = {x[t],y[t],z[t]}
f[x_,y_] = {
 3/(1+x^2) ,
 2/(1+y^2) }
x[t_] = Cos[t]
y[t_] = Sin[t]
z[t_] = 0
{a,b} = {0,Pi};
v[t_] = r'[t]
integrand = f[x[t],y[t],z[t]] . v[t]
integrand = Simplify[ integrand ]
```

13.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$

8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x,y,z) = (y+z)x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y,z) = zy + h(z)$
$\Rightarrow f(x,y,z) = (y+z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$
$= (y+z)x + zy + C$

9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x,y,z) = xe^{y+2z} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x,y,z)$
$= xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{y+2z} + C$

10. $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x,y,z) = xy \sin z + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$
$\Rightarrow f(x,y,z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) + C \Rightarrow f(x,y,z)$
$= xy \sin z + C$

11. $\frac{\partial f}{\partial z} = \frac{z}{y^2 + z^2} \Rightarrow f(x,y,z) = \frac{1}{2} \ln(y^2 + z^2) + g(x,y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2(x+y) \Rightarrow g(x,y)$
$= (x \ln x - x) + \tan(x+y) + h(y) \Rightarrow f(x,y,z) = \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x+y) + h(y)$
$\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2 + z^2} + \sec^2(x+y) + h'(y) = \sec^2(x+y) + \frac{y}{y^2 + z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x,y,z)$
$= \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x+y) + C$

12. $\frac{\partial f}{\partial x} = \frac{y}{1 + x^2y^2} \Rightarrow f(x,y,z) = \tan^{-1}(xy) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1 + x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1 + x^2y^2} + \frac{z}{\sqrt{1 - y^2z^2}}$
$\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1 - y^2z^2}} \Rightarrow g(y,z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1 - y^2z^2}} + h'(z) = \frac{y}{\sqrt{1 - y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C$
$\Rightarrow f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$

13. Let $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y,z) = y^2 + h(z) \Rightarrow f(x,y,z) = x^2 + y^2 = h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x,y,z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$
$= f(2,3,-6) - f(0,0,0) = 2^2 + 3^2 + (-6)^2 = 49$

14. Let $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = yz \Rightarrow f(x,y,z) = xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z)$
$= xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xyz + C$
$\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = f(3,5,0) - f(1,1,2) = 0 - 2 = -2$

15. Let $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y,z) = x^2y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$
$\Rightarrow g(y,z) = -yz^2 + h(z) \Rightarrow f(x,y,z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x,y,z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz = f(1,2,3) - f(0,0,0) = 2 - 2(3)^2 = -16$

16. Let $\mathbf{F}(x,y,z) = 2x\mathbf{i} - y^2\mathbf{j} - \left(\frac{4}{1+z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y,z) = -\frac{y^3}{3} + h(z)$

$\Rightarrow f(x,y,z) = x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4\tan^{-1} z + C \Rightarrow f(x,y,z)$

$= x^2 - \frac{y^3}{3} - 4\tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1-z^2}\,dz = f(3,3,1) - f(0,0,0)$

$= \left(9 - \frac{27}{3} - 4\cdot\frac{\pi}{4} + C\right) - (0 - 0 - 0 + C) = -\pi$

17. Let $\mathbf{F}(x,y,z) = (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x,y,z) = \sin y \sin x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y}$

$= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C$

$\Rightarrow f(x,y,z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz = f(0,1,1) - f(1,0,0)$

$= (0 + 1 + C) - (0 + 0 + C) = 1$

18. Let $\mathbf{F}(x,y,z) = (2\cos y)\mathbf{i} + \left(\frac{1}{y} - 2x\sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x,y,z) = 2x\cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y}$

$= \frac{1}{y} - 2x\sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y,z) = \ln|y| + h(z) \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z}$

$\Rightarrow h(z) = \ln|z| + C \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C$

$\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz = f\left(1,\frac{\pi}{2},2\right) - f(0,2,1)$

$= \left(2\cdot 0 + \ln\frac{\pi}{2} + \ln 2 + C\right) - (0\cdot\cos 2 + \ln 2 + \ln 1 + C) = \ln\frac{\pi}{2}$

19. Let $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z\ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x,y,z) = x^3 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y,z) = z^2\ln y + h(z)$

$\Rightarrow f(x,y,z) = x^3 + z^2\ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z\ln y + h'(z) = 2z\ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$

$= x^3 + z^2 \ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z \ln y\,dz = f(1,2,3) - f(1,1,1)$

$= (1 + 9 \ln 2 + C) - (1 + 0 + C) = 9 \ln 2$

20. Let $\mathbf{F}(x,y,z) = (2x \ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \ln y - yz \Rightarrow f(x,y,z) = x^2 \ln y - xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y}$

$= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2 \ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2 \ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

$= f(2,1,1) - f(1,2,1) = (4 \ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2$

21. Let $\mathbf{F}(x,y,z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x,y,z) = \frac{x}{y} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}$

$\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y,z) = \frac{y}{z} + h(z) \Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz = f(2,2,2) - f(1,1,1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right)$

$= 0$

22. Let $\mathbf{F}(x,y,z) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2 + y^2 + z^2}\left(\text{and let } \rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho}\right)$

$\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact;

$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} + h'(z)$

$= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + C$

$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = f(2,2,2) - f(-1,-1,-1) = \ln 12 - \ln 3 = \ln 4$

23. $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = (1 + t)\mathbf{i} + (1 + 2t)\mathbf{j} + (1 - 2t)\mathbf{k} \Rightarrow dx = dt,\ dy = 2\,dt,\ dz = -2\,dt$

$\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = \int_0^1 (2t + 1)\,dt + (t + 1)(2\,dt) + 4(-2)\,dt = \int_0^1 (4t - 5)\,dt = \left[2t^2 - 5t\right]_0^1 = -3$

24. $\mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}) \Rightarrow dx = 0,\ dy = 3\ dt,\ dz = 4\ dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2\ dx + yz\ dy + \left(\frac{y^2}{2}\right) dz$

$= \int_0^1 (12t^2)(3\ dt) + \left(\frac{9t^2}{2}\right)(4\ dt) = \int_0^1 54t^2\ dt = \left[18t^2\right]_0^1 = 18$

25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$ is exact $\Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ path independence

26. $\frac{\partial P}{\partial y} = -\frac{yz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = -\frac{xz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -\frac{xy}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial M}{\partial y}$

$\Rightarrow M\ dx + N\ dy + P\ dz$ is exact $\Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ path independence

27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$

$\Rightarrow f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^2-1}{y}\right)$

28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x,y,z) = e^x \ln y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y,z)$

$= y \sin z + h(z) \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla\left(e^x \ln y + y \sin z\right)$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y,z) = \frac{1}{3}y^3 + h(z)$

$\Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C \Rightarrow f(x,y,z)$

$= \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$

(a) $\text{work} = \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\ dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right)$

$= 1$

(b) $\text{work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) $\text{work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

Note: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is independent of the path from $(1,0,0)$ to $(1,0,1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x,y,z) = xe^{yz} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z\cos y \Rightarrow \frac{\partial g}{\partial y} = z\cos y$ $\Rightarrow g(y,z) = z\sin y + h(z) \Rightarrow f(x,y,z) = xe^{yz} + z\sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$ $\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{yz} + z\sin y + C \Rightarrow \mathbf{F} = \nabla\left(xe^{yz} + z\sin y\right)$

(a) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0) - (1+0) = 0$

(b) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

(c) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

Note: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is independent of the path from $(1,0,1)$ to $\left(1,\frac{\pi}{2},0\right)$.

31. (a) $\mathbf{F} = \nabla\left(x^3y^2\right) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from $(-1,1)$ to $(0,0) \Rightarrow x = t-1$ and $y = -t+1$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$ and $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\,\mathbf{i} - dt\,\mathbf{j} \Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 = \int_0^1 \left[3(t-1)^4 + 2(t-1)^4\right] dt$ $= \int_0^1 5(t-1)^4\, dt = \left[(t-1)^5\right]_0^1 = 1$; let C_2 be the path from $(0,0)$ to $(1,1) \Rightarrow x = t$ and $y = t$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\,\mathbf{i} + dt\,\mathbf{j} \Rightarrow \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 = \int_0^1 \left(3t^4 + 2t^4\right) dt$ $= \int_0^1 5t^4\, dt = 1 \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 = 2$

(b) Since $f(x,y) = x^3y^2$ is a potential function for $\mathbf{F}$, $\int_{(-1,1)}^{(1,1)} \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(-1,1) = 2$

32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = 2x \cos y \Rightarrow f(x,y,z) = x^2 \cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$
$\Rightarrow f(x,y,z) = x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla(x^2 \cos y)$

(a) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all $y \Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all $y \Rightarrow b = 2a$ and $c = 2a$

(b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$

34. $\mathbf{F} = \nabla f \Rightarrow g(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x,y,z) - f(0,0,0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0$, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and
$\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2.

37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B), respectively. The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is $f(x,y,z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is
$f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) = (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C)$
$= a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \overrightarrow{AB}$

38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\right]$
$\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$
$= \nabla f$ for some f; $\frac{\partial f}{\partial x} = \frac{xC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + g(y,z)$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2+y^2+z^2)^{3/2}} + h'(z) = \frac{zC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow h(z) = C_1 \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + C_1.$

Let $C_1 = 0 \Rightarrow f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is a potential function for $\mathbf{F}$.

(b) If s is the distance of (x,y,z) from the origin, then $s = \sqrt{x^2+y^2+z^2}$. The work done by the gravitational field $\mathbf{F}$ is work $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}}\right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$, as claimed.

13.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a \sin t$, $N = x = a \cos t$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

   Equation (3): $\oint_C M\, dy - N\, dx = \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)]\, dt = \int_0^{2\pi} 0\, dt = 0$;

   $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R 0\, dx\, dy = 0$, Flux

   Equation (4): $\oint_C M\, dx + N\, dy = \int_0^{2\pi} [(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)]\, dt = \int_0^{2\pi} a^2\, dt = 2\pi a^2$;

   $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 2\, dx\, dy = \int_{-a}^{a} 4\sqrt{a^2-y^2}\, dy = 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}\right]_{-a}^{a}$

   $= 2a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2a^2\pi$, Circulation

2. $M = y = a \sin t$, $N = 0$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;

   Equation (3): $\oint_C M\, dy - N\, dx = \int_0^{2\pi} a^2 \sin t \cos t\, dt = a^2\left[\frac{1}{2} \sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\, dx\, dy = 0$, Flux

   Equation (4): $\oint_C M\, dx + N\, dy = \int_0^{2\pi} \left(-a^2 \sin^2 t\right) dt = -a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^a -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Circulation

3. $M = 2x = 2a\cos t$, $N = -3y = -3a\sin t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 2$, $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} [(2a\cos t)(a\cos t) + (3a\sin t)(-a\sin t)]\,dt$

$= \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right)dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^a -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi} [(2a\cos t)(-a\sin t) + (-3a\sin t)(a\cos t)]\,dt$

$= \int_0^{2\pi} \left(-2a^2\sin t\cos t - 3a^2\sin t\cos t\right)dt = -5a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\,dx\,dy = 0$, Circulation

4. $M = -x^2y = -a^3\cos^2 t\sin t$, $N = xy^2 = a^3\cos t\sin^2 t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$ $\Rightarrow \frac{\partial M}{\partial x} = -2a^2\cos t\sin t$, $\frac{\partial M}{\partial y} = -a^2\cos^2 t$, $\frac{\partial N}{\partial x} = a^2\sin^2 t$, and $\frac{\partial N}{\partial y} = 2a^2\cos t\sin t$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} \left(-a^4\cos^3 t\sin t + a^4\cos t\sin^3 t\right) = \left[\frac{a^4}{4}\cos^4 t + \frac{a^4}{4}\sin^4 t\right]_0^{2\pi} = 0$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint_R (-2xy + 2xy)\,dx\,dy = 0$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi} \left(a^4\cos^2 t\sin^2 t + a^4\cos^2 t\sin^2 t\right)dt = \int_0^{2\pi} \left(2a^4\cos^2 t\sin^2 t\right)dt$

$= \int_0^{2\pi} \frac{1}{2}a^4\sin^2 2t\,dt = \frac{a^4}{4}\int_0^{4\pi} \sin^2 u\,du = \frac{a^4}{4}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{\pi a^4}{2}$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_R \left(y^2 + x^2\right)dx\,dy$

$= \int_0^{2\pi}\int_0^a r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{a^4}{4}\,d\theta = \frac{\pi a^4}{2}$, Circulation

5. $M = x - y$, $N = y - x \Rightarrow \frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$, $\frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R 2\,dx\,dy = \int_0^1\int_0^1 2\,dx\,dy = 2$;

$\text{Circ} = \iint_R [-1 - (-1)]\,dx\,dy = 0$

6. $M = x^2 + 4y,\ N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x,\ \frac{\partial M}{\partial y} = 4,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (2x + 2y)\, dx\, dy$

$= \int_0^1 \int_0^1 (2x + 2y)\, dx\, dy = \int_0^1 \left[x^2 + 2xy\right]_0^1 dy = \int_0^1 (1 + 2y)\, dy = \left[y + y^2\right]_0^1 = 2;\ \text{Circ} = \iint_R (1 - 4)\, dx\, dy$

$= \int_0^1 \int_0^1 -3\, dx\, dy = -3$

7. $M = y^2 - x^2,\ N = x^2 + y^2 \Rightarrow \frac{\partial M}{\partial x} = -2x,\ \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (-2x + 2y)\, dx\, dy$

$= \int_0^3 \int_0^x (-2x + 2y)\, dy\, dx = \int_0^3 \left(-2x^2 + x^2\right) dx = \left[-\frac{1}{3}x^3\right]_0^3 = -9;\ \text{Circ} = \iint_R (2x - 2y)\, dx\, dy$

$= \int_0^3 \int_0^x (2x - 2y)\, dy\, dx = \int_0^3 x^2\, dx = 9$

8. $M = x + y,\ N = -\left(x^2 + y^2\right) \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -2x,\ \frac{\partial N}{\partial y} = -2y \Rightarrow \text{Flux} = \iint_R (1 - 2y)\, dx\, dy$

$= \int_0^1 \int_0^x (1 - 2y)\, dy\, dx = \int_0^1 \left(x - x^2\right) dx = \frac{1}{6};\ \text{Circ} = \iint_R (-2x - 1)\, dx\, dy = \int_0^1 \int_0^x (-2x - 1)\, dy\, dx$

$= \int_0^1 \left(-2x^2 - x\right) dx = -\frac{7}{6}$

9. $M = x + e^x \sin y,\ N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y,\ \frac{\partial M}{\partial y} = e^x \cos y,\ \frac{\partial N}{\partial x} = 1 + e^x \cos y,\ \frac{\partial N}{\partial y} = -e^x \sin y$

$\Rightarrow \text{Flux} = \iint_R dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2};$

$\text{Circ} = \iint_R \left(1 + e^x \cos y - e^x \cos y\right) dx\, dy = \iint_R dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}$

10. $M = \tan^{-1}\frac{y}{x},\ N = \ln\left(x^2 + y^2\right) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial M}{\partial y} = \frac{x}{x^2 + y^2},\ \frac{\partial N}{\partial x} = \frac{2x}{x^2 + y^2},\ \frac{\partial N}{\partial y} = \frac{2y}{x^2 + y^2}$

$\Rightarrow \text{Flux} = \iint_R \left(\frac{-y}{x^2 + y^2} + \frac{2y}{x^2 + y^2}\right) dx\, dy = \int_0^\pi \int_1^2 \left(\frac{r \sin \theta}{r^2}\right) r\, dr\, d\theta = \int_0^\pi \sin \theta\, d\theta = 2;$

$\text{Circ} = \iint_R \left(\frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2}\right) dx\, dy = \int_0^\pi \int_1^2 \left(\frac{r \cos \theta}{r^2}\right) r\, dr\, d\theta = \int_0^\pi \cos \theta\, d\theta = 0$

11. $M = xy,\ N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y,\ \frac{\partial M}{\partial y} = x,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y)\, dy\, dx = \int_0^1 \int_{x^2}^{x} 3y\, dy\, dx$

$= \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2}\right) dx = \frac{1}{5};\ \text{Circ} = \iint_R -x\, dy\, dx = \int_0^1 \int_{x^2}^{x} -x\, dy\, dx = \int_0^1 \left(-x^2 + x^3\right) dx = -\frac{1}{12}$

12. $M = -\sin y,\ N = x\cos y \Rightarrow \frac{\partial M}{\partial x} = 0,\ \frac{\partial M}{\partial y} = -\cos y,\ \frac{\partial N}{\partial x} = \cos y,\ \frac{\partial N}{\partial y} = -x\sin y$

$\Rightarrow \text{Flux} = \iint_R (-x\sin y)\, dx\, dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x\sin y)\, dx\, dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8}\sin y\right) dy = -\frac{\pi^2}{8};$

$\text{Circ} = \iint_R [\cos y - (-\cos y)]\, dx\, dy = \int_0^{\pi/2} \int_0^{\pi/2} 2\cos y\, dx\, dy = \int_0^{\pi/2} \pi\cos y\, dy = \left[\pi \sin y\right]_0^{\pi/2} = \pi$

13. $M = 3xy - \frac{x}{1+y^2},\ N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2},\ \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$

$\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\, dy = \iint_R 3y\, dx\, dy = \int_0^{2\pi} \int_0^{a(1+\cos\theta)} (3r\sin\theta)\, r\, dr\, d\theta$

$= \int_0^{2\pi} a^3(1+\cos\theta)^3(\sin\theta)\, d\theta = \left[-\frac{a^3}{4}(1+\cos\theta)^4\right]_0^{2\pi} = -4a^3 - \left(-4a^3\right) = 0$

14. $M = y + e^x \ln y,\ N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y},\ \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y}\right)\right] dx\, dy = \iint_R (-1)\, dx\, dy$

$= \int_{-1}^{1} \int_{x^4+1}^{3-x^2} -\, dy\, dx = -\int_{-1}^{1} \left[\left(3 - x^2\right) - \left(x^4 + 1\right)\right] dx = \int_{-1}^{1} \left(x^4 + x^2 - 2\right) dx = -\frac{23}{15}$

15. $M = 2xy^3,\ N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2,\ \frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3\, dx + 4x^2y^2\, dy = \iint_R \left(8xy^2 - 6xy^2\right) dx\, dy$

$= \int_0^1 \int_0^{x^3} 2xy^2\, dy\, dx = \int_0^1 \frac{2}{3}x^{10}\, dx = \frac{2}{33}$

16. $M = 4x - 2y,\ N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y)\, dx + (2x - 4y)\, dy$

$= \iint_R [2 - (-2)]\, dx\, dy = 4 \iint_R dx\, dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$

17. $M = y^2,\ N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2\, dx + x^2\, dy = \iint_R (2x - 2y)\, dy\, dx$

$= \int_0^1 \int_0^{1-x} (2x - 2y)\, dy\, dx = \int_0^1 \left(-3x^2 + 4x - 1\right) dx = \left[-x^3 + 2x^2 - x\right]_0^1 = -1 + 2 - 1 = 0$

18. $M = 3y,\ N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y\, dx + 2x\, dy = \iint_R (2-3)\, dx\, dy = \int_0^x \int_0^{\sin x} -1\, dy\, dx$

$= -\int_0^\pi \sin x\, dx = -2$

19. $M = 6y + x,\ N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x)\, dx + (y + 2x)\, dy = \iint_R (2-6)\, dy\, dx$

$= -4(\text{Area of the circle}) = -16\pi$

20. $M = 2x + y^2,\ N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C \left(2x + y^2\right) dx + (2xy + 3y)\, dy = \iint_R (2y - 2y)\, dx\, dy = 0$

21. $M = x = a \cos t,\ N = y = a \sin t \Rightarrow dx = -a \sin t\, dt,\ dy = a \cos t\, dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_0^{2\pi} \left(a^2 \cos^2 t + a^2 \sin^2 t\right) dt = \frac{1}{2} \int_0^{2\pi} a^2\, dt = \pi a^2$

22. $M = x = a \cos t,\ N = y = b \sin t \Rightarrow dx = -a \sin t\, dt,\ dy = b \cos t\, dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_0^{2\pi} \left(ab \cos^2 t + ab \sin^2 t\right) dt = \frac{1}{2} \int_0^{2\pi} ab\, dt = \pi ab$

23. $M = x = a \cos^3 t,\ N = y = \sin^3 t \Rightarrow dx = -3 \cos^2 t \sin t\, dt,\ dy = 3 \sin^2 t \cos t\, dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_0^{2\pi} \left(3 \sin^2 t \cos^2 t\right)\left(\cos^2 t + \sin^2 t\right) dt = \frac{1}{2} \int_0^{2\pi} \left(3 \sin^2 t \cos^2 t\right) dt = \frac{3}{8} \int_0^{2\pi} \sin^2 2t\, dt = \frac{3}{16} \int_0^{4\pi} \sin^2 u\, du$

$= \frac{3}{16}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{3}{8}\pi$

24. $M = x = t^2,\ N = y = \frac{t^3}{3} - t \Rightarrow dx = 2t\, dt,\ dy = \left(t^2 - 1\right) dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \left[t^2\left(t^2 - 1\right) - \left(\frac{t^3}{3} - t\right)(2t)\right] dt = \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{1}{3}t^4 + t^2\right) dt = \frac{1}{2}\left[\frac{1}{15}t^5 + -\frac{1}{3}t^3\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{15}\left(9\sqrt{3} + 15\sqrt{3}\right)$

$= \frac{8}{5}\sqrt{3}$

25. (a) $M = f(x),\ N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x)\, dx + g(y)\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R 0\, dx\, dy = 0$

(b) $M = ky,\ N = hx \Rightarrow \frac{\partial M}{\partial y} = k,\ \frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky\, dx + hx\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R (h - k)\, dx\, dy = (h - k)(\text{Area of the region})$

26. $M = xy^2,\ N = x^2y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2\, dx + (x^2y + 2x)\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R (2xy + 2 - 2xy)\, dx\, dy = 2 \iint_R dx\, dy = 2$ times the area of the square

27. The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_C 4x^3y\, x + x^4\, dy = \iint_R \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right] dx\, dy$

$= \iint_R \underbrace{(4x^3 - 4x^3)}_{0}\, dx\, dy = 0.$

28. The integral is 0 for any simple closed curve C. The reasoning: By the normal form of Green's theorem, with $M = -y^3$ and $N = x^3$, $\oint_C -y^3\, dx + x^3\, dy = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_{0} + \underbrace{\frac{\partial}{\partial y}(x^3)}_{0}\right] dx\, dy = 0.$

29. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy \Rightarrow \oint_C x\, dy$

$= \iint_R (1 + 0)\, dx\, dy \Rightarrow \text{Area of } R = \iint_R dx\, dy = \oint_C x\, dy$; similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and

$\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dy\, dx \Rightarrow \oint_C y\, dx = \iint_R (0 - 1)\, dy\, dx \Rightarrow -\oint_C y\, dx$

$= \iint_R dx\, dy = \text{Area of } R$

30. $\int_a^b f(x)\, dx = \text{Area of } R = -\oint_C y\, dx$, from Exercise 29

31. Let $\delta(x,y) = 1 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\iint_R x\,dA}{A} \Rightarrow A\bar{x} = \iint_R x\,dA = \iint_R (x+0)\,dx\,dy$

$= \oint_C \frac{x^2}{2}\,dy$, $A\bar{x} = \iint_R x\,dA = \iint_R (0+x)\,dx\,dy = -\oint_C xy\,dx$, and $A\bar{x} = \iint_R x\,dA = \iint_R \left(\frac{2}{3}x + \frac{1}{3}x\right)dx\,dy$

$= \oint_C \frac{1}{3}x^2\,dy - \frac{1}{3}xy\,dx \Rightarrow \frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\bar{x}$

32. If $\delta(x,y) = 1$, then $I_y = \iint_R x^2\,\delta(x,y)\,dA = \iint_R x^2\,dA = \iint_R (x^2+0)\,dy\,dx = \frac{1}{3}\oint_C x^3\,dy$,

$\iint_R x^2\,dA = \iint_R (0+x^2)\,dy\,dx = -\oint_C x^2y\,dx$, and $\iint_R x^2\,dA = \iint_R \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right)dy\,dx$

$= \oint_C \frac{1}{4}x^3\,dy - \frac{1}{4}x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx \Rightarrow \frac{1}{3}\oint_C x^3\,dy = -\oint_C x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx = I_y$

33. $M = \frac{\partial f}{\partial y}$, $N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = \iint_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)dx\,dy = 0$ for such curves C

34. $M = \frac{1}{4}x^2y + \frac{1}{3}y^3$, $N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$, $\frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow \text{work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(1 - \frac{1}{4}x^2 - y^2\right)dx\,dy$ will be maximized on the region $R = \{(x,y)\,|\,\text{curl } \mathbf{F}\} \geq 0$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$

35. (a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}$, $N = \frac{2y}{x^2+y^2}$; since M, N are discontinuous at $(0,0)$, we cannot apply Green's Theorem over C. Thus, let C_h be the circle $x = h\cos\theta$, $y = h\sin\theta$, $0 < h \leq a$ and let C_1 be the circle $x = a\cos t$, $y = a\sin t$, $a > 0$. Then $\oint_C \nabla f\cdot\mathbf{n}\,ds = \oint_{C_1} M\,dy - N\,dx + \oint_{C_h} M\,dy - N\,dx$

$= \oint_{C_1} \frac{2x}{x^2+y^2}\,dx - \frac{2y}{x^2+y^2}\,dx + \oint_{C_h} \frac{2x}{x^2+y^2}\,dy - \frac{2y}{x^2+y^2}\,dx$. In the first integral, let $x = a\cos t$, $y = a\sin t$ $\Rightarrow dx = -a\sin t\,dt$, $dy = a\cos t\,dt$, $M = 2a\cos t$, $N = 2a\sin t$, $0 \leq t \leq 2\pi$. In the second integral, let $x = h\cos\theta$, $y = h\sin\theta \Rightarrow dx = -h\sin\theta\,d\theta$, $dy = h\cos\theta\,d\theta$, $M = 2h\cos\theta$, $N = 2h\sin\theta$, $0 \leq \theta \leq 2\pi$.

Then $\oint_C \nabla f\cdot\mathbf{n}\,ds = \oint_{C_1} \frac{2x}{x^2+y^2}\,dy - \frac{2y}{x^2+y^2}\,dx + \oint_{C_h} \frac{2x}{x^2+y^2}\,dy - \frac{2y}{x^2+y^2}\,dx$

$$= \oint_{C_1} \frac{(2a\cos t)(a\cos t)\,dt}{a^2} - \frac{(2a\sin t)(-a\sin t)\,dt}{a^2} + \oint_{C_h} \frac{(2h\cos\theta)(h\cos\theta)\,d\theta}{h^2} - \frac{(2h\sin\theta)(-h\sin\theta)\,d\theta}{h^2}$$

$$= \int_0^{2\pi} 2\,dt + \int_{2\pi}^{0} 2\,d\theta = 0 \text{ for every h}$$

(b) If K is any simple closed curve surrounding C_h (K contains (0,0)), then $\oint_C \nabla f \cdot \mathbf{n}\,ds$

$= \oint_{C_1} M\,dy - N\,dx + \oint_{C_h} M\,dy - N\,dx$, and in polar coordinates, $\nabla f \cdot \mathbf{n} = M\,dy - N\,dx$

$= \left(\frac{2r\cos\theta}{r^2}\right)(r\cos\theta\,d\theta + \sin\theta\,dr) - \left(\frac{2r\sin\theta}{r^2}\right)(-r\sin\theta\,d\theta + \cos\theta\,dr) = \frac{2r^2}{r^2}\,d\theta = 2\,d\theta$. Now,

2θ increases by 4π as K is traversed once counterclockwise from $\theta = 0$ to $\theta = 2\pi \Rightarrow \oint_C \nabla f \cdot \mathbf{n}\,ds = 0$

(since $\oint_{C_h} M\,dy - N\,dx = -4\pi$) when (0,0) is in the region, but $\oint_K \nabla f \cdot \mathbf{n}\,ds = 4\pi$ when (0,0) is not in the region.

36. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow$ C_1 encloses a simply connected region $R_1 \Rightarrow C_1$ is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\,ds = 0$, since the velocity vectors $\mathbf{F}$ are tangent to C_1. But $0 = \oint_{C_1} \mathbf{F} \cdot \mathbf{n}\,ds = \oint_{C_1} M\,dy - N\,dx = \iint_{R_1} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \Rightarrow M_x + N_y = 0$, which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

37. $\int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\,dx\,dy = N(g_2(y),y) - N(g_1(y),y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x}\,dx\right) dy = \int_c^d [N(g_2(y),y) - N(g_1(y),y)]\,dy$

$= \int_c^d N(g_2(y),y)\,dy - \int_c^d N(g_1(y),y)\,dy = \int_c^d N(g_2(y),y)\,dy + \int_d^c N(g_1(y),y)\,dy = \int_{C_2} N\,dy + \int_{C_1} N\,dy$

$= \oint_C N\,dy \Rightarrow \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$

38. $\int_a^b \int_c^d \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b [M(x,d) - M(x,c)]\,dx = \int_a^b M(x,d)\,dx + \int_b^a M(x,c)\,dx = -\int_{C_2} M\,dx - \int_{C_1} M\,dx.$

Because x is constant along C_2 and C_4, $\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$

$\Rightarrow -\left(\int_{C_1} M\,dx + \int_{C_2} M\,dx + \int_{C_3} M\,dx + \int_{C_4} M\,dx\right) = -\oint_C M\,dx \Rightarrow \int_a^b \int_c^d \frac{\partial M}{\partial y}\,dy\,dx = -\oint_C M\,dx.$

39. The curl of a conservative two-dimensional field is zero. The reasoning: A two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy-plane of a three-dimensional field whose k component is zero, and whose **i** and **j** components are independent of z. For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 14.3 $\Rightarrow$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$.

40. Green's theorem tells us that the circulation of a conservative two-dimensional field around any simple closed curve in the xy-plane is zero. The reasoning: For a conservative field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, we have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ (component test for conservative fields, Section 14.3, Eq. (3)), so curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$. By Green's theorem, the counterclockwise circulation around a simple closed plane curve C must equal the integral of curl **F** over the region R enclosed by C. Since curl $\mathbf{F} = 0$, the latter integral is zero and, therefore, so is the circulation. The circulation $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ is the same as the work $\oint_C \mathbf{F} \cdot d\mathbf{r}$ done by **F** around C, so our observation that circulation of a conservative two-dimensional field is zero agrees with the fact that the work done by a conservative field around a closed curve is always 0.

41-44. Example CAS commands:

Maple:

```
with(plots):
implicitplot({y=4 - 2*x}, x = 0..3, y = 0..5, scaling=CONSTRAINED);
M:= (x,y) -> x*exp(y);
N:= (x,y) -> 4*x^2*ln(y);
My:= diff(M(x,y),y);
Nx:= diff(N9x,y),x);
int(int(Nx - My, y = 0..4 - 2*x), x= 0..2);
evalf(%);
```

Mathematica:

```
<< Graphics`ImplicitPlot`
SetOptions[ ImplicitPlot, AspectRatio -> Automatic ];
Clear[x,y]
f[x_,y_] = {
 x E^y ,
 4 x^2 Log[y] }
y1 = 0
y2 = -2 x + 4
Plot[ {y1,y2}, {x,0,2}, AspectRatio -> Automatic ]
integrand = D[f[x,y] [[2]],x] - D[f[x,y][[1]],y]
Solve[ c, y ]
{y1,y2} = y /. %
Integrate[ integrand, {x,0,2}, {y,y1,y2} ]
Simplify[%]
N[%}
```

13.5 SURFACE AREA AND SURFACE INTEGRALS

1. $\mathbf{p}=\mathbf{k},\ \nabla f = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2+(2y)^2+(-1)^2} = \sqrt{4x^2+4y^2+1}$ and $|\nabla f\cdot\mathbf{p}| = 1$;

$z=2 \Rightarrow x^2+y^2=2$; thus $S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot \mathbf{p}|}\,dA = \iint_R \sqrt{4x^2+4y^2+1}\,dx\,dy$

$$= \iint_R \sqrt{4r^2\cos^2\theta+4r^2\sin^2\theta+1}\ r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\ r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2+1\right)^{3/2}\right]_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \tfrac{13}{6}\,d\theta = \tfrac{13}{3}\pi$$

2. $\mathbf{p}=\mathbf{k},\ \nabla f = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+1}$ and $|\nabla f\cdot\mathbf{p}|=1$; $2 \le x^2+y^2 \le 6$

$$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint_R \sqrt{4x^2+4y^2+1}\,dx\,dy = \iint_R \sqrt{4r^2+1}\ r\,dr\,d\theta = \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2+1}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2+1\right)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}} d\theta = \int_0^{2\pi} \tfrac{49}{6}\,d\theta = \tfrac{49}{3}\pi$$

3. $\mathbf{p}=\mathbf{k},\ \nabla f = \mathbf{i}+2\mathbf{j}+2\mathbf{k} \Rightarrow |\nabla f| = 3$ and $|\nabla f\cdot\mathbf{p}|=2$; $x=y^2$ and $x=2-y^2$ intersect at $(1,1)$ and $(1,-1)$

$$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint_R \tfrac{3}{2}\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{2-y^2} \tfrac{3}{2}\,dx\,dy = \int_{-1}^{1}\left(3-3y^2\right)dy = 4$$

4. $\mathbf{p}=\mathbf{k},\ \nabla f = 2x\mathbf{i}-2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4} = 2\sqrt{x^2+1}$ and $|\nabla f\cdot\mathbf{p}| = 2 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA$

$$= \iint_R \frac{2\sqrt{x^2+1}}{2}\,dx\,dy = \int_0^{\sqrt{3}}\int_0^{x} \sqrt{x^2+1}\,dy\,dx = \int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx = \left[\tfrac{1}{3}\left(x^2+1\right)^{3/2}\right]_0^{\sqrt{3}} = \tfrac{1}{3}(4)^{3/2}-\tfrac{1}{3} = \tfrac{7}{3}$$

5. $\mathbf{p}=\mathbf{k},\ \nabla f = 2x\mathbf{i}-2\mathbf{j}-2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2+(-2)^2+(-2)^2} = \sqrt{4x^2+8} = 2\sqrt{x^2+2}$ and $|\nabla f\cdot\mathbf{p}| = 2$

$$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint_R \frac{2\sqrt{x^2+2}}{2}\,dx\,dy = \int_0^{2}\int_0^{3x} \sqrt{x^2+2}\,dy\,dx = \int_0^{2} 3x\sqrt{x^2+2}\,dx = \left[\left(x^2+2\right)^{3/2}\right]_0^{2}$$

$$= 6\sqrt{6}-2\sqrt{2}$$

6. $\mathbf{p}=\mathbf{k}$, $\nabla f = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 2z$; $x^2+y^2+z^2=2$ and $z=\sqrt{x^2+y^2} \Rightarrow x^2+y^2=1$; thus, $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot \mathbf{p}|}\,dA = \iint\limits_R \frac{2\sqrt{2}}{2z}\,dA = \sqrt{2}\iint\limits_R \frac{1}{z}\,dA$

$= \sqrt{2}\iint\limits_R \frac{1}{\sqrt{2-(x^2+y^2)}}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{2-r^2}} = \sqrt{2}\int_0^{2\pi}\left(-1+\sqrt{2}\right)d\theta = 2\pi\left(2-\sqrt{2}\right)$

7. $\mathbf{p}=\mathbf{k}$, $\nabla f = c\mathbf{i}-\mathbf{k} \Rightarrow |\nabla f| = \sqrt{c^2+1}$ and $|\nabla f\cdot\mathbf{p}| = 1 \Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \sqrt{c^2+1}\,dx\,dy$

$= \int_0^{2\pi}\int_0^1 \sqrt{c^2+1}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{c^2+1}}{2}\,d\theta = \pi\sqrt{c^2+1}$

8. $\mathbf{p}=\mathbf{k}$, $\nabla f = 2x\mathbf{i}+2z\mathbf{j} \Rightarrow |\nabla f| = \sqrt{(2x)^2+(2z)^2} = 2$ and $|\nabla f\cdot\mathbf{p}| = 2z$ for the upper surface, $z \ge 0$

$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \frac{2}{2z}\,dA = \iint\limits_R \frac{1}{\sqrt{1-x^2}}\,dy\,dx = 2\int_{-1/2}^{1/2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dy\,dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

$= \left[\sin^{-1}x\right]_{-1/2}^{1/2} = \frac{\pi}{6}-\left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

9. $\mathbf{p}=\mathbf{i}$, $\nabla f = \mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{1^2+(2y)^2+(2z)^2} = \sqrt{1+4y^2+4z^2}$ and $|\nabla f\cdot\mathbf{p}| = 1$; $1 \le y^2+z^2 \le 4$

$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \sqrt{1+4y^2+4z^2}\,dy\,dz = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17}-5\sqrt{5}\right)d\theta = \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$

10. $\mathbf{p}=\mathbf{j}$, $\nabla f = 2x\mathbf{i}+\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4z^2+1}$ and $|\nabla f\cdot\mathbf{p}| = 1$; $y=0$ and $x^2+y+z^2=2 \Rightarrow x^2+z^2=2$;

thus, $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \sqrt{4x^2+4z^2+1}\,dx\,dz = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{13}{6}\,d\theta = \frac{13}{3}\pi$

11. $\mathbf{p}=\mathbf{k}$, $\nabla f = \left(2x-\frac{2}{x}\right)\mathbf{i}+\sqrt{15}\,\mathbf{j}-\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\left(2x-\frac{2}{x}\right)^2+\left(\sqrt{15}\right)^2+(-1)^2} = \sqrt{4x^2+8+\frac{4}{x^2}} = \sqrt{\left(2x+\frac{2}{x}\right)^2}$

$= 2x+\frac{2}{x}$, on $1 \le x \le 2$ and $|\nabla f\cdot\mathbf{p}| = 1 \Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \left(2x+2x^{-1}\right)dx\,dy$

$= \int_0^1\int_1^2 \left(2x+2x^{-1}\right)dx\,dy = \int_0^1 \left[x^2+2\ln x\right]_1^2 dy = \int_0^1 (3+2\ln 2)\,dy = 3+2\ln 2$

12. $\mathbf{p}=\mathbf{k},\ \nabla f = 3\sqrt{x}\,\mathbf{i}+3\sqrt{y}\,\mathbf{j}-3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x+9y+9} = 3\sqrt{x+y+1}$ and $|\nabla f\cdot\mathbf{p}| = 3$

$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint_R \sqrt{x+y+1}\,dx\,dy = \int_0^1\int_0^1 \sqrt{x+y+1}\,dx\,dy = \int_0^1 \left[\frac{2}{3}(x+y+1)^{3/2}\right]_0^1 dy$

$= \int_0^1 \left[\frac{2}{3}(y+2)^{3/2}-\frac{2}{3}(y+1)^{3/2}\right]dy = \left[\frac{4}{15}(y+2)^{5/2}-\frac{4}{15}(y+1)^{5/2}\right]_0^1 = \frac{4}{15}\left[(3)^{5/2}-(2)^{5/2}-(2)^{5/2}+1\right]$

$= \frac{4}{15}\left(9\sqrt{3}-8\sqrt{2}+1\right)$

13. The bottom face S of the cube is in the xy-plane $\Rightarrow z=0 \Rightarrow g(x,y,0)=x+y$ and $f(x,y,z)=z=0 \Rightarrow \mathbf{p}=\mathbf{k}$ and $\nabla f=\mathbf{k} \Rightarrow |\nabla f|=1$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S g\,d\sigma = \iint_R (x+y)\,dx\,dy$

$= \int_0^a\int_0^a (x+y)\,dx\,dy = \int_0^a \left(\frac{a^2}{2}+ay\right)dy = a^3$. Because of symmetry, we also get a^3 over the face of the cube in the xz-plane and a^3 over the face of the cube in the yz-plane. Next, on the top of the cube, $g(x,y,z)$ $= g(x,y,a) = x+y+a$ and $f(x,y,z)=z=a \Rightarrow \mathbf{p}=\mathbf{k}$ and $\nabla f=\mathbf{k} \Rightarrow |\nabla f|=1$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow d\sigma = dx\,dy$

$\iint_S g\,d\sigma = \iint_R (x+y+a)\,dx\,dy = \int_0^a\int_0^a (x+y+a)\,dx\,dy = \int_0^a\int_0^a (x+y)\,dx\,dy + \int_0^a\int_0^a a\,dx\,dy = 2a^3.$

Because of symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore,

$\iint_{\text{cube}} (x+y+z)\,d\sigma = 3\left(a^3+2a^3\right) = 9a^3.$

14. On the face S in the xz-plane, we have $y=0 \Rightarrow f(x,y,z)=y=0$ and $g(x,y,z)=g(x,0,z)=z \Rightarrow \mathbf{p}=\mathbf{j}$ and $\nabla f=\mathbf{j} \Rightarrow |\nabla f|=1$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow d\sigma = dx\,dz \Rightarrow \iint_S g\,d\sigma = \iint_S (y+z)\,d\sigma = \int_0^1\int_0^2 z\,dx\,dz = \int_0^1 2z\,dz$ $=1.$

On the face in the xy-plane, we have $z=0 \Rightarrow f(x,y,z)=z=0$ and $g(x,y,z)=g(x,y,0)=y \Rightarrow \mathbf{p}=\mathbf{k}$ and $\nabla f=\mathbf{k} \Rightarrow |\nabla f|=1$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S y\,d\sigma = \int_0^1\int_0^2 y\,dx\,dy = 1.$

On the triangular face in the plane $x=2$ we have $f(x,y,z)=x=2$ and $g(x,y,z)=g(2,y,z)=y+z \Rightarrow \mathbf{p}=\mathbf{i}$ and $\nabla f=\mathbf{i} \Rightarrow |\nabla f|=1$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S (y+z)\,d\sigma = \int_0^1\int_0^{1-y} (y+z)\,dz\,dy$ $= \int_0^1 \frac{1}{2}\left(1-y^2\right)dy = \frac{1}{3}.$

On the triangular face in the yz-plane, we have $x = 0 \Rightarrow f(x,y,z) = x = 0$ and $g(x,y,z) = g(0,y,z) = y + z$

$\Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S (y+z)\,d\sigma$

$= \int_0^1 \int_0^{1-y} (y+z)\,dz\,dy = \frac{1}{3}.$

Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x,y,z) = y + z = 1$ and $g(x,y,z) = y + z = 1 \Rightarrow \mathbf{p} = \mathbf{k}$ and

$\nabla f = \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2}\,dx\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S (y+z)\,d\sigma$

$= \int_0^1 \int_0^2 \sqrt{2}\,dx\,dy = 2\sqrt{2}.$ Therefore, $\iint_{\text{wedge}} g(x,y,z)\,d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

15. On the faces in the coordinate planes, $g(x,y,z) = 0 \Rightarrow$ the integral over these faces is 0.

On the face $x = a$, we have $f(x,y,z) = x = a$ and $g(x,y,z) = g(a,y,z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$

and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\,dz \Rightarrow \iint_S g\,d\sigma = \iint_S ayz\,d\sigma = \int_0^c \int_0^b ayz\,dy\,dz = \frac{ab^2c^2}{4}.$

On the face $y = b$, we have $f(x,y,z) = y = b$ and $g(x,y,z) = g(x,b,z) = bxz \Rightarrow \mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$

and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dz \Rightarrow \iint_S g\,d\sigma = \iint_S bxz\,d\sigma = \int_0^c \int_0^a bxz\,dz\,dx = \frac{a^2bc^2}{4}.$

On the face $z = c$, we have $f(x,y,z) = z = c$ and $g(x,y,z) = g(x,y,c) = cxy \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$

and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\,dx \Rightarrow \iint_S g\,d\sigma = \iint_S cxy\,d\sigma = \int_0^b \int_0^a cxy\,dx\,dy = \frac{a^2b^2c}{4}.$ Therefore,

$\iint_S g(x,y,z)\,d\sigma = \frac{abc(ab + ac + bc)}{4}.$

16. On the face $x = a$, we have $f(x,y,z) = x = a$ and $g(x,y,z) = g(a,y,z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$

and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S ayz\,d\sigma = \int_{-b}^b \int_{-c}^c ayz\,dz\,dy = 0.$ Because of the symmetry

of g on all the other faces, all the integrals are 0, and $\iint_S g(x,y,z)\,d\sigma = 0.$

17. $f(x,y,z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $g(x,y,z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k}$,

$|\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3\,dy\,dx$; $z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S g\,d\sigma = \iint_S (2 - x - y)\,d\sigma$

$$= 3\int_0^1\int_0^{1-x} (2-x-y)\,dy\,dx = 3\int_0^1 \left[(2-x)(1-x)-\tfrac{1}{2}(1-x)^2\right] dx = 3\int_0^1 \left(\tfrac{3}{2}-2x+\tfrac{x^2}{2}\right) dx = 2$$

18. $f(x,y,z) = y^2+4z = 16 \Rightarrow \nabla f = 2y\mathbf{j}+4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+16} = 2\sqrt{y^2+4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$

$$\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4}\,dx\,dy \Rightarrow \iint_S g\,d\sigma = \int_{-4}^4\int_0^1 \left(x\sqrt{y^2+4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dx\,dy = \int_{-4}^4\int_0^1 \frac{x(y^2+4)}{2}\,dx\,dy$$

$$= \int_{-4}^4 \tfrac{1}{4}(y^2+4)\,dy = \tfrac{1}{2}\left[\frac{y^3}{3}+4y\right]_0^4 = \tfrac{1}{2}\left(\tfrac{64}{3}+16\right) = \tfrac{56}{3}$$

19. $g(x,y,z) = z,\ \mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R (\mathbf{F}\cdot\mathbf{k})\,dA$

$$= \int_0^2\int_0^3 3\,dy\,dx = 18$$

20. $g(x,y,z) = y,\ \mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R (\mathbf{F}\cdot -\mathbf{j})\,dA$

$$= \int_{-1}^2\int_2^7 2\,dz\,dx = \int_{-1}^2 2(7-2)\,dx = 10(2+1) = 30$$

21. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 2a;\ \mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{z^2}{a};$

$$|\nabla g\cdot\mathbf{k}| = 2z \Rightarrow d\sigma = \tfrac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint_R \left(\tfrac{z^2}{a}\right)\left(\tfrac{a}{z}\right) dA = \iint_R z\,dA = \iint_R \sqrt{a^2-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = \frac{\pi a^3}{6}$$

22. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 2a;\ \mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-xy}{a}+\frac{xy}{a}$

$$= 0;\ |\nabla g\cdot\mathbf{k}| = 2z \Rightarrow d\sigma = \tfrac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$$

23. From Exercise 21, $\mathbf{n} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{xy}{a}-\frac{xy}{a}+\frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint_R \left(\tfrac{z}{a}\right)\left(\tfrac{a}{z}\right) dA$

$$= \iint_R 1\,dA = \frac{\pi a^2}{4}$$

24. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z\left(\frac{x^2+y^2+z^2}{a}\right) = az$

$$\Rightarrow \text{Flux} = \iint_R (za)\left(\tfrac{a}{z}\right) dx\,dy = \iint_R a^2\,dx\,dy = a^2(\text{Area of R}) = \tfrac{1}{4}\pi a^4$$

25. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a \Rightarrow$ Flux

$$= \iint_R a\left(\tfrac{a}{z}\right) dA = \iint_R \tfrac{a^2}{z}\,dA = \iint_R \frac{a^2}{\sqrt{a^2-(x^2+y^2)}}\,dA = \int_0^{\pi/2}\int_0^a \frac{a^2}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} a^2\left[-\sqrt{a^2-r^2}\right]_0^a d\theta = \tfrac{\pi a^3}{2}$$

26. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{\left(\frac{x^2}{a}\right)+\left(\frac{y^2}{a}\right)+\left(\frac{z^2}{a}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$

$$\Rightarrow \text{Flux} = \iint_R \tfrac{a}{z}\,dx\,dy = \iint_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy = \int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = \tfrac{\pi a^2}{2}$$

27. $g(x,y,z) = y^2 + z = 4 \Rightarrow \nabla g = 2y\mathbf{j} + \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2+1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$

$\Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy-3z}{\sqrt{4y^2+1}}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\,dA \Rightarrow$ Flux

$= \iint_R \left(\frac{2xy-3z}{\sqrt{4y^2+1}}\right)\sqrt{4y^2+1}\,dA = \iint_R (2xy-3z)\,dA$; $z = 0$ and $z = 4 - y^2 \Rightarrow y^2 = 4$

$$\Rightarrow \text{Flux} = \iint_R \left[2xy - 3\left(4-y^2\right)\right] dA = \int_0^1\int_{-2}^2 \left(2xy - 12 + 3y^2\right) dy\,dx = \int_0^1 \left[xy^2 - 12y + y^3\right]_{-2}^2 dx$$

$$= \int_0^1 -32\,dx = -32$$

28. $g(x,y,z) = x^2 + y^2 - z = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+1} = \sqrt{4\left(x^2+y^2\right)+1}$

$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4\left(x^2+y^2\right)+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{8x^2+8y^2-2}{\sqrt{4\left(x^2+y^2\right)+1}}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4\left(x^2+y^2\right)+1}\,dA$

$\Rightarrow \text{Flux} = \iint_R \left(\frac{8x^2+8y^2-2}{\sqrt{4\left(x^2+y^2\right)+1}}\right)\sqrt{4\left(x^2+y^2\right)+1}\,dA = \iint_R \left(8x^2+8y^2-2\right) dA$; $z = 1$ and $x^2 + y^2 = z$

$\Rightarrow x^2+y^2=1 \Rightarrow \text{Flux} = \int_0^{2\pi}\int_0^1 (8r^2-2)\, r\, dr\, d\theta = 2\pi$

29. $g(x,y,z) = y - e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x}+1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i}-\mathbf{j}}{\sqrt{e^{2x}+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-2e^x-2y}{\sqrt{e^{2x}+1}}$; $\mathbf{p} = \mathbf{i}$

$\Rightarrow |\nabla g\cdot\mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x}+1}}{e^x}\, dA \Rightarrow \text{Flux} = \iint_R \left(\frac{-2e^x-2y}{\sqrt{e^{2x}+1}}\right)\left(\frac{\sqrt{e^{2x}+1}}{e^x}\right) dA = \iint_R \frac{-2e^x-2e^x}{e^x}\, dA$

$= \iint_R -4\, dA = \int_0^1\int_1^2 -4\, dy\, dz = -4$

30. $g(x,y,z) = y - \ln x = 0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{1+x^2}}{x}$ since $1 \le x \le e$

$\Rightarrow \mathbf{n} = \frac{\left(-\frac{1}{x}\mathbf{i}+\mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i}+x\mathbf{j}}{\sqrt{1+x^2}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy}{\sqrt{1+x^2}}$; $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x}\, dA$

$\Rightarrow \text{Flux} = \iint_R \left(\frac{2xy}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}}{x}\right) dA = \int_0^1\int_1^e 2y\, dx\, dz = \int_1^e\int_0^1 2\ln x\, dz\, dx = \int_1^e 2\ln x\, dx$

$= 2[x\ln x - x]_1^e = 2(e-e) - 2(0-1) = 2$

31. On the face $z = a$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xz = 2ax$ since $z = a$;

$d\sigma = dx\, dy \Rightarrow \text{Flux} = \iint_R 2ax\, dx\, dy = \int_0^a\int_0^a 2ax\, dx\, dy = a^4$.

On the face $z = 0$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xz = 0$ since $z = 0$;

$d\sigma = dx\, dy \Rightarrow \text{Flux} = \iint_R 0\, dx\, dy = 0$.

On the face $x = a$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xy = 2ay$ since $x = a$;

$d\sigma = dy\, dz \Rightarrow \text{Flux} = \int_0^a\int_0^a 2ay\, dy\, dz = a^4$.

On the face $x = 0$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xy = 0$ since $x = 0$

$\Rightarrow \text{Flux} = 0$.

On the face $y = a$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2yz = 2az$ since $y = a$;

$d\sigma = dz\, dx \Rightarrow \text{Flux} = \int_0^a\int_0^a 2az\, dz\, dx = a^4$.

On the face $y = 0$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2yz = 0$ since $y = 0$

$\Rightarrow \text{Flux} = 0$. Therefore, Total Flux $= 3a^4$.