31. $\sin^2 x = \left(\frac{1-\cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) = \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} - \ldots$
$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2\cdot(2n)!}$

32. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right) = x^2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + 2^3x^5 + \ldots$

33. $x \ln(1+2x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^{n+1}}{n} = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{3} - \frac{2^4x^5}{4} + \ldots$

34. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$
$= \sum_{n=0}^{\infty} (n+1)x^n$

35. By the Alternating Series Estimation Theorem, the error is less than $\frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!)(5\times 10^{-4})$
$\Rightarrow |x|^5 < 600\times 10^{-4} \Rightarrow |x| < \sqrt[5]{6\times 10^{-2}} \approx 0.56968$

36. If $\cos x = 1 - \frac{x^2}{2}$ and $|x| < 0.5$, then the $|\text{error}| = |R_3(x)| = \left|\frac{\cos c}{4!}x^4\right| < \left|\frac{(.5)^4}{24}\right| = 0.0026$, where c is between 0 and x; since the next term in the series is positive, the approximation $1 - \frac{x^2}{2}$ is too small, by the Alternating Series Estimation Theorem

37. If $\sin x = x$ and $|x| < 10^{-3}$, then the $|\text{error}| = |R_2(x)| = \left|\frac{-\cos c}{3!}x^3\right| < \frac{(10^{-3})^3}{3!} \approx 1.67\times 10^{-10}$, where c is between 0 and x. The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x \Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0$.

38. $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$. By the Alternating Series Estimation Theorem the $|\text{error}| < \left|\frac{-x^2}{8}\right| < \frac{(0.01)^2}{8}$
$= 1.25\times 10^{-5}$

39. (a) $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{3^{(0.1)}(0.1)^3}{3!} < 1.87\times 10^{-4}$, where c is between 0 and x

(b) $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{(0.1)^3}{3!} = 1.67\times 10^{-4}$, where c is between 0 and x

40. $|R_4(x)| < \left|\frac{\cosh c}{5!}x^5\right| = \left|\frac{e^c + e^{-c}}{2}\frac{x^5}{5!}\right| < \frac{1.65 + \frac{1}{1.65}}{2}\cdot\frac{(0.5)^5}{5!} = (1.3)\frac{(0.5)^5}{5!} \approx 0.000293653$

41. If we approximate e^h with $1+h$ and $0 \le h \le 0.01$, then $|\text{error}| < \left|\frac{e^c h^2}{2}\right| \le \frac{e^{0.01}h\cdot h}{2} = \left(\frac{e^{0.01}(0.01)}{2}\right)h$

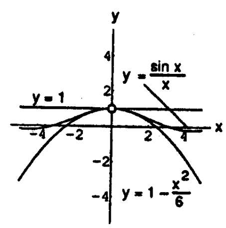
$= 0.005005\text{h} < 0.006\text{h} = (0.6\%)\text{h}$, where c is between 0 and h.

42. $|R_1| = \left|\frac{1}{(1+c)^2}\frac{x^2}{2!}\right| < \frac{x^2}{2} = \left|\frac{x}{2}\right| |x| < .01\,|x| = (1\%)\,|x| \Rightarrow \left|\frac{x}{2}\right| < .01 \Rightarrow 0 < |x| < .02$

43. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \Rightarrow \frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$; $|\text{error}| < \frac{1}{2n+1} < .01$
$\Rightarrow 2n + 1 > 100 \Rightarrow n > 49$

44. (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$, $s_1 = 1$ and $s_2 = 1 - \frac{x^2}{6}$; if L is the sum of the series representing $\frac{\sin x}{x}$, then by the Alternating Series Estimation Theorem, $L - s_1 = \frac{\sin x}{x} - 1 < 0$ and $L - s_2 = \frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right) > 0$. Therefore $1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$

(b) The graph of $y = \frac{\sin x}{x}$, $x \neq 0$, is bounded below by the graph of $y = 1 - \frac{x^2}{6}$ and above by the graph of $y = 1$ as derived in part (a).



45. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$; $f(0) = 0$, $f'(0) = 0$, $f''(0) = -1$
$\Rightarrow L(x) = 0$ and $Q(x) = -\frac{x^2}{2}$

46. $f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x}$ and $f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}$; $f(0) = 1$, $f'(0) = 1$,
$f''(0) = 1 \Rightarrow L(x) = 1 + x$ and $Q(x) = 1 + x + \frac{x^2}{2}$

47. $f(x) = (1 - x^2)^{-1/2} \Rightarrow f'(x) = x(1 - x^2)^{-3/2}$ and $f''(x) = (1 - x^2)^{-3/2} + 3x^2(1 - x^2)^{-5/2}$; $f(0) = 1$,
$f'(0) = 0$, $f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 = \frac{x^2}{2}$

48. $f(x) = \cosh x \Rightarrow f'(x) = \sinh x$ and $f''(x) = \cosh x$; $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

49. A special case of Taylor's Formula is $f(x) = f(a) + f'(c)(x - a)$. Let $x = b$ and this becomes
$f(b) - f(a) = f'(c)(b - a)$, the Mean Value Theorem

50. If f(x) is twice differentiable and at $x = a$ there is a point of inflection, then $f''(a) = 0$. Therefore,

$L(x) = Q(x) = f(a) + f'(a)(x-a)$.

51. (a) $f'' \le 0$, $f'(a) = 0$ and $x = a$ interior to the interval $I \Rightarrow f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \le 0$ throughout I $\Rightarrow f(x) \le f(a)$ throughout I $\Rightarrow$ f has a local maximum at $x = a$

(b) similar reasoning gives $f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \ge 0$ throughout I $\Rightarrow f(x) \ge f(a)$ throughout I $\Rightarrow$ f has a local minimum at $x = a$

52. (a) $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f^{(3)}(x) = 6(1-x)^{-4}$ $\Rightarrow f^{(4)}(x) = 24(1-x)^{-5}$; therefore $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$

(b) $|x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5 \Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4\left(\frac{10}{9}\right)^5 \Rightarrow$ the error

$$e_3 \le \left|\frac{\max f^{(4)}(x)\, x^4}{4!}\right| < (0.1)^4\left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017, \text{ since } \left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|.$$

53. Let $P = x + \pi \Rightarrow |x| = |P - \pi| < .5 \times 10^{-n}$ since P approximates π accurate to n decimals. Then, $P + \sin P = (\pi + x) + \sin(\pi + x) = (\pi + x) - \sin x = \pi + (x - \sin x) \Rightarrow |(P + \sin P) - \pi|$ $= |\sin x - x| \le \frac{|x|^3}{3!} < \frac{0.125}{3!} \times 10^{-3n} < .5 \times 10^{-3n} \Rightarrow P + \sin P$ gives an approximation to π correct to 3n decimals.

54. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n x^{n-k}$ and $f^{(k)}(0) = k!\, a_k$ $\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$ for k a nonnegative integer. Therefore, the coefficients of f(x) are identical with the corresponding coefficients in the Maclaurin series of f(x) and the statement follows.

55. Note: f even $\Rightarrow f(-x) = f(x) \Rightarrow -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ odd;
f odd $\Rightarrow f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ even;
also, f odd $\Rightarrow f(-0) = f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$

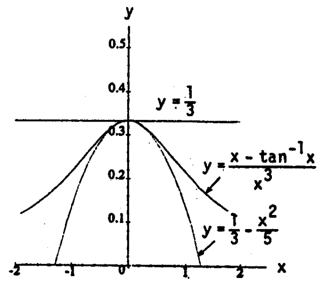
(a) If f(x) is even, then any odd-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_1 = a_3 = a_5 = \ldots = 0$; that is, the Maclaurin series for f contains only even powers.

(b) If f(x) is odd, then any even-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_0 = a_2 = a_4 = \ldots = 0$; that is, the Maclaurin series for f contains only odd powers.

56. (a) Suppose f(x) is a continuous periodic function with period p. Let x_0 be an arbitrary real number. Then f assumes a minimum m_1 and a maximum m_2 in the interval $[x_0, x_0 + p]$; i.e., $m_1 \le f(x) \le m_2$ for all x in $[x_0, x_0 + p]$. Since f is periodic it has exactly the same values on all other intervals $[x_0 + p, x_0 + 2p]$, $[x_0 + 2p, x_0 + 3p]$, …, and $[x_0 - p, x_0]$, $[x_0 - 2p, x_0 - p]$, …, and so forth. That is, for all real numbers $-\infty < x < \infty$ we have $m_1 \le f(x) \le m_2$. Now choose $M = \max\{|m_1|, |m_2|\}$. Then $-M \le -|m_1| \le m_1 \le f(x) \le m_2 \le |m_2| \le M \Rightarrow |f(x)| \le M$ for all x.

(b) The dominate term in the nth order Taylor polynomial generated by cos x about $x = a$ is $\frac{\sin(a)}{n!}(x-a)^n$ or $\frac{\cos(a)}{n!}(x-a)^n$. In both cases, as $|x|$ increases the absolute value of these dominate terms tends to ∞, causing the graph of $P_n(x)$ to move away from cos x.

57. (a)



(b) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \Rightarrow \frac{x - \tan^{-1}x}{x^3} = \frac{1}{3} - \frac{x^2}{5} + \ldots$; from the Alternating Series Estimation Theorem, $\frac{x - \tan^{-1}x}{x^3} - \frac{1}{3} < 0$ $\Rightarrow \frac{x - \tan^{-1}x}{x^3} - \left(\frac{1}{3} - \frac{x^2}{5}\right) > 0 \Rightarrow \frac{1}{3} < \frac{x - \tan^{-1}x}{x^3} < \frac{1}{3} - \frac{x^2}{5}$; therefore, the $\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} = \frac{1}{3}$

58. $E(x) = f(x) - b_0 - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n$

$\Rightarrow 0 = E(a) = f(a) - b_0 \Rightarrow b_0 = f(a)$; from condition (b),

$$\lim_{x\to a} \frac{f(x) - f(a) - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n}{(x-a)^n} = 0$$

$$\Rightarrow \lim_{x\to a} \frac{f'(x) - b_1 - 2b_2(x-a) - 3b_3(x-a)^2 - \ldots - nb_n(x-a)^{n-1}}{n(x-a)^{n-1}} = 0$$

$$\Rightarrow b_1 = f'(a) \Rightarrow \lim_{x\to a} \frac{f''(x) - 2b_2 - 3!\,b_3(x-a) - \ldots - n(n-1)b_n(x-a)^{n-2}}{n(n-1)(x-a)^{n-2}} = 0$$

$$\Rightarrow b_2 = \frac{1}{2}f''(a) \Rightarrow \lim_{x\to a} \frac{f'''(x) - 3!\,b_3 - \ldots - n(n-1)(n-2)b_n(x-a)^{n-3}}{n(n-1)(n-2)(x-a)^{n-3}} = 0$$

$$= b_3 = \frac{1}{3!}f'''(a) \Rightarrow \lim_{x\to a} \frac{f^{(n)}(x) - n!\,b_n}{n!} = 0 \Rightarrow b_n = \frac{1}{n!}f^{(n)}(a); \text{ therefore,}$$

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = P_n(x)$$

59-64. Example CAS commands:

Maple:

```
f:= x -> (1+x)^(3/2);
plot(f(x), x = -1..2);
mp:=proc(n):
convert(series(f(x),x=0,n),polynom) end:
p1:= mp(2); p2:= mp(3); p3:=mp(4);
der:=proc(n):
simplify(subs(x=z,diff(f(x),x$(n+1)))) end:
der(2); der(3); der(4);
plot(der(3),z=0..2, title = `3rd Derivative`);
Max:= 0.56: r:= (x,n) -> Max*x^(n+1)/(n+1)!;
r(x,2);
plot(r(x,2),x=0..2, title = `Maximum Remainder Term Using P2`);
plot({f(x),mp(3)}, x = -1..2, title = `Function and Taylor Polynomial P2`);
```

```
plot(f(x) - mp(3), x=-1..2, title = `Maximum Error Function `);
R:= (x,z,n) -> der(n)*x^(n+1)/(n+1)!;
R(x,z,3);
with(plots):
plot3d(R(x,z,3), x=-1..2, z=0..2);
```

Mathematica:

```
Clear[f,x,c]
f[x_] = (1+x) ^ (3/2)
{a,b} = {-1/2,2};
Plot[ f[x], {x,a,b} ]
p1[x_] = Series[ f[x], {x,0,1} ] // Normal
p2[x_] = Series[ f[x], {x,0,2} ] // Normal
p3[x_] = Series[ f[x], {x,0,3} ] // Normal
f''[c]
Plot[ f''[c], {c,a,b} ]
m1 = f''[a]
f'''[c]
Plot[ f'''[c], {c,a,b} ]
m2 = -f'''[a]
f''''[c]
Plot[ f''''[c], {c,a,b} ]
m3 = f''''[a]
r1 [x_] = m1 x^2/2!
Plot[ r1[x], {x,a,b} ]
r2 [x_] = m2 x^3/3!
Plot[ r2[x], {x,a,b} ]
r3 [x_] = m3 x^4/4!
Plot[ r3[x], {x,a,b} ]
```

```
Note: In estimating Rn from these graphs, consider only the portions where c
is between 0 and x.  (Mathematica has no simple way to plot only that
portion.)
```

```
Plot3D[ f''[c] x^2/2!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot3D[ f'''[c] x^3/3!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot3D[ f''''[c] x^4/4!, {x,a,b}, {c,a,b}, PlotRange -> All ]
Plot[ {f[x],p1[x],p2[x],p3[x]}, {x,a,b} ]
```

8.8 APPLICATIONS OF POWER SERIES

1. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!} + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$

2. $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)x^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3}{3!} + \ldots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots$

3. $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!} + \ldots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$

4. $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!} + \ldots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$

5. $\left(1+\frac{x}{2}\right)^{-2} = 1 - 2\left(\frac{x}{2}\right) + \frac{(-2)(-3)\left(\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(\frac{x}{2}\right)^3}{3!} + \ldots = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$

6. $\left(1-\frac{x}{2}\right)^{-2} = 1 - 2\left(-\frac{x}{2}\right) + \frac{(-2)(-3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \ldots = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \ldots$

7. $\left(1+x^3\right)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(x^3\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(x^3\right)^3}{3!} + \ldots = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \ldots$

8. $\left(1+x^2\right)^{-1/3} = 1 - \frac{1}{3}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(x^2\right)^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(x^2\right)^3}{3!} + \ldots = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \ldots$

9. $\left(1+\frac{1}{x}\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{1}{x}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{x}\right)^3}{3!} + \ldots = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$

10. $\left(1-\frac{2}{x}\right)^{1/3} = 1 + \frac{1}{3}\left(-\frac{2}{x}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{x}\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{2}{x}\right)^3}{3!} + \ldots = 1 - \frac{2}{3x} - \frac{4}{9x^2} - \frac{40}{81x^3} - \ldots$

11. $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$

12. $\left(1+x^2\right)^3 = 1 + 3x^2 + \frac{(3)(2)\left(x^2\right)^2}{2!} + \frac{(3)(2)(1)\left(x^2\right)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

13. $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$

14. $\left(1-\frac{x}{2}\right)^4 = 1 + 4\left(-\frac{x}{2}\right) + \frac{(4)(3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{2}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{2}\right)^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 0$

$\Rightarrow a_1 + a_0 = 0$, $2a_2 + a_1 = 0$, $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore $a_1 = -1$, $a_2 = \frac{-a_1}{2\cdot 1} = \frac{1}{2}$, $a_3 = \frac{-a_2}{3} = -\frac{1}{3\cdot 2}$, …, $a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^nx^n}{n!} = e^{-x}$

16. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - 2y = (a_1 - 2a_0) + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \ldots + (na_n - 2a_{n-1})x^{n-1} + \ldots = 0$

$\Rightarrow a_1 - 2a_0 = 0$, $2a_2 - 2a_1 = 0$, $3a_3 - 2a_2 = 0$ and in general $na_n - 2a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore $a_1 = 2a_0 = 2(1) = 2$, $a_2 = \frac{2}{2}a_1 = \frac{2}{2}(2) = \frac{2^2}{2}$, $a_3 = \frac{2}{3}a_2 = \frac{2}{3}\left(\frac{2^2}{2}\right) = \frac{2^3}{3\cdot 2}, \ldots,$

$a_n = \left(\frac{2}{n}\right)a_{n-1} = \left(\frac{2}{n}\right)\left(\frac{2^{n-1}}{n-1}\right)a_{n-2} = \frac{2^n}{n!} \Rightarrow y = 1 + 2x + \frac{2^2}{2}x^2 + \frac{2^3}{3!}x^3 + \ldots + \frac{2^n}{n!}x^n + \ldots$

$= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots + \frac{(2x)^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$

17. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 1$

$\Rightarrow a_1 - a_0 = 1$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore $a_1 = 1$, $a_2 = \frac{a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{3\cdot 2}$, $a_4 = \frac{a_3}{4} = \frac{1}{4\cdot 3\cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 0 + 1x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = e^x - 1$

18. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 1$

$\Rightarrow a_1 + a_0 = 1$, $2a_2 + a_1 = 0$, $3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 2$ when $x = 0$ we have $a_0 = 2$. Therefore $a_1 = 1 - a_0 = -1$, $a_2 = \frac{-a_1}{2\cdot 1} = \frac{1}{2}$, $a_3 = \frac{-a_2}{3} = -\frac{1}{3\cdot 2}, \ \ldots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 2 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots \ = 1 + \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right)$

$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + e^{-x}$

19. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = x$

$\Rightarrow a_1 - a_0 = 0$, $2a_2 - a_1 = 1$, $3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore $a_1 = 0$, $a_2 = \frac{1 + a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{3\cdot 2}$, $a_4 = \frac{a_3}{4} = \frac{1}{4\cdot 3\cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 0 + 0x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 - x = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = e^x - x - 1$

20. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 2x$

$\Rightarrow a_1 + a_0 = 0,\ 2a_2 + a_1 = 2,\ 3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore $a_1 = 1,\ a_2 = \frac{2 - a_1}{2} = \frac{1}{2},\ a_3 = \frac{-a_2}{3} = -\frac{1}{3\cdot 2},\ \ldots,\ a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = -1 + 1x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots$

$= \left(1 - 1x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right) - 2 + 2x = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 2 + 2x = e^{-x} + 2x - 2$

21. $y' - xy = a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x + \ldots + (na_n - a_{n-2})x^{n-1} + \ldots = 0 \Rightarrow a_1 = 0,\ 2a_2 - a_0 = 0,\ 3a_3 - a_1 = 0,$ $4a_4 - a_2 = 0$ and in general $na_n - a_{n-2} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_2 = \frac{a_0}{2} = \frac{1}{2}$, $a_3 = \frac{a_1}{3} = 0,\ a_4 = \frac{a_2}{4} = \frac{1}{2\cdot 4},\ a_5 = \frac{a_3}{5} = 0,\ \ldots,\ a_{2n} = \frac{1}{2\cdot 4\cdot 6\cdots 2n}$ and $a_{2n+1} = 0$

$\Rightarrow y = 1 + \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1}{2\cdot 4\cdot 6}x^6 + \ldots + \frac{1}{2\cdot 4\cdot 6\cdots 2n}x^{2n} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = e^{x^2/2}$

22. $y' - x^2y = a_1 + 2a_2x + (3a_3 - a_0)x^2 + (4a_4 - a_1)x^3 + \ldots + (na_n - a_{n-3})x^{n-1} + \ldots = 0 \Rightarrow a_1 = 0,\ a_2 = 0,$ $3a_3 - a_0 = 0,\ 4a_4 - a_1 = 0$ and in general $na_n - a_{n-3} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_3 = \frac{a_0}{3} = \frac{1}{3},\ a_4 = \frac{a_1}{4} = 0,\ a_5 = \frac{a_2}{5} = 0,\ a_6 = \frac{a_3}{6} = \frac{1}{3\cdot 6},\ \ldots,\ a_{3n} = \frac{1}{3\cdot 6\cdot 9\cdots 3n},\ a_{3n+1} = 0$ and $a_{3n+2} = 0$

$\Rightarrow y = 1 + \frac{1}{3}x^3 + \frac{1}{3\cdot 6}x^6 + \frac{1}{3\cdot 6\cdot 9}x^9 + \ldots + \frac{1}{3\cdot 6\cdot 9\cdots 3n}x^{3n} + \ldots = \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!} = e^{x^3/3}$

23. $(1 - x)y' - y = (a_1 - a_0) + (2a_2 - a_1 - a_1)x + (3a_3 - 2a_2 - a_2)x^2 + (4a_4 - 3a_3 - a_3)x^3 + \ldots$ $+ (na_n - (n-1)a_{n-1} - a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 - a_0 = 0,\ 2a_2 - 2a_1 = 0,\ 3a_3 - 3a_2 = 0$ and in general $(na_n - na_{n-1}) = 0$. Since $y = 2$ when $x = 0$, we have $a_0 = 2$. Therefore $a_1 = 2,\ a_2 = 2,\ \ldots,\ a_n = 2 \Rightarrow y = 2 + 2x + 2x^2 + \ldots = \sum_{n=0}^{\infty} 2x^n = \frac{2}{1 - x}$

24. $(1 + x^2)y' + 2xy = a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1 + a_1)x^2 + (4a_4 + 2a_2 + 2a_2)x^3 + \ldots + (na_n + na_{n-2})x^{n-1} + \ldots$ $= 0 \Rightarrow a_1 = 0,\ 2a_2 + 2a_0 = 0,\ 3a_3 + 3a_1 = 0,\ 4a_4 + 4a_2 = 0$ and in general $na_n + na_{n-2} = 0$. Since $y = 3$ when $x = 0$, we have $a_0 = 3$. Therefore $a_2 = -3,\ a_3 = 0,\ a_4 = 3,\ \ldots,\ a_{2n+1} = 0,\ a_{2n} = (-1)^n 3$

$\Rightarrow y = 3 - 3x^2 + 3x^4 - \ldots = \sum_{n=0}^{\infty} 3(-1)^n x^{2n} = \sum_{n=0}^{\infty} 3\left(-x^2\right)^n = \frac{3}{1 + x^2}$

25. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)x + (4\cdot 3a_4 - a_2)x^2 + \ldots + (n(n-1)a_n - a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 - a_0 = 0,$
$3\cdot 2a_3 - a_1 = 0$, $4\cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when $x = 0$,
we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 0$, $a_3 = \frac{1}{3\cdot 2}$, $a_4 = 0$, $a_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2}$, $\ldots$, $a_{2n+1} = \frac{1}{(2n+1)!}$ and
$a_{2n} = 0 \Rightarrow y = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$

26. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' + y$
$= (2a_2 + a_0) + (3\cdot 2a_3 + a_1)x + (4\cdot 3a_4 + a_2)x^2 + \ldots + (n(n-1)a_n + a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 + a_0 = 0,$
$3\cdot 2a_3 + a_1 = 0$, $4\cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 0$ and $y = 1$ when $x = 0$,
we have $a_0 = 1$ and $a_1 = 0$. Therefore $a_2 = -\frac{1}{2}$, $a_3 = 0$, $a_4 = \frac{1}{4\cdot 3\cdot 2}$, $a_5 = 0$, $\ldots$, $a_{2n+1} = 0$ and $a_{2n} = \frac{(-1)^n}{(2n)!}$
$\Rightarrow y = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \ldots = \sum \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$

27. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' + y$
$= (2a_2 + a_0) + (3\cdot 2a_3 + a_1)x + (4\cdot 3a_4 + a_2)x^2 + \ldots + (n(n-1)a_n + a_{n-2})x^{n-2} + \ldots = x \Rightarrow 2a_2 + a_0 = 0,$
$3\cdot 2a_3 + a_1 = 1$, $4\cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 1$ and $y = 2$ when $x = 0$,
we have $a_0 = 2$ and $a_1 = 1$. Therefore $a_2 = -1$, $a_3 = 0$, $a_4 = \frac{1}{4\cdot 3}$, $a_5 = 0$, $\ldots$, $a_{2n} = -2\cdot\frac{(-1)^{n+1}}{(2n)!}$ and
$a_{2n+1} = 0 \Rightarrow y = 2 + x - x^2 + 2\cdot\frac{x^4}{4!} + \ldots = 2 + x - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{(2n)!}$

28. $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \Rightarrow y'' = 2a_2 + 3\cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)x + (4\cdot 3a_4 - a_2)x^2 + \ldots + (n(n-1)a_n - a_{n-2})x^{n-2} + \ldots = x \Rightarrow 2a_2 - a_0 = 0,$
$3\cdot 2a_3 - a_1 = 1$, $4\cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 2$ and $y = -1$ when $x = 0$,
we have $a_0 = -1$ and $a_1 = 2$. Therefore $a_2 = \frac{-1}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{-1}{2\cdot 3\cdot 4}$, $a_5 = \frac{1}{5\cdot 4\cdot 2} = \frac{3}{5!}$, $\ldots$, $a_{2n} = \frac{-1}{(2n)!}$
and $a_{2n+1} = \frac{3}{(2n+1)!} \Rightarrow y = -1 + 2x - \frac{1}{2}x^2 + \frac{3}{3!}x^3 - \ldots = -1 + 2x - \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{3x^{2n+1}}{(2n+1)!}$

29. $y = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots + a_n(x-2)^n + \ldots$
$\Rightarrow y'' = 2a_2 + 3\cdot 2a_3(x-2) + \ldots + n(n-1)a_n(x-2)^{n-2} + \ldots \Rightarrow y'' - y$
$= (2a_2 - a_0) + (3\cdot 2a_3 - a_1)(x-2) + (4\cdot 3a_4 - a_2)(x-2)^2 + \ldots + (n(n-1)a_n - a_{n-2})(x-2)^{n-2} + \ldots$
$= -2 - (x-2) \Rightarrow 2a_2 - a_0 = -2$, $3\cdot 2a_3 - a_1 = -1$, $4\cdot 3a_4 - a_2$ and in general $n(n-1)a_n - a_{n-2} = 0$.
Since $y' = -2$ and $y = 0$ when $x = 2$, we have $a_0 = 0$ and $a_1 = -2$. Therefore $a_2 = \frac{-2}{2} = -1$,
$a_3 = \frac{-2-1}{3\cdot 2} = -\frac{3}{3\cdot 2}$, $a_4 = -\frac{2}{4\cdot 3\cdot 2}$, $a_5 = -\frac{3}{5\cdot 4\cdot 3\cdot 2}$, $\ldots$, $a_{2n} = -\frac{2}{(2n)!}$, $a_{2n+1} = -\frac{3}{(2n+1)!}$
$\Rightarrow y = -2(x-2) - \frac{2}{2!}(x-2)^2 - \frac{3}{3!}(x-2)^3 - \frac{2}{4!}(x-2)^4 - \frac{3}{5!}(x-2)^5 - \ldots$

$$= -2(x-2) - \sum_{n=1}^{\infty}\left[\frac{2(x-2)^{2n}}{(2n)!} + \frac{3(x-2)^{2n+1}}{(2n+1)!}\right]$$

30. $y'' - x^2y = 2a_2 + 6a_3x + (4\cdot 3a_4 - a_0)x^2 + \ldots + (n(n-1)a_n - a_{n-4})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 = 0,\ 6a_3 = 0,$ $4\cdot 3a_4 - a_0 = 0,\ 5\cdot 4a_5 - a_1 = 0$, and in general $n(n-1)a_n - a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$, we have $a_0 = a$, $a_1 = b$, $a_2 = 0$, $a_3 = 0$, $a_4 = \frac{a}{3\cdot 4}$, $a_5 = \frac{b}{4\cdot 5}$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{a}{3\cdot 4\cdot 7\cdot 8}$, $a_9 = \frac{b}{4\cdot 5\cdot 8\cdot 9}$
$\Rightarrow y = a + bx + \frac{a}{3\cdot 4}x^4 + \frac{b}{4\cdot 5}x^5 + \frac{a}{3\cdot 4\cdot 7\cdot 8}x^8 + \frac{b}{4\cdot 5\cdot 8\cdot 9}x^9 + \ldots$

31. $y'' + x^2y = 2a_2 + 6a_3x + (4\cdot 3a_4 + a_0)x^2 + \ldots + (n(n-1)a_n + a_{n-4})x^{n-2} + \ldots = x \Rightarrow 2a_2 = 0,\ 6a_3 = 1,$ $4\cdot 3a_4 + a_0 = 0,\ 5\cdot 4a_5 + a_1 = 0$, and in general $n(n-1)a_n + a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$, we have $a_0 = a$ and $a_1 = b$. Therefore $a_2 = 0$, $a_3 = \frac{1}{2\cdot 3}$, $a_4 = -\frac{a}{3\cdot 4}$, $a_5 = -\frac{b}{4\cdot 5}$, $a_6 = 0$, $a_7 = \frac{1}{2\cdot 3\cdot 6\cdot 7}$
$\Rightarrow y = a + bx + \frac{1}{2\cdot 3}x^3 - \frac{a}{3\cdot 4}x^4 - \frac{b}{4\cdot 5}x^5 - \frac{1}{2\cdot 3\cdot 6\cdot 7}x^7 + \frac{ax^8}{3\cdot 4\cdot 7\cdot 8} + \frac{bx^9}{4\cdot 5\cdot 8\cdot 9} + \ldots$

32. $y'' - 2y' + y = (2a_2 - 2a_1 + a_0) + (2\cdot 3a_3 - 4a_2 + a_1)x + (3\cdot 4a_4 - 2\cdot 3a_3 + a_2)x^2 + \ldots$ $+ ((n-1)na_n - 2(n-1)a_{n-1} + a_{n-2})x^{n-2} + \ldots = 0 \Rightarrow 2a_2 - 2a_1 + a_0 = 0,\ 2\cdot 3a_3 - 4a_2 + a_1 = 0,$ $3\cdot 4a_4 - 2\cdot 3a_3 + a_2 = 0$ and in general $(n-1)na_n - 2(n-1)a_{n-1} + a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when when $x = 0$, we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 1$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{6}$, $a_5 = \frac{1}{24}$ and $a_n = \frac{1}{(n-1)!}$
$\Rightarrow y = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \ldots = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = xe^x$

33. $F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots\right)dt = \left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \ldots\right]_0^x \approx \frac{x^3}{3} - \frac{x^7}{7\cdot 3!}$
$\Rightarrow |\text{error}| < \frac{1}{11\cdot 5!} \approx 0.0008$

34. $F(x) = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \ldots\right)dt = \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7\cdot 2!} - \frac{t^9}{9\cdot 3!} + \frac{t^{11}}{11\cdot 4!} - \frac{t^{13}}{13\cdot 5!} + \ldots\right]_0^x$
$\approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7\cdot 2!} - \frac{x^9}{9\cdot 3!} + \frac{x^{11}}{11\cdot 4!} \Rightarrow |\text{error}| < \frac{1}{13\cdot 5!} \approx 0.00064$

35. (a) $F(x) = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right)dt = \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{30} - \ldots\right]_0^x \approx \frac{x^2}{2} - \frac{x^4}{12} \Rightarrow |\text{error}| < \frac{(0.5)^6}{30} \approx .00052$

(b) $|\text{error}| < \frac{1}{33\cdot 34} \approx .00089$ so $F(x) \approx \frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \ldots + (-1)^{15}\frac{x^{32}}{31\cdot 32}$

36. (a) $F(x) = \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots\right)dt = \left[t - \frac{t^2}{2\cdot 2} + \frac{t^3}{3\cdot 3} - \frac{t^4}{4\cdot 4} + \frac{t^5}{5\cdot 5} - \ldots\right]_0^x \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2}$

$\Rightarrow |\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$

(b) $|\text{error}| < \frac{1}{32^2} \approx .00097$ so $F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots + (-1)^{31}\frac{x^{31}}{31^2}$

37. $\frac{1}{x^2}(e^x - (1+x)) = \frac{1}{x^2}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots\right) - 1 - x\right) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots \Rightarrow \lim\limits_{x\to 0} \frac{e^x - (1+x)}{x^2}$

$= \lim\limits_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\right) = \frac{1}{2}$

38. $\frac{1}{t^4}\left(1 - \cos t - \frac{t^2}{2}\right) = \frac{1}{t^4}\left[1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right)\right] = -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots \Rightarrow \lim\limits_{t\to 0} \frac{1 - \cos t - \left(\frac{t^2}{2}\right)}{t^4}$

$= \lim\limits_{t\to 0}\left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\right) = -\frac{1}{24}$

39. $x^2\left(-1 + e^{-1/x^2}\right) = x^2\left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \ldots\right) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots \Rightarrow \lim\limits_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$

$= \lim\limits_{x\to\infty}\left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\right) = -1$

40. $\frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{y^3}{6} + \frac{23y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y}$

$\Rightarrow \lim\limits_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \lim\limits_{y\to 0} \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y} = -\frac{1}{6}$

41. $\frac{\ln(1+x^2)}{1 - \cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \ldots\right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} \Rightarrow \lim\limits_{x\to 0} \frac{\ln(1+x^2)}{1 - \cos x} = \lim\limits_{x\to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} = 2!$

$= 2$

42. $(x+1)\sin\left(\frac{1}{x+1}\right) = (x+1)\left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \ldots\right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots$

$\Rightarrow \lim\limits_{x\to\infty} (x+1)\sin\left(\frac{1}{x+1}\right) = \lim\limits_{x\to\infty}\left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots\right) = 1$

43. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

44. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^n}{n}\right| = \frac{1}{n10^n}$ when $x = 0.1$;

$\frac{1}{n10^n} < \frac{1}{10^8} \Rightarrow n10^n > 10^8$ when $n \geq 8 \Rightarrow 7$ terms

45. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\right| = \frac{1}{2n-1}$ when $x = 1$;

$\frac{1}{2n-1} < \frac{1}{10^3} \Rightarrow n > \frac{1001}{2} = 500.5 \Rightarrow$ the first term not used is the $501^{\text{st}} \Rightarrow$ we must use 500 terms

46. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and $\lim_{n\to\infty} \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = x^2 \lim_{n\to\infty} \left|\frac{2n-1}{2n+1}\right| = x^2$

$\Rightarrow \tan^{-1} x$ converges for $|x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ which is a convergent series; when $x = 1$

we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ which is a convergent series $\Rightarrow$ the series representing $\tan^{-1} x$ diverges for $|x| > 1$

47. (a) $\left(1-x^2\right)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$;

$$\lim_{n\to\infty} \left|\frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)x^{2n+3}}{2\cdot4\cdot6\cdots(2n)(2n+2)(2n+3)} \cdot \frac{2\cdot4\cdot6\cdots(2n)(2n+1)}{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim_{n\to\infty} \left|\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\right| < 1$$

$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1

(b) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\left(1-x^2\right)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

48. $\left(1-x^2\right)^{-1/2} = \left(1+\left(-x^2\right)\right)^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}\left(-x^2\right)^2}{2!}$

$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}\left(-x^2\right)^3}{3!} + \ldots = 1 + \frac{x^2}{2} + \frac{1\cdot3x^4}{2^2\cdot2!} + \frac{1\cdot3\cdot5x^6}{2^3\cdot3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)x^{2n}}{2^n\cdot n!}$

$\Rightarrow \sin^{-1} x = \int_0^x \left(1-t^2\right)^{-1/2} dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)x^{2n}}{2^n\cdot n!}\right) dt = x + \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}{2\cdot4\cdots(2n)(2n+1)}$,

where $|x| < 1$

49. $\left[\tan^{-1} t\right]_x^{\infty} = \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)}\right] dt = \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \ldots\right) dt$

$= \int_x^{\infty} \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \ldots\right) dt = \lim_{b\to\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots$

$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,\ x > 1$; $\left[\tan^{-1} t\right]_{-\infty}^{x} = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^{x} \frac{dt}{1+t^2}$

$= \lim_{b \to -\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots \right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,$
$x < -1$

50. (a) $\tan\left(\tan^{-1}(n+1) - \tan^{-1}(n-1)\right) = \frac{\tan\left(\tan^{-1}(n+1)\right) - \tan\left(\tan^{-1}(n-1)\right)}{1 + \tan\left(\tan^{-1}(n+1)\right)\tan\left(\tan^{-1}(n-1)\right)} = \frac{(n+1)-(n-1)}{1+(n+1)(n-1)} = \frac{2}{n^2}$

(b) $\sum_{n=1}^{N} \tan^{-1}\left(\frac{2}{n^2}\right) = \sum_{n=1}^{N} \left[\tan^{-1}(n+1) - \tan^{-1}(n-1)\right] = \left(\tan^{-1} 2 - \tan^{-1} 0\right) + \left(\tan^{-1} 3 - \tan^{-1} 1\right)$
$+ \left(\tan^{-1} 4 - \tan^{-1} 2\right) + \ldots + \left(\tan^{-1}(N+1) - \tan^{-1}(N-1)\right) = \tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4}$

(c) $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) = \lim_{N\to\infty} \left[\tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4}\right] = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}$

8.9 FOURIER SERIES

1. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1\, dx = \left(\frac{1}{\pi}\right) x \Big|_{-\pi}^{\pi} = 2$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{\pi n} \sin nx \Big|_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx\, dx = -\frac{1}{n\pi} \cos nx \Big|_{-\pi}^{\pi} = \frac{1}{nx}[\cos(-n\pi) - \cos(n\pi)] = 0$

Therefore,

$f(x) = \frac{a_0}{2} = 1.$

2. $a_0 = \frac{1}{\pi} \int_{-\pi}^{0} -dx + \frac{1}{\pi} \int_{0}^{\pi} dx = \left(\frac{1}{\pi}\right)(-x) \Big|_{\pi}^{0} + \left(\frac{1}{\pi}\right) x \Big|_{0}^{\pi} = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{0} -\cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos nx\, dx = \left(-\frac{1}{n\pi}\right) \sin nx \Big|_{\pi}^{0} + \left(\frac{1}{n\pi}\right) \sin nx \Big|_{0}^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx\, dx = \left(\frac{1}{n\pi}\right) \cos nx \Big|_{-\pi}^{0} + \left(-\frac{1}{n\pi}\right) \cos nx \Big|_{0}^{\pi}$

$= \frac{1}{nx}[\cos 0 - \cos(-n\pi)] + \left(-\frac{1}{n\pi}\right)(\cos n\pi - \cos 0)$

$= \frac{1}{n\pi}[1 - (-1)^n] - \frac{1}{n\pi}[(-1)^n - 1] = \frac{2}{n\pi}[1 - (-1)^n]$

Therefore,

$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - (-1)^n] \sin nx.$

3. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big[_{-\pi}^{\pi} = 0.$ (Note: x is an odd function)

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0.$ (because x cos nx is an odd function)

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$ (because x sin nx is even)

$= \frac{2}{\pi}\left(-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\Big[_0^{\pi}\right) = -\frac{2}{n}\cos nx = \frac{2}{n}(-1)^{n+1}$

Therefore,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx.$$

4. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\,dx = \frac{1}{\pi}\left(x - \frac{1}{2}x^2\right)\Big[_{-\pi}^{\pi} = 2$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\cos nx\,dx = \frac{1}{\pi}\left[\frac{1}{n}(1-x)\sin nx - \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\sin nx\,dx = -\frac{1}{\pi}\left[\frac{1}{n}(1-x)\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = \frac{2\pi}{n\pi}\cos n\pi = \frac{2}{n}(-1)^n.$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx = 1 - \sum_{n=}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$$

(Compare this result with the Fourier series found in problems 1 and 3.)

5. $a_0 = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\int_{0}^{\pi} x^2\,dx = \frac{\pi^2}{6}$

$a_n = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{1}{2\pi}\int_{0}^{\pi} x^2\cos nx\,dx$ (even function)

$= \frac{1}{2\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{1}{n^2}\cos n\pi = \frac{(-1)^n}{n^2}$

$b^n = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$ (odd function)

Therefore,

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

6. $a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{2}{n^2}\cos n\pi = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos x\right]_0^{\pi} = \frac{2}{\pi n^3}[(-1)^n - 1] + \frac{\pi}{n}(-1)^{n+1}$$

Therefore,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left\{\frac{2}{\pi n^3}[(-1)^n - 1] + \frac{\pi}{n}(-1)^{n+1}\right\}\sin nx.$$

7. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2}{\pi}\sinh \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx = \frac{1}{\pi}\left[\frac{n^2e^x}{1+n^2}\left(\frac{1}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2\cos n\pi}{(1+n^2)}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = \frac{2(-1)^n}{\pi(n^2+1)}\sinh \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{1}{\pi}\left[\frac{n^2e^x}{1+n^2}\left(-\frac{1}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2n\cos n\pi}{(1+n^2)}\left(\frac{e^{-\pi} - e^{\pi}}{2}\right) = \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh \pi$$

Therefore,

$$f(x) = \frac{1}{\pi}\sinh \pi + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(n^2+1)}\sinh \pi\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh \pi\sin nx$$

$$= \frac{\sinh \pi}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{n^2+1}\sin nx\right]$$

$$= \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right].$$

8. $a_0 = \frac{1}{\pi}\int_0^{\pi} e^x\,dx = \frac{1}{\pi}e^{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^x\cos nx\,dx = \frac{1}{\pi}\left[\frac{n^2e^x}{1+n^2}\left(\frac{1}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{n^2}{1+n^2}\right)\left(\frac{e^{\pi}}{n^2}\cos n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi(n^2+1)}[e^{\pi}(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} e^x\sin nx\,dx = \frac{1}{\pi}\left[\frac{n^2e^x}{1+n^2}\left(-\frac{1}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{n^2}{1+n^2}\right)\left(-\frac{e^{\pi}}{n}\cos n\pi + \frac{1}{n}\right) = \frac{n}{\pi(n^2+1)}\left[e^{\pi}(-1)^{n+1}+1\right]$$

Therefore,

$$f(x) = \frac{e^{\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi(n^2+1)}\left[e^{\pi}(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty} \frac{n}{\pi(n^2+1)}\left[e^{\pi}(-1)^{n+1}+1\right]\sin nx.$$

9. $a_0 = \frac{1}{\pi}\int_0^{\pi} \cos x\, dx = \frac{1}{\pi}\sin x\Big|_0^{\pi} = 0$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos x \cos nx\, dx = \begin{cases} 0, & n \neq 1 \\ \frac{1}{2}, & n = 1 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \cos x \sin nx\, dx = \begin{cases} \frac{1}{2\pi}\sin^2 x\Big|_0^{\pi} = 0, & n = 1 \\ \left(-\frac{\cos(n-1)x}{2\pi(n-1)} - \frac{\cos(n+1)x}{2\pi(n+1)}\right|_0^{\pi}, & n \neq 1 \end{cases} = \begin{cases} 0, & n = 1 \\ (1+(-1)^n)\frac{n}{\pi(n^2-1)}, & n \neq 1 \end{cases}$$

Therefore,

$$f(x) = \frac{1}{2}\cos x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{n(1+(-1)^n)}{n^2-1}\sin nx.$$

10. $a_0 = \frac{1}{2}\int_{-2}^{0} -x\, dx + \frac{1}{2}\int_0^2 2\, dx = 3$

$$a_n = \frac{1}{2}\int_{-2}^{0} -x\cos\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_0^2 2\cos\frac{n\pi x}{2}\, dx = \frac{1}{2}\left[-\frac{2x}{n\pi}\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_{-2}^{0} + \left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] + 0$$

$$b_n = \frac{1}{2}\int_{-2}^{0} -x\sin\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_0^2 2\sin\frac{n\pi x}{2}\, dx = \frac{1}{2}\left[\frac{2x}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{-2}^{0} - \left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}$$

Therefore,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2}\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi x}{2}.$$

11. $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx \, dx = \frac{1}{n\pi}\sin nx \Big]_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx \, dx = 0$$

Therefore,

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx.$$

Note: $\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2k \quad \text{(even)} \\ (-1)^k, & n = 2k+1 \quad \text{(odd)} \end{cases}$

Thus we can write f(x) in the form:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos(2k+1)x.$$

12. $a_0 = \int_{-1}^{1} |x| \, dx = 2\int_0^1 x \, dx = 1$

$$a_n = \int_{-1}^{1} |x|\cos(n\pi x) \, dx = 2\int_0^1 x\cos(n\pi x) \, dx = 2\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1 = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \int_{-1}^{1} |x|\sin(n\pi x) \, dx = 0 \qquad \text{(because } |x|\sin n\pi x \text{ is odd)}$$

Therefore,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x).$$

13. $a_0 = \int_{-1}^{1/2} -(2x-1) \, dx + \int_{1/2}^{1} (2x-1) \, dx = (x - x^2\big]_{-1}^{1/2} + (x^2 - x\big]_{1/2}^{1} = \frac{5}{2}$

$$a_n = \int_{-1}^{1/2} -(2x-1)\cos(n\pi x) \, dx + \int_{1/2}^{1} (2x-1)\cos(n\pi x) \, dx$$

$$= \left[\frac{(1-2x)}{n\pi}\sin(n\pi x) - \frac{2}{n^2\pi^2}\cos(n\pi x)\right]_{-1}^{1/2} + \left[\frac{(2x-1)}{n\pi}\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]_{1/2}^{1}$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] + \frac{2}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$b_n = \int_{-1}^{1/2} -(2x-1)\sin(n\pi x) \, dx + \int_{1/2}^{1} (2x-1)\sin(n\pi x) \, dx$$

$$= \left[-\frac{(1-2x)}{n\pi}\cos(n\pi x) - \frac{2}{n^2\pi^2}\sin(n\pi x)\right]_{-1}^{1/2} + \left[\frac{(2x-1)}{n\pi}\cos(n\pi x) + \frac{2}{n^2\pi^2}\sin(n\pi x)\right]_{1/2}^{1}$$

$$= \left[\frac{3}{n\pi}\cos n\pi - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \left[-\frac{1}{n\pi}\cos n\pi - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}(-1)^n - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{2}{n\pi}\left[(-1)^n - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

Therefore,

$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos(n\pi x) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^n - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\sin(n\pi x).$$

14. $f(x) = x|x|$ is an odd function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x|x|\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x|x|\cos nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x|x|\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{2}{n^3}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{(2-\pi^2n^2)(-1)^n - 2}{n^3}\right]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(2-\pi^2n^2)(-1)^n - 2}{n^3}\sin nx.$$

15. From exercise #5,

$$\frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

Setting $x = \pi$,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$\frac{3\pi^2}{12} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2},$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots.$$

16. From Exercise #6, $f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \left\{ \frac{2[(-1)^n - 1]}{\pi n^3} + \frac{\pi}{n}(-1)^{n+1} \right\} \sin nx$. Setting $x = 0$ and multiplying both sides by $\frac{1}{2}$ gives $\frac{\pi^2}{12} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$. Note: The Fourier series will converge to 0 at $x = 0$ because the discontinuity in f at $x = 0$ is removable.

17. $\int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx = \frac{L}{m\pi} \sin \frac{m\pi x}{L}\Big|_{-L}^{L} = \frac{L}{m\pi}[\sin m\pi - \sin(-m\pi)] = \frac{L}{m\pi}(0 - 0) = 0.$

18. $\int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx = -\frac{L}{m\pi} \cos \frac{m\pi x}{L}\Big|_{-L}^{L} = -\frac{L}{m\pi}[\cos m\pi - \cos(-m\pi)] = -\frac{L}{m\pi}(\cos m\pi - \cos m\pi) = 0.$

19. $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$

$$\begin{aligned}\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left[\cos \frac{(m+n)\pi x}{L} + \cos \frac{(n-m)\pi x}{L}\right] dx \\ &= 0, \quad \text{if } m \neq n, \text{ by exercise 17}\end{aligned}$$

If $m = n$,

$$\begin{aligned}\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left(\cos \frac{2m\pi x}{L} + 1\right) dx = \frac{1}{2} \int_{-L}^{L} \cos \frac{2m\pi x}{L}\, dx + \frac{1}{2} \int_{-L}^{L} dx \\ &= 0 + L, \quad \text{by exercise 17} \\ &= L.\end{aligned}$$

20. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

$$\begin{aligned}\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left[\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}\right] dx \\ &= 0, \quad \text{if } m \neq n, \text{ by exercise 17}\end{aligned}$$

If $m = n$,

$$\begin{aligned}\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{mx\pi}{L}\, dx &= \frac{1}{2} \int_{-L}^{L} \left(1 - \cos \frac{2m\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} dx - \frac{1}{2} \int_{-L}^{L} \cos \frac{2m\pi x}{L}\, dx \\ &= L - 0, \quad \text{by exercise 17} \\ &= L.\end{aligned}$$

21. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

If $m \neq n$,

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{(n+m)\pi x}{L} + \sin\frac{(n-m)\pi x}{L}\right]dx = 0, \text{ by exercise 18}$$

If $m = n$,

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(\sin\frac{2m\pi x}{L} + 0\right)dx = 0, \text{ by exercise 18}$$

22. If two functions, f and g, are piecewise continuous on an interval f, then so is $f+g$. This is true because of the properties of limits: $\lim\limits_{x\to c^+}[f(x)+g(x)] = \lim\limits_{x\to c^+} f(x) + \lim\limits_{x\to c^+} g(x) = f(c^+) + g(c^+)$ and $\lim\limits_{x\to c^-}[f(x)+g(x)]$

$= \lim\limits_{x\to c^-} f(x) + \lim\limits_{x\to c^-} g(x) = f(c^-) + g(c^-)$. Therefore, if f and g are piecewise continuous on I, then so is $f+g$.

This result also applies to the functions f' and g', that is, if f' and g' are piecewise continuous on I, then so is $f' + g' = (f+g)'$. Consequently, Theorem 18 applies to $f+g$, and $f+g$ is equal to its Fourier series at all points of continuity, and at jump discontinuities in $f+g$, the Fourier series converges to the average $\frac{(f+g)(c^+)+(f+g)(c^-)}{2} = \frac{f(c^+)+f(c^-)}{2} + \frac{g(c^+)+g(c^-)}{2}$ where $f(c^+)$, $f(c^-)$, $g(c^+)$, and $g(c^-)$ denote the right and left limits of f and g at c.

23. (a) Since the function $f(x) = x$ and its derivative $f'(x) = 1$ are continuous on $-\pi < x < \pi$, the function f satisfies conditions of Theorem 18, and $f(x) = x = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)$.

(b) $\frac{d}{dx}\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx) = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\frac{d}{dx}(\sin(nx)) = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\,2\cos(nx)$

This series diverges by the n^{th} term test because $\lim\limits_{x\to\infty}\left((-1)^{n+1}\,2\cos(nx)\right) \neq 0$.

(c) We cannot be assured that term-by-term differentiation of the Fourier series of a piecewise continuous function gives a Fourier series that converges on the derivative of the function and, in fact, the series might not converge at all.

24. $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ since f is piecewise continuous on $-\pi < x < \pi$. Therefore,

$$\int_{-\pi}^{x} f(s)\,ds = \int_{-\pi}^{x}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(ns) + b_n\sin(ns)]\right]ds = \int_{-\pi}^{x}\frac{a_0}{2}\,ds + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{x}\cos(ns)\,ds + b_n\int_{-\pi}^{x}\sin(ns)\,ds\right]$$

$$= \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty}\left[\frac{a_n}{n}(\sin(nx) - \sin(-n\pi)) - \frac{b_n}{n}(\cos(nx) - \cos(-n\pi)\right]$$

$$= \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty}\frac{1}{n}(a_n\sin(nx) - b_n(\cos(nx) - \cos(n\pi)))$$

8.10 FOURIER COSINE AND SINE SERIES

1. $a_0 = \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{1}{\pi} x^2 \Big|_0^{\pi} = \pi$

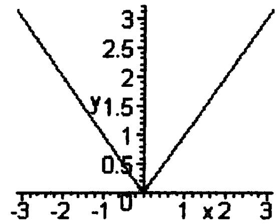
$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx = \frac{2}{\pi}\left[\frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2} \cos n\pi - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$b_n = 0$

Therefore,

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos nx.$$



Even extension of $f(x)$

2. $a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x\, dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{4}{\pi}$; $a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x\, dx = 0$;

For $n = \geq 2$, $a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx\, dx$

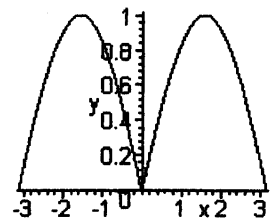
$$= \frac{2}{\pi}\left(\frac{n^2}{n^2 - 1}\right)\left[\frac{1}{n} \sin x \sin nx + \frac{1}{n^2} \cos x \cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{n^2}{n^2 - 1}\right)\left(\frac{1}{n^2} \cos \pi \cos n\pi - + \frac{1}{n^2}\right) = \frac{2}{\pi(n^2 - 1)}[(-1)^{n+1} - 1]$$

$b_n = 0$

Therefore

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{[(-1)^n + 1]}{1 - n^2} \cos nx.$$



Even extension of $f(x)$

3. $a_0 = 2 \int_0^{1} e^x\, dx = 2e^x \Big|_0^1 = 2(e - 1)$

$$a_n = 2 \int_0^{1} e^x \cos n\pi x\, dx$$

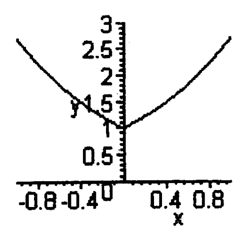
$$= 2\left(\frac{n^2\pi^2}{1 + n^2\pi^2}\right)\left[e^x\left(\frac{1}{n\pi} \sin n\pi x + \frac{1}{n^2\pi^2} \cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1 + n^2\pi^2}(e \cos n\pi - 1) = \frac{2\left[e(-1)^n - 1\right]}{1 + n^2\pi^2}$$

$b_n = 0$

Therefore

$$f(x) = (e - 1) + 2 \sum_{n=1}^{\infty} \frac{\left[e(-1)^n - 1\right]}{1 + n^2\pi^2} \cos n\pi x.$$



Even extension of $f(x)$

4. $a_0 = \frac{2}{\pi}\int_0^{\pi} \cos x \, dx = \frac{2}{\pi}\sin x\Big|_0^{\pi} = 0$

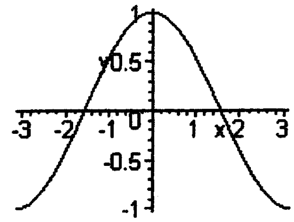
$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \ \cos nx \, dx$$
$$= \frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left[\frac{1}{n}\cos x \sin nx - \frac{1}{n^2}\sin x \cos nx\right]_0^{\pi}$$
$$= 0, \ \text{if } n \neq 1.$$

For $n = 1$:

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \cos^2 x \, dx = \frac{2}{\pi}\left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{\pi} = 1$$

Therefore,

$$f(x) = a_1 \cos \tfrac{\pi x}{\pi} = \cos x.$$



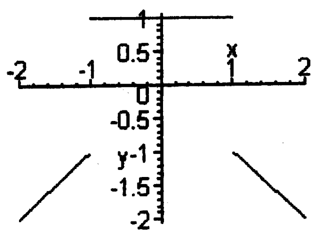
Even extension of $f(x)$

5. $a_0 = \frac{2}{2}\int_0^1 dx + \frac{2}{2}\int_1^2 -x \, dx = 1 + \left(-\frac{1}{2}x^2\right|_1^2 = -\frac{1}{2}$

$$a_n = \int_0^1 \cos\frac{n\pi x}{2} \, dx + \int_1^2 -x \cos\frac{n\pi x}{2} \, dx$$
$$= \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 - \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_1^2$$
$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\sin n\pi - \frac{4}{n^2\pi^2}\cos n\pi + \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$
$$= \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left[(-1)^{n+1} + \cos\frac{n\pi}{2}\right]$$

Therefore,

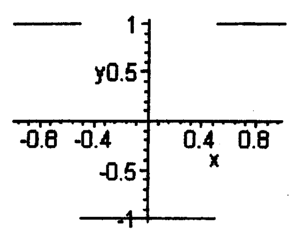
$$f(x) = -\frac{1}{4} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{\pi n^2}\left((-1)^{n+1} + \cos\frac{n\pi}{2}\right)\right]\cos\frac{n\pi x}{2}.$$



Even extension of $f(x)$

6. $a_0 = \frac{2}{1}\int_0^{1/2} -dx + \frac{2}{1}\int_{1/2}^1 dx = 0$

$$a_n = 2\int_0^{1/2} -\cos n\pi x \, dx + 2\int_{1/2}^1 \cos n\pi x \, dx$$
$$= -\frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} + \frac{2}{n\pi}\sin n\pi x\Big|_{1/2}^1$$
$$= \frac{2}{n\pi}\left(-\sin\frac{n\pi}{2} + \sin n\pi - \sin\frac{n\pi}{2}\right)$$
$$= -\frac{4}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} -\frac{4}{n\pi}(-1)^k, & \text{if } n = 2k+1 \quad \text{(odd)} \\ 0, & \text{if } n = 2k \quad \text{(even)} \end{cases}$$

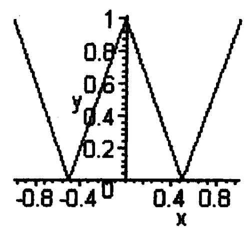


Even extension of $f(x)$

Thus,

$$f(x) = \sum_{k=0}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k+1)} \cos(2k+1)\pi x.$$

7. $a_0 = 2\int_0^{1/2} -(2x-1)\,dx + 2\int_{1/2}^{1} (2x-1)\,dx = 1$



Even extension of $f(x)$

$$a_n = 2\int_0^{1/2} -(2x-1)\cos n\pi x\,dx + 2\int_{1/2}^{1} (2x-1)\cos n\pi x\,dx$$

$$= 2\left[\frac{(1-2x)}{n\pi}\sin n\pi x - \frac{2}{n^2\pi^2}\cos n\pi x\right]_0^{1/2}$$

$$+ 2\left[\frac{(2x-1)}{n\pi}\sin n\pi x + \frac{2}{n^2\pi^2}\cos n\pi x\right]_{1/2}^{1}$$

$$= 2\left[0 - \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2} - 0 + \frac{2}{n^2\pi^2}\right]$$

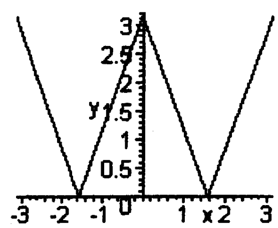
$$+ 2\left[0 + \frac{2}{n^2\pi^2}\cos n\pi - 0 - \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1 - 2\cos\frac{n\pi}{2} + (-1)^n\right]$$

Therefore,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\right]\cos n\pi x.$$

8. $a_0 = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (2x-\pi)\,dx = \pi$



Even extension of $f(x)$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (2x-\pi)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi-2x)}{n}\sin nx - \frac{2}{n^2}\cos nx\right]_0^{\pi/2}$$

$$+ \frac{2}{\pi}\left[\frac{(2x-\pi)}{n}\sin nx + \frac{2}{n^2}\cos nx\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n^2}\cos\frac{n\pi}{2} - 0 + \frac{2}{n^2}\right] + \frac{2}{\pi}\left[0 + \frac{2}{n^2}\cos n\pi - 0 - \frac{2}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n^2}[1 + (-1)^n] - \frac{2}{n^2}\cos\frac{n\pi}{2}\right]$$

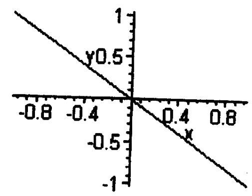
Therefore,

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[(-1)^n + 1 - 2\cos\frac{n\pi}{2}\right]\cos nx.$$

9. $b_n = 2\int_0^1 -x \sin n\pi x\,dx = 2\Big[\frac{x}{n\pi}\cos n\pi x$
$-\frac{1}{n^2\pi^2}\sin n\pi x\Big]_0^1 = \frac{2}{n\pi}\cos n\pi = \frac{2(-1)^n}{n\pi}$

Therefore,

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi x.$$



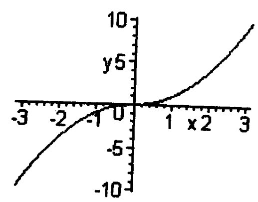
Even extension of $f(x)$

10. $b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx = \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^\pi$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}\cos n\pi - \frac{2}{n^3}\right] = \frac{2}{\pi}\left[(-1)^n\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right) - \frac{2}{n^3}\right]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[(-1)^n\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right) - \frac{2}{n^3}\right]\sin nx.$$



Even extension of $f(x)$

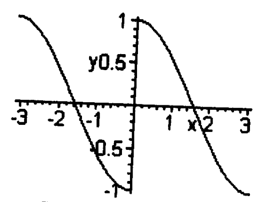
11. $b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx$

$$= \frac{2}{\pi}\left[\frac{n^2}{n^2-1}\left(-\frac{1}{n}\cos x\cos nx - \frac{1}{n^2}\sin x \sin nx\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{n}{n^2-1}(-\cos\pi\cos n\pi + 1)\right] = \frac{2n}{\pi(n^2-1)}[(-1)^n+1]$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n[(-1)^n+1]}{n^2-1}\sin nx = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{k}{4k^2-1}\sin 2kx.$$



Even extension of $f(x)$

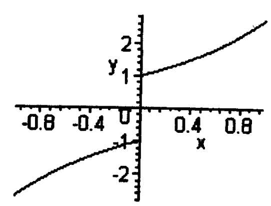
12. $b_n = 2\int_0^1 e^x \sin n\pi x\,dx$

$$= 2\left[\frac{n^2\pi^2}{n^2\pi^2+1}\cdot e^x\left(-\frac{1}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= \frac{2n^2\pi^2}{n^2\pi^2+1}\left(-\frac{e}{n\pi}\cos n\pi + \frac{1}{n\pi}\right) = \frac{2n\pi}{1+n^2\pi^2}[e(-1)^{n+1}+1]$$

Therefore,

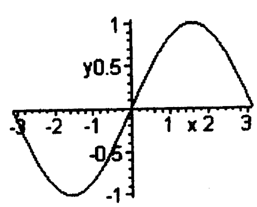
$$f(x) = 2\pi\sum_{n=1}^{\infty}\frac{[e(-1)^{n+1}+1]n}{1+n^2\pi^2}\sin n\pi x.$$



Even extension of $f(x)$

13. $b_n = \frac{2}{\pi}\int_0^\pi \sin x \sin nx\,dx$

$= 0$, if $n \neq 1$



Even extension of $f(x)$

$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{2}{\pi}\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1.$

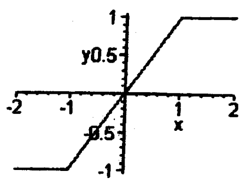
Therefore, $f(x) = b_1 \sin x = \sin x$.

14. $b_n = \int_0^1 x \sin\frac{n\pi x}{2}\, dx + \int_1^2 \sin\frac{n\pi x}{2}\, dx$

$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^1 - \left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2$

$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2}$

$= \frac{2}{n\pi}(-1)^{n+1} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}.$



Even extension of $f(x)$

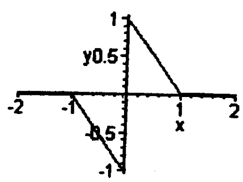
Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}.$

15. $b_n = \int_0^1 (1-x)\sin\frac{n\pi x}{2}\, dx$

$= \left[-\frac{2(1-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^1 = \frac{-4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}.$

Therefore,

$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}.$



Even extension of $f(x)$

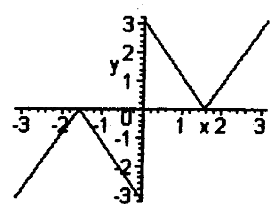
16. $b_n = \frac{2}{\pi}\int_0^{\pi/2} -(2x-\pi)\sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (2x-\pi)\sin nx\, dx$

$= \frac{2}{\pi}\left[\frac{(2x-\pi)}{n}\cos nx - \frac{2}{n^2}\sin nx\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{(\pi-2x)}{n}\cos nx + \frac{2}{n^2}\sin nx\right]_{\pi/2}^{\pi}$

$= \frac{2}{\pi}\left[-\frac{2}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n} - \frac{\pi}{n}\cos n\pi - \frac{2}{n^2}\sin\frac{n\pi}{2}\right]$

$= \frac{2}{\pi}\left[\frac{\pi}{n}[(-1)^{n+1}+1] - \frac{4}{n^2}\sin\frac{n\pi}{2}\right].$

Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\pi}{n}[(-1)^{n+1}+1] - \frac{4}{n^2}\sin\frac{n\pi}{2}\right]\sin nx.$



Even extension of $f(x)$

17. (a) $b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{n\pi}[-\cos nx]_0^{\pi} = \frac{2}{n\pi}(-\cos n\pi + 1) = \frac{2}{n\pi}[1-(-1)^n] \Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin nx$

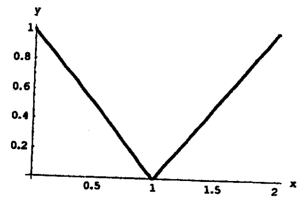
$= \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin[(2n-1)x] = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots$

$\Rightarrow f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\right] \Rightarrow \frac{\pi}{4}f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots.$

(b) Evaluate $f(x)$ at $x = \frac{\pi}{2} \Rightarrow \frac{\pi}{4}\cdot 1 = \sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \ldots$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

18. (a)



(b) Use the even extension of f(x) over the interval $-2 < x < 2$.

$$a_0 = \frac{2}{2} \int_0^2 f(x)\,dx = \frac{2}{2} \cdot 1 = 1;$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^1 (1-x) \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (x-1) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx - \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 x \cos\left(\frac{n\pi x}{2}\right) dx$$

Evaluate the two integrals:

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\int x \cos\left(\frac{n\pi x}{2}\right) dx = \begin{bmatrix} u = x;\ dv = \cos\left(\frac{n\pi x}{2}\right) dx \\ du = dx;\ v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \end{bmatrix} = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi} \int \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

Therefore, $a_n = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^1$

$$+\left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_1^2 = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right)$$

$$+ \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2} \cos n\pi - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\left[1 + \cos n\pi - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

The b_n's are all zero since the Fourier series is for an even extension of f(x).

Table of coefficient values:

n	0	1	2	3	4	5	6	7	8	9	10	...
a_n	1	0	$\frac{4}{\pi^2}$	0	0	0	$\frac{4}{9\pi^2}$	0	0	0	$\frac{4}{25\pi^2}$	...

Therefore, the Fourier series representation of f(x) is:

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(4m-2)\pi x}{2} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos((2m-1)\pi x)$$

(c) Same answer as in part (b).

19. $f(x) = \sin x = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n + 1]}{1-n^2}\cos nx = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx)$ for $0 < x < \pi$

Evaluate the function and its series representation at $x = \frac{\pi}{2}$.

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(n\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi} - 1\right) = \frac{1}{2} - \frac{\pi}{4}$$

20. Any piecewise continuous extension of f(x) over the interval $-2 < x < 2$ will give a Fourier series representation that will converge to f(x) in the interval $0 < x < 2$. For example, the function $g(x) = 2 - x$ for $-2 < x < 2$ will work.

CHAPTER 8 PRACTICE EXERCISES

1. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1$

2. converges to 0, since $0 \le a_n \le \frac{2}{\sqrt{n}}$, $\lim\limits_{n\to\infty} 0 = 0$, $\lim\limits_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences

3. converges to -1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{1-2^n}{2^n}\right) = \lim\limits_{n\to\infty}\left(\frac{1}{2^n} - 1\right) = -1$

4. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left[1 + (0.9)^n\right] = 1 + 0 = 1$

5. diverges, since $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \ldots\}$

6. converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \ldots\}$

7. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{\ln n^2}{n} = 2\lim\limits_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{\ln(2n+1)}{n} = \lim\limits_{n\to\infty}\frac{\left(\frac{2}{2n+1}\right)}{1} = 0$

9. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{n + \ln n}{n}\right) = \lim\limits_{n\to\infty}\frac{1 + \left(\frac{1}{n}\right)}{1} = 1$

10. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{\ln\left(2n^3+1\right)}{n} = \lim\limits_{n\to\infty}\frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim\limits_{n\to\infty}\frac{12n}{6n^2} = \lim\limits_{n\to\infty}\frac{2}{n} = 0$

11. converges to e^{-5}, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n-5}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{(-5)}{n}\right)^n = e^{-5}$ by Table 8.1

12. converges to $\frac{1}{e}$, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ by Table 8.1

13. converges to 3, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3^n}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Table 8.1

14. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Table 8.1

15. converges to ln 2, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n\left(2^{1/n}-1\right) = \lim\limits_{n\to\infty} \frac{2^{1/n}-1}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{\left[\frac{\left(-2^{1/n}\ln 2\right)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim\limits_{n\to\infty} 2^{1/n}\ln 2$
$= 2^0 \cdot \ln 2 = \ln 2$

16. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n+1} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$

17. diverges, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty$

18. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Table 8.1

19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \ldots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$
$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$

20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2}+\frac{2}{3}\right) + \left(\frac{-2}{3}+\frac{2}{4}\right) + \ldots + \left(\frac{-2}{n}+\frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim\limits_{n\to\infty} s_n$
$= \lim\limits_{n\to\infty} \left(-1+\frac{2}{n+1}\right) = -1$

21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2}-\frac{3}{5}\right) + \left(\frac{3}{5}-\frac{3}{8}\right) + \left(\frac{3}{8}-\frac{3}{11}\right) + \ldots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$
$= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\frac{3}{2} - \frac{3}{3n+2}\right) = \frac{3}{2}$

22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9}+\frac{2}{13}\right) + \left(\frac{-2}{13}+\frac{2}{17}\right) + \left(\frac{-2}{17}+\frac{2}{21}\right) + \ldots + \left(\frac{-2}{4n-3} + \frac{2}{4n+1}\right)$
$= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(-\frac{2}{9} + \frac{2}{4n+1}\right) = -\frac{2}{9}$

23. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1-\left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)\left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is

$$\frac{\left(-\frac{3}{4}\right)}{1-\left(\frac{-1}{4}\right)} = -\frac{3}{5}$$

25. diverges, a p-series with $p = \frac{1}{2}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n} = -5 \sum_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series

27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the nth term of a convergent p-series

29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the nth term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

30. $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^{1} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test

31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the nth term of a convergent p-series

32. diverges by the Direct Comparison Test for $e^{n^n} > n \Rightarrow \ln\left(e^{n^n}\right) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n)$ $\Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the nth term of the divergent harmonic series

33. $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test

34. Since $f(x) = \dfrac{3x^2}{x^3+1} \Rightarrow f'(x) = \dfrac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Rightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\lim\limits_{n\to\infty} \dfrac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim\limits_{n\to\infty} \dfrac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

35. converges absolutely by the Ratio Test since $\lim\limits_{n\to\infty} \left[\dfrac{n+2}{(n+1)!} \cdot \dfrac{n!}{n+1}\right] = \lim\limits_{n\to\infty} \dfrac{n+2}{(n+1)^2} = 0 < 1$

36. diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist

37. converges absolutely by the Ratio Test since $\lim\limits_{n\to\infty} \left[\dfrac{3^{n+1}}{(n+1)!} \cdot \dfrac{n!}{3^n}\right] = \lim\limits_{n\to\infty} \dfrac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2^n 3^n}{n^n}} = \lim\limits_{n\to\infty} \dfrac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)}$

$= \sqrt{\lim\limits_{n\to\infty} \dfrac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n^2(n^2-1)}{n^4}} = 1$

41. $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\dfrac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \dfrac{n3^n}{(x+4)^n}\right| < 1 \Rightarrow \dfrac{|x+4|}{3} \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right) < 1 \Rightarrow \dfrac{|x+4|}{3} < 1$

$\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 3^n}{n3^n} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{n3^n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \leq x < -1$

(b) the interval of absolute convergence is $-7 < x < -1$

(c) the series converges conditionally at $x = -7$

42. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Rightarrow (x-1)^2 \lim\limits_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

43. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n}\right| < 1 \Rightarrow |3x-1| \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$

$\Rightarrow -1 < 3x - 1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$

$= -\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent; at $x = \frac{2}{3}$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$

(b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$

(c) there are no values for which the series converges conditionally

44. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n}\right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim\limits_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1}\right| < 1$

$\Rightarrow \frac{|2x+1|}{2}(1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have

$\sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the nth-Term Test for Divergence since $\lim\limits_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \ne 0$; at $x = \frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum\limits_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the nth-Term Test

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

45. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}\right| < 1 \Rightarrow |x| \lim\limits_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right| < 1 \Rightarrow \frac{|x|}{e} \lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$

$\Rightarrow \frac{|x|}{e} \cdot 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

46. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}$;

the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm\sqrt{3}$, both diverge

(a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$

(b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$

(c) there are no values for which the series converges conditionally

48. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n+3}}{2n+3} \cdot \frac{2n+1}{(x-1)^{2n+1}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right) < 1 \Rightarrow (x-1)^2(1) < 1$

$\Rightarrow (x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges; at $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$, which also converges conditionally

(a) the radius is 1; the interval of convergence is $0 \le x \le 2$

(b) the interval of absolute convergence is $0 < x < 2$

(c) the series converges conditionally at $x = 0$ and $x = 2$

49. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{\left(\frac{2}{e^{n+1} - e^{-n-1}}\right)}{\left(\frac{2}{e^n - e^{-n}}\right)}\right| < 1$

$\Rightarrow |x| \lim_{n\to\infty} \left|\frac{e^{-1} - e^{-2n-1}}{1 - e^{-2n-2}}\right| < 1 \Rightarrow \frac{|x|}{e} < 1 \Rightarrow -e < x < e$; the series $\sum_{n=1}^{\infty} (\pm e)^n \operatorname{csch} n$, obtained with $x = \pm e$, both diverge since $\lim_{n\to\infty} (\pm e)^n \operatorname{csch} n \neq 0$

(a) the radius is e; the interval of convergence is $-e < x < e$

(b) the interval of absolute convergence is $-e < x < e$

(c) there are no values for which the series converges conditionally

50. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1} \coth(n+1)}{x^n \coth(n)}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{1+e^{-2n-2}}{1-e^{-2n-2}} \cdot \frac{1-e^{-2n}}{1+e^{-2n}}\right| < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < x$; the series $\sum_{n=1}^{\infty} (\pm 1)^n \coth n$, obtained with $x = \pm 1$, both diverge since $\lim_{n\to\infty} (\pm 1)^n \coth n \neq 0$

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

51. The given series has the form $1 - x + x^2 - x^3 + \ldots + (-x)^n + \ldots = \frac{1}{1+x}$, where $x = \frac{1}{4}$; the sum is $\frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

52. The given series has the form $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \ln(1+x)$, where $x = \frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right) \approx 0.510825624$

53. The given series has the form $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sin x$, where $x = \pi$; the sum is $\sin \pi = 0$

54. The given series has the form $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \cos x$, where $x = \frac{\pi}{3}$; the sum is $\cos \frac{\pi}{3} = \frac{1}{2}$

55. The given series has the form $1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x$, where $x = \ln 2$; the sum is $e^{\ln(2)} = 2$

56. The given series has the form $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \ldots = \tan^{-1} x$, where $x = \frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a = 1$ and $r = 2x \Rightarrow \frac{1}{1-2x}$ $= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ where $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with $a = 1$ and $r = -x^3 \Rightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$ $= 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1$

59. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$

60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$

61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(x^{5/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$

62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{5x} = \cos\left((5x)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left((5x)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{(2n)!}$

63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$

64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

65. $f(x) = \sqrt{3+x^2} = \left(3+x^2\right)^{1/2} \Rightarrow f'(x) = x\left(3+x^2\right)^{-1/2} \Rightarrow f''(x) = -x^2\left(3+x^2\right)^{-3/2} + \left(3+x^2\right)^{-1/2}$
$\Rightarrow f'''(x) = 3x^3\left(3+x^2\right)^{-5/2} - 3x\left(3+x^2\right)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$,
$f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2 \cdot 1!} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!} + \ldots$

66. $f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$; $f(2) = -1$, $f'(2) = 1$,
$f''(2) = -2$, $f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \ldots$

67. $f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}$; $f(3) = \frac{1}{4}$,
$f'(3) = -\frac{1}{4^2}$, $f''(3) = \frac{2}{4^3}$, $f'''(2) = \frac{-6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \ldots$

68. $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$; $f(a) = \frac{1}{a}$, $f'(a) = -\frac{1}{a^2}$, $f''(a) = \frac{2}{a^3}$,
$f'''(a) = \frac{-6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \ldots$

69. Assume the solution has the form $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1} + a_n x^n + \ldots$
$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$
$= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 + a_0 = 0$, $2a_2 + a_1 = 0$,
$3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore $a_1 = 1$,
$a_2 = \frac{-a_1}{2 \cdot 1} = -\frac{1}{2}$, $a_3 = \frac{-a_2}{3} = \frac{1}{3 \cdot 2}$, $a_4 = \frac{-a_3}{4} = -\frac{1}{4 \cdot 3 \cdot 2}$, …, $a_n = \frac{-a_{n-1}}{n} = \frac{-1}{n} \frac{(-1)^n}{(n-1)!} = \frac{(-1)^{n+1}}{n!}$
$\Rightarrow y = -1 + x - \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 - \ldots + \frac{(-1)^{n+1}}{n!}x^n + \ldots = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = -e^{-x}$

70. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 0 \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 0,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = -3$ when $x = 0$ we have $a_0 = -3$. Therefore $a_1 = -3$,

$a_2 = \frac{a_1}{2} = \frac{-3}{2},\ a_3 = \frac{a_2}{3} = \frac{-3}{3\cdot 2},\ a_n = \frac{a_{n-1}}{n} = \frac{-3}{n!} \Rightarrow y = -3 - 3x - \frac{3}{2\cdot 1}x^2 - \frac{3}{3\cdot 2}x^3 - \ldots - \frac{-3}{n!}x^n + \ldots$

$= -3\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right) = -3\sum_{n=0}^{\infty} \frac{x^n}{n!} = -3e^x$

71. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + 2y$

$= (a_1 + 2a_0) + (2a_2 + 2a_1)x + (3a_3 + 2a_2)x^2 + \ldots + (na_n + 2a_{n-1})x^{n-1} + \ldots = 0$. Since $y = 3$ when $x = 0$ we have $a_0 = 3$. Therefore $a_1 = -2a_0 = -2(3) = -3(2),\ a_2 = -\frac{2}{2}a_1 = -\frac{2}{2}(-2\cdot 3) = 3\left(\frac{2^2}{2}\right),\ a_3 = -\frac{2}{3}a_2$

$= -\frac{2}{3}\left[3\left(\frac{2^2}{2}\right)\right] = -3\left(\frac{2^3}{3\cdot 2}\right),\ \ldots,\ a_n = \left(-\frac{2}{n}\right)a_{n-1} = \left(-\frac{2}{n}\right)\left(3\left(\frac{(-1)^{n-1}2^{n-1}}{(n-1)!}\right)\right) = 3\left(\frac{(-1)^n 2^n}{n!}\right)$

$\Rightarrow y = 3 - 3(2x) + 3\frac{(2)^2}{2}x^2 - 3\frac{(2)^3}{3\cdot 2}x^3 + \ldots + 3\frac{(-1)^n 2^n}{n!}x^n + \ldots$

$= 3\left[1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \ldots + \frac{(-1)^n(2x)^n}{n!} + \ldots\right] = 3\sum_{n=0}^{\infty} \frac{(-1)^n(2x)^n}{n!} = 3e^{-2x}$

72. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$

$= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = 1 \Rightarrow a_1 + a_0 = 1,\ 2a_2 + a_1 = 0,$

$3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$ for $n > 1$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore

$a_1 = 1 - a_0 = 1,\ a_2 = \frac{-a_1}{2\cdot 1} = -\frac{1}{2},\ a_3 = \frac{-a_2}{3} = \frac{1}{3\cdot 2},\ a_4 = \frac{-a_3}{4} = -\frac{1}{4\cdot 3\cdot 2},\ \ldots,\ a_n$

$= \frac{-a_{n-1}}{n} = \left(\frac{-1}{n}\right)\frac{(-1)^n}{(n-1)!} = \frac{(-1)^{n+1}}{n!} \Rightarrow y = 0 + x - \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 - \ldots + \frac{(-1)^{n+1}}{n!}x^n + \ldots$

$= -1\left[1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 - \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right] + 1 = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + 1 = 1 - e^{-x}$

73. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = 3x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 3,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = -1$ when $x = 0$ we have $a_0 = -1$. Therefore

$a_1 = -1,\ a_2 = \frac{3 + a_1}{2} = \frac{2}{2},\ a_3 = \frac{a_2}{3} = \frac{2}{3\cdot 2},\ a_4 = \frac{a_3}{4} = \frac{2}{4\cdot 3\cdot 2},\ \ldots,\ a_n = \frac{a_{n-1}}{n} = \frac{2}{n!}$

$\Rightarrow y = -1 - x + \left(\frac{2}{2}\right)x^2 + \frac{3}{3\cdot 2}x^3 + \frac{2}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{2}{n!}x^n + \ldots$

$= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 3 - 3x = 2\sum_{n=0}^{\infty}\frac{x^n}{n!} - 3 - 3x = 2e^x - 3x - 3$

74. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} + y$

$= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \ldots + (na_n + a_{n-1})x^{n-1} + \ldots = x \Rightarrow a_1 + a_0 = 0,\ 2a_2 + a_1 = 1,$

$3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$ for $n > 2$. Since $y = 0$ when $x = 0$ we have $a_0 = 0$. Therefore

$a_1 = 0,\ a_2 = \frac{1 - a_1}{2} = \frac{1}{2},\ a_3 = \frac{-a_2}{3} = -\frac{1}{3\cdot 2}, \ldots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 0 - 0x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots\right) - 1 + x$

$= \sum_{n=0}^{\infty}\frac{(-1)^nx^n}{n!} - 1 + x = e^{-x} + x - 1$

75. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = 1,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = 1$ when $x = 0$ we have $a_0 = 1$. Therefore

$a_1 = 1,\ a_2 = \frac{1 + a_1}{2} = \frac{2}{2},\ a_3 = \frac{a_2}{3} = \frac{2}{3\cdot 2},\ a_4 = \frac{a_3}{4} = \frac{2}{4\cdot 3\cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{2}{n!}$

$\Rightarrow y = 1 + x + \left(\frac{2}{2}\right)x^2 + \frac{2}{3\cdot 2}x^3 + \frac{2}{4\cdot 2\cdot 2}x^4 + \ldots + \frac{2}{n!}x^n + \ldots$

$= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) - 1 - x = 2\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1 - x = 2e^x - x - 1$

76. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \Rightarrow \frac{dy}{dx} - y$

$= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots + (na_n - a_{n-1})x^{n-1} + \ldots = -x \Rightarrow a_1 - a_0 = 0,\ 2a_2 - a_1 = -1,$

$3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$ for $n > 2$. Since $y = 2$ when $x = 0$ we have $a_0 = 2$. Therefore

$a_1 = 2,\ a_2 = \frac{-1 + a_1}{2} = \frac{1}{2},\ a_3 = \frac{a_2}{3} = \frac{1}{3\cdot 2},\ a_4 = \frac{a_3}{4} = \frac{1}{4\cdot 3\cdot 2}, \ldots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$

$\Rightarrow y = 2 + 2x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$

$= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots + \frac{1}{n!}x^n + \ldots\right) + 1 + x = \sum_{n=0}^{\infty}\frac{x^n}{n!} + 1 + x = e^x + x + 1$

77. $\lim_{x\to 0}\frac{7\sin x}{e^{2x} - 1} = \lim_{x\to 0}\frac{7\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{\left(2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right)} = \lim_{x\to 0}\frac{7\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)}{\left(2 + \frac{2^2x}{2!} + \frac{2^3x^2}{3!} + \ldots\right)} = \frac{7}{2}$

78. $\lim\limits_{\theta\to 0} \frac{e^{\theta}-e^{-\theta}-2\theta}{\theta-\sin\theta} = \lim\limits_{\theta\to 0} \frac{\left(1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\ldots\right)-\left(1-\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\ldots\right)-2\theta}{\theta-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)} = \lim\limits_{\theta\to 0} \frac{2\left(\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots\right)}{\left(\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+\ldots\right)}$

$= \lim\limits_{\theta\to 0} \frac{2\left(\frac{1}{3!}+\frac{\theta^2}{5!}+\ldots\right)}{\left(\frac{1}{3!}-\frac{\theta^2}{5!}+\ldots\right)} = 2$

79. $\lim\limits_{t\to 0} \left(\frac{1}{2-2\cos t}-\frac{1}{t^2}\right) = \lim\limits_{t\to 0} \frac{t^2-2+2\cos t}{2t^2(1-\cos t)} = \lim\limits_{t\to 0} \frac{t^2-2+2\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\ldots\right)}{2t^2\left(1-1+\frac{t^2}{2}-\frac{t^4}{4!}+\ldots\right)} = \lim\limits_{t\to 0} \frac{2\left(\frac{t^4}{4!}-\frac{t^6}{6!}+\ldots\right)}{\left(t^4-\frac{2t^6}{4!}+\ldots\right)}$

$= \lim\limits_{t\to 0} \frac{2\left(\frac{1}{4!}-\frac{t^2}{6!}+\ldots\right)}{\left(1-\frac{2t^2}{4!}+\ldots\right)} = \frac{1}{12}$

80. $\lim\limits_{h\to 0} \frac{\left(\frac{\sin h}{h}\right)-\cos h}{h^2} = \lim\limits_{h\to 0} \frac{\left(1-\frac{h^2}{3!}+\frac{h^4}{5!}-\ldots\right)-\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\ldots\right)}{h^2}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{h^2}{2!}-\frac{h^2}{3!}+\frac{h^4}{5!}-\frac{h^4}{4!}+\frac{h^6}{6!}-\frac{h^6}{7!}+\ldots\right)}{h^2} = \lim\limits_{h\to 0} \left(\frac{1}{2!}-\frac{1}{3!}+\frac{h^2}{5!}-\frac{h^2}{4!}+\frac{h^4}{6!}-\frac{h^4}{7!}+\ldots\right) = \frac{1}{3}$

81. $\lim\limits_{z\to 0} \frac{1-\cos^2 z}{\ln(1-z)+\sin z} = \lim\limits_{z\to 0} \frac{1-\left(1-z^2+\frac{z^4}{3}-\ldots\right)}{\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots\right)+\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right)} = \lim\limits_{z\to 0} \frac{\left(z^2-\frac{z^4}{3}+\ldots\right)}{\left(-\frac{z^2}{2}-\frac{2z^3}{3}-\frac{z^4}{4}-\ldots\right)}$

$= \lim\limits_{z\to 0} \frac{\left(1-\frac{z^2}{3}+\ldots\right)}{\left(-\frac{1}{2}-\frac{2z}{3}-\frac{z^2}{4}-\ldots\right)} = -2$

82. $\lim\limits_{y\to 0} \frac{y^2}{\cos y-\cosh y} = \lim\limits_{y\to 0} \frac{y^2}{\left(1-\frac{y^2}{2}+\frac{y^4}{4!}-\frac{y^6}{6!}+\ldots\right)-\left(1+\frac{y^2}{2!}+\frac{y^4}{4!}+\frac{y^6}{6!}+\ldots\right)} = \lim\limits_{y\to 0} \frac{y^2}{\left(-\frac{2y^2}{2}-\frac{2y^6}{6!}-\ldots\right)}$

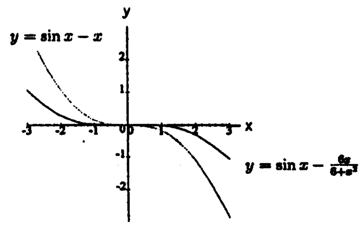
$= \lim\limits_{y\to 0} \frac{1}{\left(-1-\frac{2y^4}{6!}-\ldots\right)} = -1$

83. $\lim\limits_{x\to 0} \left(\frac{\sin 3x}{x^3}+\frac{r}{x^2}+s\right) = \lim\limits_{x\to 0} \left[\frac{\left(3x-\frac{(3x)^3}{6}+\frac{(3x)^5}{120}-\ldots\right)}{x^3}+\frac{r}{x^2}+s\right] = \lim\limits_{x\to 0} \left(\frac{3}{x^2}-\frac{9}{2}+\frac{81x^2}{40}+\ldots+\frac{r}{x^2}+s\right) = 0$

$\Rightarrow \frac{r}{x^2}+\frac{3}{x^2} = 0$ and $s-\frac{9}{2} = 0 \Rightarrow r = -3$ and $s = \frac{9}{2}$

84. (a) $\csc x \approx \frac{1}{x} + \frac{x}{6} \Rightarrow \csc x \approx \frac{6+x^2}{6x} \Rightarrow \sin x \approx \frac{6x}{6+x^2}$

(b) The approximation $\sin x \approx \frac{6x}{6+x^2}$ is better than $\sin x \approx x$.



85. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 1$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos nx\,dx = -\frac{1}{n\pi}\sin nx\Big|_{-\pi}^{0} + \frac{2}{nx}\sin nx\Big|_{0}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nx\,dx = \frac{1}{n\pi}\cos nx\Big|_{-\pi}^{0} - \frac{2}{nx}\cos nx\Big|_{0}^{\pi}$

$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{2}{n\pi}(\cos n\pi - 1) = \frac{3}{n\pi}(1-\cos n\pi) = \frac{3}{n\pi}(1-(-1)^n)$

Therefore, $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}(1-(-1)^n)\sin nx = \frac{1}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin[(2n-1)x]$

86. $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} x\,dx = \frac{1}{2}$

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{0}^{1} x\cos n\pi x\,dx = \left[\frac{\cos n\pi x}{n^2\pi^2} + \frac{x\sin n\pi x}{n\pi}\right]\Big|_{0}^{1} = \frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} = -\frac{1}{n^2\pi^2}(1-(-1)^n)$

$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{0}^{1} x\sin n\pi x\,dx = \left[\frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi}\right]\Big|_{0}^{1} = -\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}(-1)^n$

Therefore, $f(x) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi x - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin n\pi x$

87. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\,dx = \frac{1}{\pi}(2\pi^2) = 2\pi$

$a_n = 0$ because $f(x) - \pi$ is an odd function

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n^2} - \frac{(x+\pi)\cos nx}{n}\right]\Big|_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi = -\frac{2(-1)^n}{n}$

Therefore, $f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$

88. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{2}{\pi}$

$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx = 0 \qquad b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]\Big|_0^{\pi} = \frac{1}{2}$

For $n \geq 2$

$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \frac{1}{\pi}\left[\frac{\cos[(n-1)x]}{2(n-1)} - \frac{\cos[(n+1)x]}{2(n+1)}\right]\Big|_0^{\pi}$

$= \frac{1}{\pi}\left[\frac{1}{2(n-1)} + \frac{1}{2(n+1)} + \frac{\cos[(n-1)\pi]}{2(n-1)} - \frac{\cos[(n+1)\pi]}{2(n+1)}\right] = \frac{1+\cos n\pi}{(1-n^2)\pi} = \frac{1+(-1)^n}{(1-n^2)\pi}$

$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin[(n-1)x]}{2(n-1)} - \frac{\sin[(n+1)x]}{2(n+1)}\right]\Big|_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin[(n-1)\pi]}{2(n-1)} - \frac{\sin[(n+1)\pi]}{2(n+1)}\right] = 0$

Therefore,

$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos nx = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-(2n)^2}\cos[(2n)x]$

89. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}[2+4] = 3$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right)dx + \int_0^{2} (1+x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$

$= \frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2(1+x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^{2} = \frac{2}{n^2\pi^2}[\cos n\pi - 1] = \frac{2((-1)^n - 1)}{n^2\pi^2}$

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right)dx + \int_0^{2} (1+x)\sin\left(\frac{n\pi x}{2}\right)dx\right]$

$= -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \left[-\frac{(1+x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^{2} = \frac{1}{n\pi}(\cos n\pi - 1) + \frac{1}{n\pi}(1 - 3\cos n\pi)$

$= \frac{(-1)^n - 1}{n\pi} + \frac{1-3(-1)^n}{n\pi} = -\frac{2(-1)^n}{n\pi}$

Therefore, $f(x) = \frac{3}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$

90. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 dx\right] = \frac{1}{2}\left[\frac{1}{2}+1\right] = \frac{3}{4}$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx\right]$

$= \left[\frac{2}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^1 + \frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_1^2 = \frac{2}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right)-1\right)$

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx\right]$

$= \left[\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right]\Bigg|_0^1 - \frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_1^2 = \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos n\pi$

Therefore,

$f(x) = \frac{3}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)-1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos n\pi\right)\sin\left(\frac{n\pi x}{2}\right)$

91. (a) $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1$; $a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx = 2\int_0^{1/2}\cos n\pi x\,dx$

$= \frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$. Therefore, $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos n\pi x$

(b) $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2}\sin n\pi x\,dx = -\frac{2}{n\pi}\cos n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$

Therefore, $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin n\pi x$

92. (a) $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_1^2 x\,dx = \frac{3}{2}$; $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_1^2 x\cos\left(\frac{n\pi x}{2}\right)dx$

$= \left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right]\Bigg|_1^2 = \frac{4}{n^2\pi^2}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right] - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

Therefore, $f(x) = \frac{3}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$

(b) $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_1^2 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$.

Therefore, $f(x) = \frac{2}{\pi}\sum\left(-\frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n}\cos n\pi + \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{2}\right)$

93. (a) $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 \sin \pi x\,dx = -\frac{2}{\pi}\cos \pi x\Big|_0^1 = \frac{4}{\pi}$; $a_1 = \frac{2}{1}\int_0^1 f(x)\cos \pi x\,dx = 2\int_0^1 \sin \pi x \cos \pi x\,dx$

$= -\frac{1}{2\pi}\cos^2 \pi x\Big|_0^1 = 0.$ For $n \ge 2$, $a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = 2\int_0^1 \sin \pi x \cos n\pi x\,dx$

$$= \left[\frac{\cos[(n-1)\pi x]}{n-1} - \frac{\cos[(n+1)\pi x]}{n+1}\right]\Bigg|_0^1 = \frac{1}{\pi}\left[\frac{1}{n-1} + \frac{1}{n+1} + \frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}\right]$$

Therefore, $f(x) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}\right]\cos n\pi x$

(b) $b_1 = \frac{2}{1}\int_0^1 f(x)\sin \pi x\,dx = 2\int_0^1 \sin^2 \pi x\,dx = \left[x - \frac{\sin 2\pi x}{2\pi}\right]\Big|_0^1 = 1$; $b_n = 0$ for $n \ge 2$.

Therefore, $f(x) = \sin \pi x$, as expected.

94. (a) $a_0 = \frac{4}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - 0\right] = \frac{4}{\pi}$; $a_n = \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos 2nx\,dx$

$$= \frac{2}{\pi}\left[\frac{\sin[(2n-1)x]}{2n-1} + \frac{\sin[(2n+1)x]}{2n+1}\right]\Bigg|_0^{\pi/2} = \frac{2}{\pi}\left[\frac{\sin\left[\frac{(2n-1)\pi}{2}\right]}{2n-1} + \frac{\sin\left[\frac{(2n+1)\pi}{2}\right]}{2n+1}\right] = \frac{4(-1)^n}{\pi(1-4n^2)}$$

Therefore, $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-4n^2)}\cos 2nx$

(b) $b_n = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx\,dx = \,= \frac{2}{\pi}\left[-\frac{\cos[(2n-1)x]}{2n-1} - + \frac{\cos[(2n+1)x]}{2n+1}\right]\Bigg|_0^{\pi/2}$

$$= \frac{2}{\pi}\left[\frac{1}{2n-1} + \frac{1}{2n+1} - \frac{\cos\left[\frac{(2n-1)\pi}{2}\right]}{2n-1} - \frac{\cos\left[\frac{(2n+1)\pi}{2}\right]}{2n+1}\right] = \frac{8n}{(4n^2-1)\pi}$$

Therefore, $f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx$

95. (a) $a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (2x + x^2)\,dx = \frac{2}{3}\left(x^2 + \frac{x^3}{3}\right)\Big|_0^3 = \frac{2}{3}(18) = 12$

$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^3 (2x + x^2)\cos\left(\frac{n\pi x}{3}\right)dx =$ (using CAS)

$$= \frac{2}{n^3\pi^3}\left[6n\pi(1+x)\cos\left(\frac{n\pi x}{3}\right) + \left(n^2\pi^2 x(x+2) - 18\right)\sin\left(\frac{n\pi x}{3}\right)\right]\Bigg|_0^3 = \frac{12}{n^2\pi^2}[4(-1)^n - 1]$$

Therefore, $f(x) = 6 + \frac{12}{\pi^2}\sum_{n=1}^{\infty} \frac{4(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{3}\right)$

(b) $b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^3 (2x+x^2)\sin\left(\frac{n\pi x}{3}\right)dx =$ (using CAS)

$= \frac{2}{n^3\pi^3}\left[-18n\pi(1+x)\sin\left(\frac{n\pi x}{3}\right) - (n^2\pi^2 x(x+2) - 18)\cos\left(\frac{n\pi x}{3}\right)\right]\Big|_0^3 = \frac{2\left[(18-15n^2\pi^2)(-1)^n - 18\right]}{n^3\pi^3}$

Therefore, $f(x) = \frac{6}{\pi^3}\sum_{n=1}^{\infty}\frac{(6-5n^2\pi^2)(-1)^n - 6}{n^3}\sin\left(\frac{n\pi x}{3}\right)$

96. (a) $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 e^{-x}\,dx = -e^{-x}\Big|_0^2 = 1 - \frac{1}{e^2} = \frac{e^2-1}{e^2}$

$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 e^{-x}\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{4+n^2\pi^2}\left[e^{-x}\left(n\pi\sin\left(\frac{n\pi x}{2}\right) - 2\cos\left(\frac{n\pi x}{2}\right)\right)\right]\Big|_0^2$

$= \frac{4(e^2-(-1)^n)}{(4+n^2\pi^2)e^2}$. Therefore, $f(x) = \frac{e^2-1}{2e^2} + \frac{4}{e^2}\sum_{n=1}^{\infty}\frac{(e^2-(-1)^n)}{(4+n^2\pi^2)}\cos\left(\frac{n\pi x}{2}\right)$

(b) $a_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^2 e^{-x}\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2}{4+n^2\pi^2}\left[e^{-x}\left(n\pi\cos\left(\frac{n\pi x}{2}\right) + 2\sin\left(\frac{n\pi x}{2}\right)\right)\right]\Big|_0^2$

$= \frac{2n\pi(e^2-(-1)^n)}{(4+n^2\pi^2)e^2}$. Therefore, $f(x) = \frac{2\pi}{e^2}\sum_{n=1}^{\infty}\frac{n(e^2-(-1)^n)}{(4+n^2\pi^2)}\sin\left(\frac{n\pi x}{2}\right)$

97. (a) $\sum_{n=1}^{\infty}\left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right) = \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \left(\sin\frac{1}{4} - \sin\frac{1}{5}\right) + \left(\sin\frac{1}{6} - \sin\frac{1}{7}\right) + \ldots + \left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right)$

$+ \ldots = \sum_{n=2}^{\infty}(-1)^n\sin\frac{1}{n}$; $f(x) = \sin\frac{1}{x} \Rightarrow f'(x) = \frac{-\cos\left(\frac{1}{x}\right)}{x^2} < 0$ if $x \ge 2 \Rightarrow \sin\frac{1}{n+1} < \sin\frac{1}{n}$, and

$\lim_{n\to\infty}\sin\frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty}(-1)^n\sin\frac{1}{n}$ converges by the Alternating Series Test

(b) $|\text{error}| < \left|\sin\frac{1}{42}\right| \approx 0.02381$ and the sum is an underestimate because the remainder is positive

98. (a) $\sum_{n=1}^{\infty}\left(\tan\frac{1}{2n} - \tan\frac{1}{2n+1}\right) = \sum_{n=2}^{\infty}(-1)^n\tan\frac{1}{n}$ (see Exercise 89); $f(x) = \tan\frac{1}{x} \Rightarrow f'(x) = \frac{-\sec^2\left(\frac{1}{x}\right)}{x^2} < 0$

$\Rightarrow \tan\frac{1}{n+1} < \tan\frac{1}{n}$, and $\lim_{n\to\infty}\tan\frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty}(-1)^n\tan\frac{1}{n}$ converges by the Alternating Series Test

(b) $|\text{error}| < \left|\tan\frac{1}{42}\right| \approx 0.02382$ and the sum is an underestimate because the remainder is positive

105. Yes, the series $\sum_{n=1}^{\infty} a_n b_n$ converges as we now show. Since $\sum_{n=1}^{\infty} a_n$ converges it follows that $a_n \to 0 \Rightarrow a_n < 1$ for $n >$ some index $N \Rightarrow a_n b_n < b_n$ for $n > N \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges by the Direct Comparison Test with $\sum_{n=1}^{\infty} b_n$

106. No, the series $\sum_{n=1}^{\infty} a_n b_n$ might diverge (as it would if a_n and b_n both equaled n) or it might converge (as it would if a_n and b_n both equaled $\frac{1}{n}$).

107. $\sum_{n=1}^{\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} (x_{k+1} - x_k) = \lim_{n\to\infty} (x_{n+1} - x_1) = \lim_{n\to\infty} (x_{n+1}) - x_1 \Rightarrow$ both the series and sequence must either converge or diverge.

108. It converges by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{a_n}{1+a_n}\right)}{a_n} = \lim_{n\to\infty} \frac{1}{1+a_n} = 1$ because $\sum_{n=1}^{\infty} a_n$ converges

109. (a) $\sum_{n=1}^{\infty} \frac{a_n}{n} = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \ldots \ge a_1 + \left(\frac{1}{2}\right)a_2 + \left(\frac{1}{3} + \frac{1}{4}\right)a_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)a_8$

$+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}\right)a_{16} + \ldots \ge \frac{1}{2}(a_2 + a_4 + a_8 + a_{16} + \ldots)$ which is a divergent series

(b) $a_n = \frac{1}{\ln n}$ for $n \ge 2 \Rightarrow a_2 \ge a_3 \ge a_4 \ge \ldots$, and $\frac{1}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 8} + \ldots = \frac{1}{\ln 2} + \frac{1}{2\ln 2} + \frac{1}{3\ln 2} + \ldots$

$= \frac{1}{\ln 2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$ which diverges so that $1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by part (a)

110. (a) $T = \frac{\left(\frac{1}{2}\right)}{2}\left(0 + 2\left(\frac{1}{2}\right)^2 e^{1/2} + e\right) = \frac{1}{8}e^{1/2} + \frac{1}{4}e \approx 0.885660616$

(b) $x^2 e^x = x^2\left(1 + x + \frac{x^2}{2} + \ldots\right) = x^2 + x^3 + \frac{x^4}{2} + \ldots \Rightarrow \int_0^1 \left(x^2 + x^3 + \frac{x^4}{2}\right) dx = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{41}{60} = 0.6833\overline{3}$

(c) If the second derivative is positive, the curve is concave upward and the polygonal line segments used in the trapezoidal rule lie above the curve. The trapezoidal approximation is therefore greater than the actual area under the graph.

(d) All terms in the Maclaurin series are positive. If we truncate the series, we are omitting positive terms and hence the estimate is too small.

(e) $\int_0^1 x^2 e^x\, dx = \left[x^2 e^x - 2xe^x + 2e^x\right]_0^1 = e - 2e + 2e - 2 = e - 2 \approx 0.7182818285$

111. (a) $\int_0^x \frac{1}{1+t^2} dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n}{2n+1} x^{2n+1} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$$

(b) By definition,

$$R_n(x) = f(x) - P_n(x) = \tan^{-1} x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n}{2n+1} x^{2n+1} \right) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$$

If the integrand goes to zero in the limit, then so will the value of the integral.

$|x| < 1 \Rightarrow |t| < 1 \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} = \frac{1}{1+t^2} \lim_{n\to\infty} (-1)^{n+1} t^{2n+2} = 0$. If $|x| = 1$, then the value of the integrand will approach 0 for all values of t between 0 and x, while at $t = x$, it will oscillate between $\pm \frac{1}{1+t^2}$. However, the integral of a function will converge provided the function is piecewise continuous in the interval $0 < t < x$. Therefore, we would expect that convergence of $R_n(x)$ to zero would not be affected by the value of the integrand at the single value $t = x$ provided it is finite, which it is. Therefore, $\lim_{n\to\infty} R_n(x) = 0$ for $|x| \le 1$.

(c) For $|x| \le 1$, $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

(d) $\tan^{-1} 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (1)^{2n+1} = \sum_{n=0}^{\infty} 5 \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots + \frac{(-1)^n}{2n+1} + \ldots$

112. (a) Substituting x^2 for x in the Maclaurin series for sin x,

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Integrating term-by-term and observing that the constant term is 0,

$$\int_0^x \sin t^2 \, dt = \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \ldots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \ldots$$

(b) $\int_0^1 \sin x \, dx = \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \ldots + (-1)^n \frac{1}{(4n+3)(2n+1)!} + \ldots.$

Since the third term is $\frac{1}{11(5!)} = \frac{1}{1320} < 0.001$, it suffices to use the first two nonzero terms (through degree 7).

113. (a) $g(x) = 2x + 3 \Rightarrow g^{-1}(x) = \frac{x-3}{2}$ and when the iterative method is applied to $g^{-1}(x)$ we have $x_0 = 2$ $\to -2.99999881$ in 23 iterations $\Rightarrow -3$ is the fixed point

(b) $g(x) = 1 - 4x \Rightarrow g^{-1}(x) = \frac{1-x}{4}$ and when the iterative method is applied to $g^{-1}(x)$ we have $x_0 = 2$ $\to 0.199999571$ in 12 iterations $\Rightarrow 0.2$ is the fixed point

CHAPTER 8 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. converges since $\frac{1}{(3n-2)^{(2n+1)/2}} < \frac{1}{(3n-2)^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^{3/2}}$ converges by the Limit Comparison Test:
$\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{(3n-2)^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{3n-2}{n}\right)^{3/2} = 3^{3/2}$

2. converges by the Integral Test: $\int_1^{\infty} \left(\tan^{-1} x\right)^2 \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\frac{\left(\tan^{-1} x\right)^3}{3}\right]_1^b = \lim_{b\to\infty} \left[\frac{\left(\tan^{-1} b\right)^3}{3} - \frac{\pi^3}{192}\right]$
$= \left(\frac{\pi^3}{24} - \frac{\pi^3}{192}\right) = \frac{7\pi^3}{192}$

3. diverges by the nth-Term Test since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \tanh n = \lim_{b\to\infty} (-1)^n\left(\frac{1-e^{-2n}}{1+e^{-2n}}\right) = \lim_{n\to\infty} (-1)^n$ does not exist

4. converges by the Direct Comparison Test: $n! < n^n \Rightarrow \ln(n!) < n \ln(n) \Rightarrow \frac{\ln(n!)}{\ln(n)} < n$
$\Rightarrow \log_n(n!) < n \Rightarrow \frac{\log_n(n!)}{n^3} < \frac{1}{n^2}$, which is the nth-term of a convergent p-series

5. converges by the Direct Comparison Test: $a_1 = 1 = \frac{12}{(1)(3)(2)^2}$, $a_2 = \frac{1\cdot 2}{3\cdot 4} = \frac{12}{(2)(4)(3)^2}$, $a_3 = \left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right)$
$= \frac{12}{(3)(5)(4)^2}$, $a_4 = \left(\frac{3\cdot 4}{5\cdot 6}\right)\left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right) = \frac{12}{(4)(6)(5)^2}, \ldots \Rightarrow 1 + \sum_{n=1}^{\infty} \frac{12}{(n+1)(n+3)(n+2)^2}$ represents the given series and $\frac{12}{(n+1)(n+3)(n+2)^2} < \frac{12}{n^4}$, which is the nth-term of a convergent p-series

6. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{(n-1)(n+1)} = 0 < 1$

7. diverges by the nth-Term Test since if $a_n \to L$ as $n \to \infty$, then $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2}$ $\neq 0$

8. Split the given series into $\sum_{n=1}^{\infty} \frac{1}{3^{2n+1}}$ and $\sum_{n=1}^{\infty} \frac{2n}{3^{2n}}$; the first subseries is a convergent geometric series and the second converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{2n}{3^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{2}\,\sqrt[n]{n}}{9} = \frac{1\cdot 1}{9} = \frac{1}{9} < 1$

9. $f(x) = \cos x$ with $a = \frac{\pi}{3} \Rightarrow f\left(\frac{\pi}{3}\right) = 0.5$, $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{\pi}{3}\right) = -0.5$, $f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f^{(4)}\left(\frac{\pi}{3}\right) = 0.5$;
$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \ldots$

10. $f(x) = \sin x$ with $a = 2\pi \Rightarrow f(2\pi) = 0$, $f'(2\pi) = 1$, $f''(2\pi) = 0$, $f'''(2\pi) = -1$, $f^{(4)}(2\pi) = 0$, $f^{(5)}(2\pi) = 1$, $f^{(6)}(2\pi) = 0$, $f^{(7)}(2\pi) = -1$; $\sin x = (x-2\pi) - \frac{(x-2\pi)^3}{3!} + \frac{(x-2\pi)^5}{5!} - \frac{(x-2\pi)^7}{7!} + \ldots$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{2!} + \ldots$ with $a = 0$

12. $f(x) = \ln x$ with $a = 1 \Rightarrow f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f^{(4)}(1) = -6$; $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

13. $f(x) = \cos x$ with $a = 22\pi \Rightarrow f(22\pi) = 1$, $f'(22\pi) = 0$, $f''(22\pi) = -1$, $f'''(22\pi) = 0$, $f^{(4)}(22\pi) = 1$, $f^{(5)}(22\pi) = 0$, $f^{(6)}(22\pi) = -1$; $\cos x = 1 - \frac{1}{2}(x-22\pi)^2 + \frac{1}{4!}(x-22\pi)^4 - \frac{1}{6!}(x-22\pi)^6 + \ldots$

14. $f(x) = \tan^{-1} x$ with $a = 1 \Rightarrow f(1) = \frac{\pi}{4}$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2}$, $f'''(1) = \frac{1}{2}$; $\tan^{-1} x = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \ldots$

15. Yes, the sequence converges: $c_n = \left(a^n + b^n\right)^{1/n} \Rightarrow c_n = b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} \Rightarrow \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} = b$ since $0 < a < b$

16. $1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \ldots = 1 + \sum\limits_{n=1}^{\infty} \frac{2}{10^{3n-2}} + \sum\limits_{n=1}^{\infty} \frac{3}{10^{3n-1}} + \sum\limits_{n=1}^{\infty} \frac{7}{10^{3n}}$

$$= 1 + \sum_{n=0}^{\infty} \frac{2}{10^{3n+1}} + \sum_{n=0}^{\infty} \frac{3}{10^{3n+2}} + \sum_{n=0}^{\infty} \frac{7}{10^{3n+3}} = 1 + \frac{\left(\frac{2}{10}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{3}{10^2}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{7}{10^3}\right)}{1-\left(\frac{1}{10}\right)^3}$$

$$= 1 + \frac{200}{999} + \frac{30}{999} + \frac{7}{999} = \frac{999+237}{999} = \frac{412}{333}$$

17. $s_n = \sum\limits_{k=0}^{n-1} \int\limits_k^{k+1} \frac{dx}{1+x^2} \Rightarrow s_n = \int\limits_0^1 \frac{dx}{1+x^2} + \int\limits_1^2 \frac{dx}{1+x^2} + \ldots + \int\limits_{n-1}^{n} \frac{dx}{1+x^2} \Rightarrow s_n = \int\limits_0^n \frac{dx}{1+x^2}$

$\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\tan^{-1} n - \tan^{-1} 0\right) = \frac{\pi}{2}$

18. $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+2)(2x+1)^{n+1}} \cdot \frac{(n+1)(2x+1)^n}{nx^n}\right| = \lim\limits_{n\to\infty} \left|\frac{x}{2x+1} \cdot \frac{(n+1)^2}{n(n+2)}\right| = \left|\frac{x}{2x+1}\right| < 1$

$\Rightarrow |x| < |2x+1|$; if $x > 0$, $|x| < |2x+1| \Rightarrow x < 2x+1 \Rightarrow x > -1$; if $-\frac{1}{2} < x < 0$, $|x| < |2x+1|$ $\Rightarrow -x < 2x+1 \Rightarrow 3x > -1 \Rightarrow x > -\frac{1}{3}$; if $x < -\frac{1}{2}$, $|x| < |2x+1| \Rightarrow -x < -2x-1 \Rightarrow x < -1$. Therefore, the series converges absolutely for $x < -1$ and $x > -\frac{1}{3}$.

19. (a) From Fig. 8.13 in the text with $f(x) = \frac{1}{x}$ and $a_k = \frac{1}{k}$, we have $\int_1^{n+1} \frac{1}{x}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ $\le 1 + \int_1^n f(x)\,dx \Rightarrow \ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \le 1 + \ln n \Rightarrow 0 \le \ln(n+1) - \ln n$ $\le \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n \le 1$. Therefore the sequence $\left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n\right\}$ is bounded above by 1 and below by 0.

(b) From the graph in Fig. 8.13(a) with $f(x) = \frac{1}{x}$, $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln n$

$\Rightarrow 0 > \frac{1}{n+1} - [\ln(n+1) - \ln n] = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right)$.

If we define $a_n = 1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{n} - \ln n$, then $0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

20. (a) Each A_{n+1} fits into the corresponding upper triangular region, whose vertices are: $(n, f(n) - f(n+1))$, $(n+1, f(n+1))$ and $(n, f(n))$ along the line whose slope is $f(n+2) - f(n+1)$. All the A_n's fit into the first upper triangular region whose area is $\frac{f(1) - f(2)}{2} \Rightarrow \sum_{n=1}^{\infty} A_n < \frac{f(1) - f(2)}{2}$

(b) If $A_k = \frac{f(k+1) + f(k)}{2} - \int_k^{k+1} f(x)\,dx$, then

$$\sum_{k=1}^{n-1} A_k = \frac{f(1) + f(2) + f(2) + f(3) + f(3) + \ldots + f(n-1) + f(n)}{2} - \int_1^2 f(x)\,dx - \int_2^3 f(x)\,dx - \ldots - \int_{n-1}^n f(x)\,dx$$

$= \frac{f(1) + f(n)}{2} + \sum_{k=2}^{n-1} f(k) - \int_1^n f(x)\,dx \Rightarrow \sum_{k=1}^{n-1} A_k = \sum_{k=1}^{n} f(k) - \frac{f(1) + f(n)}{2} - \int_1^n f(x)\,dx < \frac{f(1) - f(2)}{2}$, from part (a). The sequence $\left\{\sum_{k=1}^{n-1} A_k\right\}$ is bounded above and increasing, so it converges and the limit in question must exist.

(c) From part (b) we have $\sum_{k=1}^{\infty} f(k) - \int_1^n f(x)\,dx < f(1) - \frac{f(2)}{2} + \frac{f(n)}{2}$

$\Rightarrow \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right] < \lim_{n\to\infty} \left[f(1) - \frac{f(2)}{2} + \frac{f(n)}{2}\right] = f(1) - \frac{f(2)}{2}$. The sequence $\left\{\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right\}$ is bounded and increasing, so it converges and the limit in question must exist.

21. The number of triangles removed at stage n is 3^{n-1}; the side length at stage n is $\frac{b}{2^{n-1}}$; the area of a triangle at stage n is $\frac{\sqrt{3}}{4}\left(\frac{b}{2^{n-1}}\right)^2$.

(a) $\frac{\sqrt{3}}{4}b^2 + 3\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^2}\right) + 3^2\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^4}\right) + 3^3\frac{\sqrt{3}}{4}\left(\frac{b^2}{2^6}\right) + \ldots = \frac{\sqrt{3}}{4}b^2 \sum_{n=0}^{\infty} \frac{3^n}{2^{2n}} = \frac{\sqrt{3}}{4}b^2 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$

(b) a geometric series with sum $\frac{\left(\frac{\sqrt{3}}{4}b^2\right)}{1-\left(\frac{3}{4}\right)} = \sqrt{3}b^2$

(c) No; for instance,the three vertices of the original triangle are not removed. However the total area removed is $\sqrt{3}b^2$ which equals the area of the original triangle. Thus the set of points not removed has area 0.

22. The sequence $\{x_n\}$ converges to $\frac{\pi}{2}$ from below so $\epsilon_n = \frac{\pi}{2} - x_n > 0$ for each n. By the Alternating Series Estimation Theorem $\epsilon_{n+1} \approx \frac{1}{3!}(\epsilon_n)^3$ with $|\text{error}| < \frac{1}{5!}(\epsilon_n)^5$, and since the remainder is negative this is an overestimate $\Rightarrow 0 < \epsilon_{n+1} < \frac{1}{6}(\epsilon_n)^3$.

23. (a) No, the limit does not appear to depend on the value of the constant a

(b) Yes, the limit depends on the value of b. The answer to part (c) shows how the limit depends on the value of (b).

(c) $s = \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)^n \Rightarrow \log s = \frac{\log\left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)}{\left(\frac{1}{n}\right)} \Rightarrow \lim_{n\to\infty} \log s = \frac{\left(\frac{1}{1-\frac{\cos\left(\frac{a}{n}\right)}{n}}\right)\left(\frac{-\frac{a}{n}\sin\left(\frac{a}{n}\right) + \cos\left(\frac{a}{n}\right)}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$

$= \lim_{n\to\infty} \frac{\frac{a}{n}\sin\left(\frac{a}{n}\right) - \cos\left(\frac{a}{n}\right)}{1 - \frac{\cos\left(\frac{a}{n}\right)}{n}} = \frac{0-1}{1-0} = -1 \Rightarrow \lim_{n\to\infty} s = e^{-1} \approx 0.3678794412$; similarly,

$\lim_{n\to\infty} \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{bn}\right)^n = e^{-1/b}$

24. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \left[\left(\frac{1+\sin a_n}{2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1+\sin a_n}{2}\right) = \frac{1+\sin\left(\lim_{n\to\infty} a_n\right)}{2} = \frac{1+\sin 0}{2}$ $= \frac{1}{2} \Rightarrow$ the series converges by the nth-Root Test

25. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{b^{n+1}x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n x^n}\right| < 1 \Rightarrow |bx| < 1 \Rightarrow -\frac{1}{b} < x < \frac{1}{b} = 5 \Rightarrow b = \pm\frac{1}{5}$

26. A polynomial has only a finite number of nonzero terms in its Taylor series, but the functions sin x, ln x and e^x have infinitely many nonzero terms in their Taylor expansions.

27. (a) $\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} \Rightarrow C = 2 > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(b) $\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} + \frac{0}{n^2} \Rightarrow C = 1 \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

28. $\frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2} = \frac{4n^2+2n}{4n^2-4n+1} = 1 + \frac{\left(\frac{6}{4}\right)}{n} + \frac{5}{4n^2-4n+1} = 1 + \frac{\left(\frac{3}{2}\right)}{n} + \frac{\left[\frac{5n^2}{(4n^2-4n+1)}\right]}{n^2}$ after long division

$\Rightarrow C = \frac{3}{2} > 1$ and $|f(n)| = \frac{5n^2}{4n^2-4n+1} = \frac{5}{\left(4 - \frac{4}{n} + \frac{1}{n^2}\right)} \le 5 \Rightarrow \sum_{n=1}^{\infty} u_n$ converges by Raabe's Test

29. (a) $\sum_{n=1}^{\infty} a_n = L \Rightarrow a_n^2 \le a_n \sum_{n=1}^{\infty} a_n = a_n L \Rightarrow \sum_{n=1}^{\infty} a_n^2$ converges by the Direct Comparison Test

(b) converges by the Limit Comparison Test: $\lim_{n\to\infty} \frac{\left(\frac{a_n}{1-a_n}\right)}{a_n} = \lim_{n\to\infty} \frac{1}{1-a_n} = 1$ since $\sum_{n=1}^{\infty} a_n$ converges and therefore $\lim_{x\to\infty} a_n = 0$

30. If $0 < a_n < 1$ then $|\ln(1-a_n)| = -\ln(1-a_n) = a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \ldots < a_n + a_n^2 + a_n^3 + \ldots = \frac{a_n}{1-a_n}$, a positive term of a convergent series, by the Limit Comparison Test and Exercise 29b

31. $(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$ where $|x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}(1-x)^{-1} = \sum_{n=1}^{\infty} nx^{n-1}$ and when $x = \frac{1}{2}$ we have $4 = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} + \ldots$

32. (a) $\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x} \Rightarrow \sum_{n=1}^{\infty} (n+1)x^n = \frac{2x-x^2}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1 - \frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}$, $|x| > 1$

(b) $x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} \Rightarrow x = \frac{2x^2}{(x-1)^3} \Rightarrow x^3 - 3x^2 + x - 1 = 0 \Rightarrow x = 1 + \left(1 + \frac{\sqrt{57}}{9}\right)^{1/3} + \left(1 - \frac{\sqrt{57}}{9}\right)^{1/3}$

≈ 2.769292, using a CAS or calculator

33. $e^{-x^2} \le e^{-x}$ for $x \ge 1$, and $\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1} \Rightarrow \int_1^{\infty} e^{-x^2}\,dx$ converges by the Comparison Test for improper integrals $\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

34. Yes, the series $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges by the Direct Comparison Test: $1+a_n < 1+a_n+\frac{a_n^2}{2!}+\frac{a_n^3}{3!}+\ldots$ $\Rightarrow 1+a_n < e^{a_n} \Rightarrow \ln(1+a_n) < a_n$

35. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1+x+x^2+x^3+\ldots) = 1+2x+3x^2+4x^3+\ldots = \sum_{n=1}^{\infty} nx^{n-1}$

(b) from part (a) we have $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\left[\frac{1}{1-\left(\frac{5}{6}\right)}\right] = 6$

(c) from part (a) we have $\sum_{n=1}^{\infty} np^{n-1}q = \frac{q}{(1-p)^2} = \frac{q}{q^2} = \frac{1}{q}$

36. (a) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 2^{-k} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k2^{-k} = \frac{1}{2}\sum_{k=1}^{\infty} k2^{1-k} = \left(\frac{1}{2}\right)\frac{1}{\left[1-\left(\frac{1}{2}\right)\right]^2} = 2$ by Exercise 35(a)

(b) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^k = \left(\frac{1}{5}\right)\left[\frac{\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)}\right] = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\frac{5^{k-1}}{6^k} = \frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$ $= \left(\frac{1}{6}\right)\frac{1}{\left[1-\left(\frac{5}{6}\right)\right]^2} = 6$

(c) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\left(\frac{1}{k(k+1)}\right)$ $= \sum_{k=1}^{\infty}\frac{1}{k+1}$, a divergent series so that E(x) does not exist

37. (a) $R_n = C_0e^{-kt_0} + C_0e^{-2kt_0} + \ldots + C_0e^{-nkt_0} = \frac{C_0e^{-kt_0}\left(1-e^{-nkt_0}\right)}{1-e^{-kt_0}} \Rightarrow R = \lim_{n\to\infty} R_n = \frac{C_0e^{-kt_0}}{1-e^{-kt_0}} = \frac{C_0}{e^{kt_0}-1}$

(b) $R_n = \frac{e^{-1}\left(1-e^{-n}\right)}{1-e^{-1}} \Rightarrow R_1 = e^{-1} \approx 0.36787944$ and $R_{10} = \frac{e^{-1}\left(1-e^{-10}\right)}{1-e^{-1}} \approx 0.58195028$; $R = \frac{1}{e-1} \approx 0.58197671$; $R - R_{10} \approx 0.00002643 \Rightarrow \frac{R-R_{10}}{R} < 0.0001$

(c) $R_n = \frac{e^{-.1}\left(1-e^{-.1n}\right)}{1-e^{-.1}}$, $\frac{R}{2} = \frac{1}{2}\left(\frac{1}{e^{.1}-1}\right) \approx 4.7541659$; $R_n > \frac{R}{2} \Rightarrow \frac{1-e^{-.1n}}{e^{.1}-1} > \left(\frac{1}{2}\right)\left(\frac{1}{e^{.1}-1}\right)$ $\Rightarrow 1-e^{-n/10} > \frac{1}{2} \Rightarrow e^{-n/10} < \frac{1}{2} \Rightarrow -\frac{n}{10} < \ln\left(\frac{1}{2}\right) \Rightarrow \frac{n}{10} > -\ln\left(\frac{1}{2}\right) \Rightarrow n > 6.93 \Rightarrow n = 7$

38. (a) $R = \frac{C_0}{e^{kt_0}-1} \Rightarrow Re^{kt_0} = R + C_0 = C_H \Rightarrow e^{kt_0} = \frac{C_H}{C_L} \Rightarrow t_0 = \frac{1}{k}\ln\left(\frac{C_H}{C_L}\right)$

(b) $t_0 = \frac{1}{0.05}\ln e = 20$ hrs

(c) Give an initial dose that produces a concentration of 2 mg/ml followed every $t_0 = \frac{1}{0.02}\ln\left(\frac{2}{0.5}\right) \approx 69.31$ hrs by a dose that raises the concentration by 1.5 mg/ml

(d) $t_0 = \frac{1}{0.2}\ln\left(\frac{0.1}{0.03}\right) = 5\ln\left(\frac{10}{3}\right) \approx 6$ hrs

39. The convergence of $\sum_{n=1}^{\infty} |a_n|$ implies that $\lim_{n\to\infty} |a_n| = 0$. Let $N > 0$ be such that $|a_n| < \frac{1}{2} \Rightarrow 1 - |a_n| > \frac{1}{2}$ $\Rightarrow \frac{|a_n|}{1-|a_n|} < 2|a_n|$ for all $n > N$. Now $|\ln(1+a_n)| = \left|a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \ldots\right| \le |a_n| + \left|\frac{a_n^2}{2}\right| + \left|\frac{a_n^3}{3}\right| + \left|\frac{a_n^4}{4}\right| + \ldots$ $< |a_n| + |a_n|^2 + |a_n|^3 + |a_n|^4 + \ldots = \frac{|a_n|}{1-|a_n|} < 2|a_n|$. Therefore $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges by the Direct Comparison Test since $\sum_{n=1}^{\infty} |a_n|$ converges.

40. $\sum_{n=3}^{\infty} \frac{1}{n \ln n(\ln(\ln n))^p}$ converges if $p > 1$ and diverges otherwise by the Integral Test: when $p = 1$ we have $\lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x(\ln(\ln x))} = \lim_{b\to\infty} [\ln(\ln(\ln x))]_3^b = \infty$; when $p \ne 1$ we have $\lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x(\ln(\ln x))^p}$

$$= \lim_{b\to\infty} \left[\frac{(\ln(\ln x))^{-p+1}}{1-p}\right]_3^b = \begin{cases} \frac{(\ln(\ln 3))^{-p+1}}{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

NOTES: