

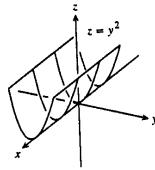
CHAPTER 11 MULTIVARIABLE FUNCTIONS AND THEIR DERIVATIVES

11.1 FUNCTIONS OF SEVERAL VARIABLES

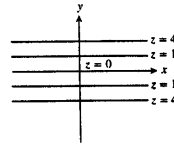
1. (a) Domain: all points in the xy -plane
(b) Range: all real numbers
(c) level curves are straight lines $y - x = c$ parallel to the line $y = x$
(d) no boundary points
(e) both open and closed
(f) unbounded
2. (a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$
(b) Range: $z \geq 0$
(c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$
(d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line
(e) closed
(f) unbounded
3. (a) Domain: all points in the xy -plane
(b) Range: $z \geq 0$
(c) level curves: for $f(x, y) = 0$, the origin; for $f(x, y) = c > 0$, ellipses with center $(0, 0)$ and major and minor axes along the x - and y -axes, respectively
(d) no boundary points
(e) both open and closed
(f) unbounded
4. (a) Domain: all points in the xy -plane
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the union of the lines $y = \pm x$; for $f(x, y) = c \neq 0$, hyperbolas centered at $(0, 0)$ with foci on the x -axis if $c > 0$ and on the y -axis if $c < 0$
(d) no boundary points
(e) both open and closed
(f) unbounded
5. (a) Domain: all points in the xy -plane
(b) Range: all real numbers
(c) level curves are hyperbolas with the x - and y -axes as asymptotes when $f(x, y) \neq 0$, and the x - and y -axes when $f(x, y) = 0$
(d) no boundary points
(e) both open and closed
(f) unbounded
6. (a) Domain: all $(x, y) \neq (0, y)$
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the x -axis minus the origin; for $f(x, y) = c \neq 0$, the parabolas $y = cx^2$ minus the origin
(d) boundary is the line $x = 0$

- [illegible]

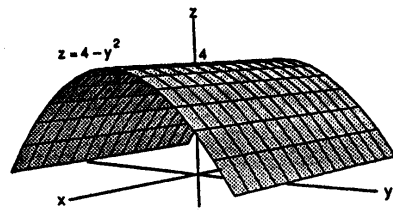
19. (a)



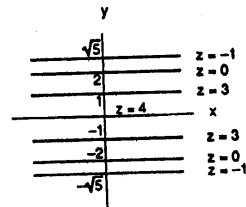
(b)



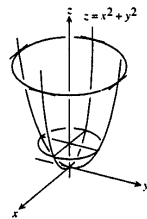
20. (a)



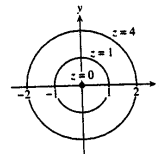
(b)



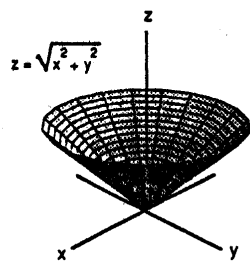
21. (a)



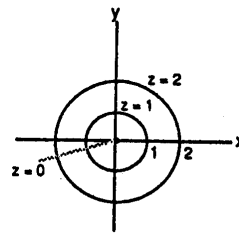
(b)



22. (a)



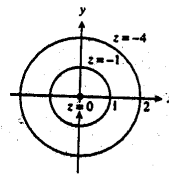
(b)



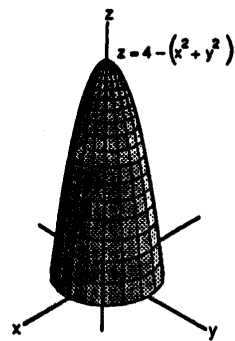
23. (a)



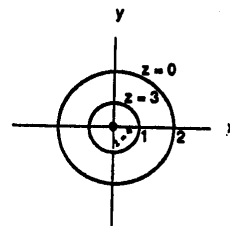
(b)



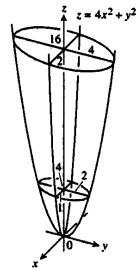
24. (a)



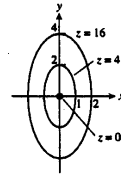
(b)



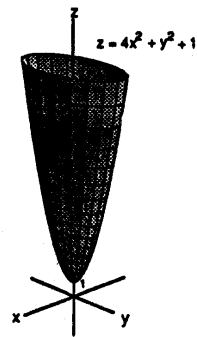
25. (a)



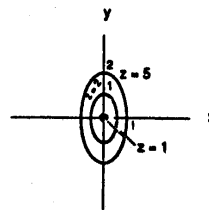
(b)



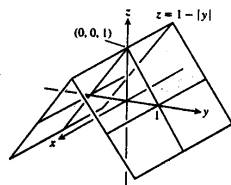
26. (a)



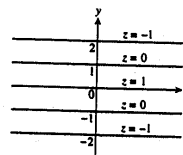
(b)



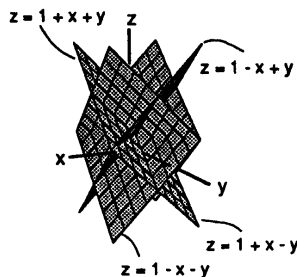
27. (a)



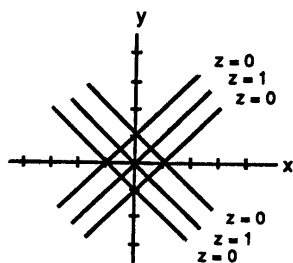
(b)



28. (a)



(b)



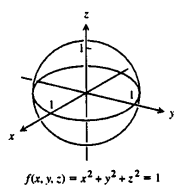
$$29. f(x, y) = 16 - x^2 - y^2 \text{ and } (2\sqrt{2}, \sqrt{2}) \Rightarrow z = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10$$

$$30. f(x, y) = \sqrt{x^2 - 1} \text{ and } (1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \text{ or } x = -1$$

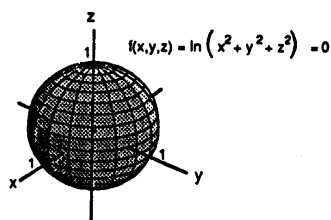
$$31. f(x, y) = \int_x^y \frac{1}{1+t^2} dt \text{ at } (-\sqrt{2}, \sqrt{2}) \Rightarrow z = \tan^{-1} y - \tan^{-1} x; \text{ at } (-\sqrt{2}, \sqrt{2}) \Rightarrow z = \tan^{-1} \sqrt{2} - \tan^{-1}(-\sqrt{2}) \\ = 2 \tan^{-1} \sqrt{2} \Rightarrow \tan^{-1} y - \tan^{-1} x = 2 \tan^{-1} \sqrt{2}$$

$$32. f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n \text{ at } (1, 2) \Rightarrow z = \frac{1}{1 - \left(\frac{x}{y}\right)} = \frac{y}{y-x}; \text{ at } (1, 2) \Rightarrow z = \frac{2}{2-1} = 2 \Rightarrow 2 = \frac{y}{y-x} \Rightarrow 2y - 2x = y \\ \Rightarrow y = 2x$$

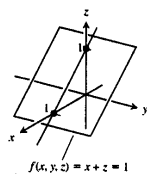
33.



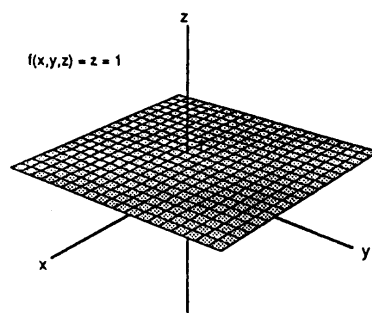
34.



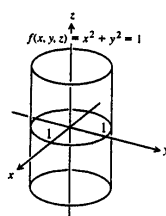
35.



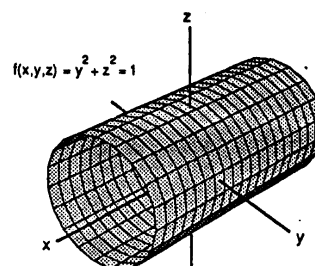
36.



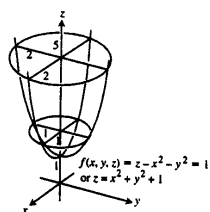
37.



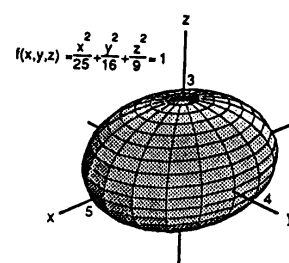
38.



39.



40.



$$41. f(x, y, z) = \sqrt{x-y} - \ln z \text{ at } (3, -1, 1) \Rightarrow w = \sqrt{x-y} - \ln z; \text{ at } (3, -1, 1) \Rightarrow w = \sqrt{3-(-1)} - \ln 1 = 2 \\ \Rightarrow \sqrt{x-y} - \ln z = 2$$

$$42. f(x, y, z) = \ln(x^2 + y + z^2) \text{ at } (-1, 2, 1) \Rightarrow w = \ln(x^2 + y + z^2); \text{ at } (-1, 2, 1) \Rightarrow w = \ln(1 + 2 + 1) = \ln 4 \\ \Rightarrow \ln 4 = \ln(x^2 + y + z^2) \Rightarrow x^2 + y + z^2 = 4$$

$$43. g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n! z^n} \text{ at } (\ln 2, \ln 4, 3) \Rightarrow w = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n! z^n} = e^{(x+y)/z}; \text{ at } (\ln 2, \ln 4, 3) \Rightarrow w = e^{(\ln 2 + \ln 4)/3} \\ = e^{(\ln 8)/3} = e^{\ln 2} = 2 \Rightarrow 2 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 2$$

$$44. g(x, y, z) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2-1}} \text{ at } \left(0, \frac{1}{2}, 2\right) \Rightarrow w = [\sin^{-1} \theta]_x^y + [\sec^{-1} t]_{\sqrt{2}}^z \\ = \sin^{-1} y - \sin^{-1} x + \sec^{-1} z - \sec^{-1}(\sqrt{2}) \Rightarrow w = \sin^{-1} y - \sin^{-1} x + \sec^{-1} z - \frac{\pi}{4}; \text{ at } \left(0, \frac{1}{2}, 2\right) \\ \Rightarrow w = \sin^{-1} \frac{1}{2} - \sin^{-1} 0 + \sec^{-1} 2 - \frac{\pi}{4} = \frac{\pi}{4} \Rightarrow \frac{\pi}{2} = \sin^{-1} y - \sin^{-1} x + \sec^{-1} z$$

$$45. f(x, y, z) = xyz \text{ and } x = 20 - t, y = t, z = 20 \Rightarrow w = (20 - t)(t)(20) \text{ along the line} \Rightarrow w = 400t - 20t^2 \\ \Rightarrow \frac{dw}{dt} = 400 - 40t; \frac{dw}{dt} = 0 \Rightarrow 400 - 40t = 0 \Rightarrow t = 10 \text{ and } \frac{d^2w}{dt^2} = -40 \text{ for all } t \Rightarrow \text{yes, maximum at } t = 10 \\ \Rightarrow x = 20 - 10 = 10, y = 10, z = 20 \Rightarrow \text{maximum of } f \text{ along the line is } f(10, 10, 20) = (10)(10)(20) = 2000$$

$$46. f(x, y, z) = xy - z \text{ and } x = t - 1, y = t - 2, z = t + 7 \Rightarrow w = (t - 1)(t - 2) - (t + 7) = t^2 - 4t - 5 \text{ along the line} \\ \Rightarrow \frac{dw}{dt} = 2t - 4; \frac{dw}{dt} = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2 \text{ and } \frac{d^2w}{dt^2} = 2 \text{ for all } t \Rightarrow \text{yes, minimum at } t = 2 \Rightarrow x = 2 - 1 = 1, \\ y = 2 - 2 = 0, \text{ and } z = 2 + 7 = 9 \Rightarrow \text{minimum of } f \text{ along the line is } f(1, 0, 9) = (1)(0) - 9 = -9$$

$$47. w = 4 \left(\frac{\text{Th}}{d} \right)^{1/2} = 4 \left[\frac{(290 \text{ K})(16.8 \text{ km})}{5 \text{ K/km}} \right]^{1/2} \approx 124.86 \text{ km} \Rightarrow \text{must be } \frac{1}{2}(124.86) \approx 63 \text{ km south of Nantucket}$$

48. The graph of $f(x_1, x_2, x_3, x_4)$ is a set in a five-dimensional space. It is the set of points $(x_1, x_2, x_3, x_4, f(x_1, x_2, x_3, x_4))$ for (x_1, x_2, x_3, x_4) in the domain of f . The graph of $f(x_1, x_2, x_3, \dots, x_n)$ is a set in an $(n + 1)$ -dimensional space. It is the set of points $(x_1, x_2, x_3, \dots, x_n, f(x_1, x_2, x_3, \dots, x_n))$ for $(x_1, x_2, x_3, \dots, x_n)$ in the domain of f .

49-52. Example CAS commands:

Maple:

```
with(plots):
f:= (x,y) -> x*sin(y/2) + y*sin(2*x);
plot3d(f(x,y), x = 0..3*Pi, y=0..3*Pi, axes=FRAMED, title = 'x sin y/2 + y sin 2x');
contourplot(f(x,y), x=0..5*Pi, y=0..5*Pi);
eq:= f(x,y) = f(3*Pi,3*Pi);
implicitplot(eq, x=0..3*Pi, y=0..10*Pi);
```


Mathematica:

```

Clear[x,y]
<< Graphics`ImplicitPlot`
SetOptions[Plot3D, PlotPoints -> 25];
SetOptions[ContourPlot, PlotPoints -> 25,
  ContourShading -> False];
f[x_,y_] = x Sin[y/2] + y Sin[2x]
{xa,xb} = {0, 5 Pi};
{ya,yb} = {0, 5 Pi};
{x0,y0} = {3Pi, 3Pi};
Plot3D[ f[x,y], {x,xa,xb}, {y,ya,yb} ]
ContourPlot[ f[x,y], {x,xa,xb}, {y,ya,yb} ]
ImplicitPlot[ f[x,y] == f[x0,y0], {x,xa,xb}, {y,ya,yb} ]

```

53-56. Example CAS commands:

Maple:

```

with(plots):
eq:= ln(x^2 + y^2 + z^2) = 0.25;
implicitplot3d(eq, x=-1..1, y=-1..1, z=-1..1, axes=BOXED,scaling=CONSTRAINED);

```

Mathematica:

```

ContourPlot3D[ 4 Log[x^2+y^2+z^2],
  {x,-1.1,1.2}, {y,-1.1,1.2}, {z,-1.1,1.2},
  Contours->{1.} ]

```

57-60. Example CAS commands:

Maple:

```

with(plots):
x:= (u,v) -> u*cos(v);
y:= (u,v) -> u*sin(v);
z:= (u,v) -> u;
plot3d([x(u,v), y(u,v), z(u,v)], u = 0..2, v = 0..2*Pi, axes=FRAMED);
contourplot([x(u,v),y(u,v),z(u,v)],u=0..2, v=0..2*Pi);

```

Mathematica:

Note: While in Maple it is trivial to get contours from any 3D surface, in Mathematica it is not obvious for parametric surfaces. In these examples, z only depends on one parameter, so we can solve for that parameter in terms of z , and substitute to get x & y in terms of z and the other parameter, then parametrically plot level curves for several equally spaced values of z (using "Table").

```

ParametricPlot3D[ {u Cos[v], u Sin[v], u},
  {u,0,2}, {v,0,2Pi} ]
ParametricPlot[ Evaluate[Table[
  {z Cos[v], z Sin[v]}, {z,0,2,1/3} ]],
  {v,0,2Pi}, AspectRatio -> Automatic ]

```

11.2 LIMITS AND CONTINUITY IN HIGHER DIMENSIONS

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - 0^2 + 5}{0^2 + 0^2 + 2} = \frac{5}{2}$
2. $\lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$
3. $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1} = \sqrt{24} = 2\sqrt{6}$
4. $\lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2 = \left[\frac{1}{2} + \left(\frac{1}{-3} \right) \right]^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36}$
5. $\lim_{(x,y) \rightarrow (0, \frac{\pi}{4})} \sec x \tan y = (\sec 0) \left(\tan \frac{\pi}{4} \right) = (1)(1) = 1$
6. $\lim_{(x,y) \rightarrow (0,0)} \cos \left(\frac{x^2 + y^3}{x + y + 1} \right) = \cos \left(\frac{0^2 + 0^3}{0 + 0 + 1} \right) = \cos 0 = 1$
7. $\lim_{(x,y) \rightarrow (0, \ln 2)} e^{x-y} = e^{0 - \ln 2} = e^{\ln \left(\frac{1}{2} \right)} = \frac{1}{2}$
8. $\lim_{(x,y) \rightarrow (1,1)} \ln |1 + x^2 y^2| = \ln |1 + (1)^2 (1)^2| = \ln 2$
9. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x} = \lim_{(x,y) \rightarrow (0,0)} (e^y) \left(\frac{\sin x}{x} \right) = e^0 \cdot \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1 \cdot 1 = 1$
10. $\lim_{(x,y) \rightarrow (1,1)} \cos \left(\sqrt[3]{|xy| - 1} \right) = \cos \left(\sqrt[3]{(1)(1) - 1} \right) = \cos 0 = 1$
11. $\lim_{(x,y) \rightarrow (1,0)} \frac{x \sin y}{x^2 + 1} = \frac{1 \cdot \sin 0}{1^2 + 1} = \frac{0}{2} = 0$
12. $\lim_{(x,y) \rightarrow \left(\frac{\pi}{2}, 0 \right)} \frac{\cos y + 1}{y - \sin x} = \frac{(\cos 0) + 1}{0 - \sin \left(\frac{\pi}{2} \right)} = \frac{1 + 1}{-1} = -2$
13. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y) \rightarrow (1,2)} (x-y) = (1-1) = 0$
14. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x+y) = (1+1) = 2$
15. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1} = \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim_{(x,y) \rightarrow (1,1)} (y-2) = (1-2) = -1$

28. (a) All (x, y) so that $x \neq y$
 (b) All (x, y)

29. (a) All (x, y) except where $x = 0$ or $y = 0$
 (b) All (x, y)

30. (a) All (x, y) so that $x^2 - 3x + 2 \neq 0 \Rightarrow (x - 2)(x - 1) \neq 0 \Rightarrow x \neq 2$ and $x \neq 1$
 (b) All (x, y) so that $y \neq x^2$

31. (a) All (x, y, z)
 (b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$

32. (a) All (x, y, z) so that $xyz > 0$
 (b) All (x, y, z)

33. (a) All (x, y, z) with $z \neq 0$
 (b) All (x, y, z) with $x^2 + z^2 \neq 1$

34. (a) All (x, y, z) except $(x, 0, 0)$
 (b) All (x, y, z) except $(0, y, 0)$ or $(x, 0, 0)$

$$35. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x > 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{x^2+x^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}|x|} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}x} = \lim_{x \rightarrow 0} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}};$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x < 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{x^2+x^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}(-x)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$36. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=0}} \frac{x^4}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4+0^2} = 1; \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x^2}} \frac{x^4}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4+(x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

$$37. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim_{x \rightarrow 0} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim_{x \rightarrow 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \frac{1-k^2}{1+k^2} \Rightarrow \text{different limits for different values of } k$$

$$38. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq 0}} \frac{xy}{|xy|} = \lim_{x \rightarrow 0} \frac{x(kx)}{|x(kx)|} = \lim_{x \rightarrow 0} \frac{kx^2}{|kx^2|} = \lim_{x \rightarrow 0} \frac{k}{|k|}; \text{ if } k > 0, \text{ the limit is } 1; \text{ but if } k < 0, \text{ the limit is } -1$$

$$39. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq -1}} \frac{x-y}{x+y} = \lim_{x \rightarrow 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \Rightarrow \text{different limits for different values of } k, k \neq -1$$

$$40. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx \\ k \neq 1}} \frac{x+y}{x-y} = \lim_{x \rightarrow 0} \frac{x+kx}{x-kx} = \frac{1+k}{1-k} \Rightarrow \text{different limits for different values of } k, k \neq 1$$

$$41. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y = kx^2 \\ k \neq 0}} \frac{x^2 + y}{y} = \lim_{x \rightarrow 0} \frac{x^2 + kx^2}{kx^2} = \frac{1+k}{k} \Rightarrow \text{different limits for different values of } k, k \neq 0$$

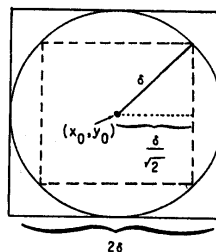
$$42. \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y = kx^2 \\ k \neq 1}} \frac{x^2}{x^2 - y} = \lim_{x \rightarrow 0} \frac{x^2}{x^2 - kx^2} = \frac{1}{1-k} \Rightarrow \text{different limits for different values of } k, k \neq 1$$

43. In Eq. (1), if the point (x, y) lies within a disk centered at (x_0, y_0) and radius less than δ , then $|f(x, y) - L| < \epsilon$; in Eq. (2), if the point (x, y) lies within a square centered at (x_0, y_0) with the side length less than 2δ , then $|f(x, y) - L| < \epsilon$. Since every circle of radius δ is circumscribed by a square of side length 2δ ,

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |x - x_0| < \delta \text{ and } |y - y_0| < \delta; \text{ likewise, every square of side length } \frac{2\delta}{\sqrt{2}} \text{ is circumscribed by a circle of radius } \delta$$

$$\delta \text{ so that } |x - x_0| < \frac{\delta}{\sqrt{2}} \text{ and } |y - y_0| < \frac{\delta}{\sqrt{2}}$$

$\Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$. Thus the requirements are equivalent: small circles give small inscribed squares, and small squares give small inscribed circles.



44. $\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} g(x,y,z) = L$ if, for every number $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all (x,y,z) in the domain of g , $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta \Rightarrow |g(x,y,z) - L| < \epsilon$. With four independent variables and $P = (x, y, z, t)$, $\lim_{P \rightarrow (x_0, y_0, z_0, t_0)} h(x, y, z, t) = L$ if, for every number $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all P in the domain of h , $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (t - t_0)^2} < \delta \Rightarrow |h(x, y, z, t) - L| < \epsilon$.

$$45. \text{ Let } \delta = 0.1. \text{ Then } \sqrt{x^2 + y^2} < \delta \Rightarrow \sqrt{x^2 + y^2} < 0.1 \Rightarrow x^2 + y^2 < 0.01 \Rightarrow |x^2 + y^2 - 0| < 0.01 \Rightarrow |f(x, y) - f(0, 0)| < 0.01 = \epsilon.$$

$$46. \text{ Let } \delta = 0.05. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{y}{x^2 + 1} - 0 \right| = \left| \frac{y}{x^2 + 1} \right| \leq |y| < 0.05 = \epsilon.$$

$$47. \text{ Let } \delta = 0.005. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x+y}{x^2+1} - 0 \right| = \left| \frac{x+y}{x^2+1} \right| \leq |x+y| < |x| + |y| < 0.005 + 0.005 = 0.01 = \epsilon.$$

$$48. \text{ Let } \delta = 0.01. \text{ Since } -1 \leq \cos x \leq 1 \Rightarrow 1 \leq 2 + \cos x \leq 3 \Rightarrow \frac{1}{3} \leq \frac{1}{2 + \cos x} \leq 1 \Rightarrow \left| \frac{x+y}{3} \right| \leq \left| \frac{x+y}{2 + \cos x} \right| \leq |x+y| \leq |x| + |y|. \text{ Then } |x| < \delta \text{ and } |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x+y}{2 + \cos x} - 0 \right| = \left| \frac{x+y}{2 + \cos x} \right| \leq |x| + |y| < 0.01 + 0.01$$

$$= 0.02 = \epsilon.$$

$$\begin{aligned} 49. \text{ Let } \delta = \sqrt{0.015}. \text{ Then } \sqrt{x^2 + y^2 + z^2} < \delta &\Rightarrow |f(x, y, z) - f(0, 0, 0)| = |x^2 + y^2 + z^2 - 0| = |x^2 + y^2 + z^2| \\ &= (\sqrt{x^2 + y^2 + z^2})^2 < (\sqrt{0.015})^2 = 0.015 = \epsilon. \end{aligned}$$

$$\begin{aligned} 50. \text{ Let } \delta = 0.2. \text{ Then } |x| < \delta, |y| < \delta, \text{ and } |z| < \delta &\Rightarrow |f(x, y, z) - f(0, 0, 0)| = |xyz - 0| = |xyz| = |x||y||z| < (0.2)^3 \\ &= 0.008 = \epsilon. \end{aligned}$$

$$\begin{aligned} 51. \text{ Let } \delta = 0.005. \text{ Then } |x| < \delta, |y| < \delta, \text{ and } |z| < \delta &\Rightarrow |f(x, y, z) - f(0, 0, 0)| = \left| \frac{x + y + z}{x^2 + y^2 + z^2 + 1} - 0 \right| \\ &= \left| \frac{x + y + z}{x^2 + y^2 + z^2 + 1} \right| \leq |x + y + z| \leq |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon. \end{aligned}$$

$$\begin{aligned} 52. \text{ Let } \delta = \tan^{-1}(0.1). \text{ Then } |x| < \delta, |y| < \delta, \text{ and } |z| < \delta &\Rightarrow |f(x, y, z) - f(0, 0, 0)| = |\tan^2 x + \tan^2 y + \tan^2 z| \\ &\leq |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2 \delta + \tan^2 \delta + \tan^2 \delta = 0.01 + 0.01 + 0.01 = 0.03 \\ &= \epsilon. \end{aligned}$$

$$\begin{aligned} 53. \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) &= \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} (x + y - z) = x_0 + y_0 - z_0 = f(x_0, y_0, z_0) \Rightarrow f \text{ is continuous at} \\ &\text{every } (x_0, y_0, z_0) \end{aligned}$$

$$\begin{aligned} 54. \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) &= \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} (x^2 + y^2 + z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0, y_0, z_0) \Rightarrow f \text{ is continuous at} \\ &\text{every point } (x_0, y_0, z_0) \end{aligned}$$

$$55. \lim_{(x, y) \rightarrow (0, 0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - (r \cos \theta)(r^2 \sin^2 \theta)}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \lim_{r \rightarrow 0} \frac{r(\cos^3 \theta - \cos \theta \sin^2 \theta)}{1} = 0$$

$$56. \lim_{(x, y) \rightarrow (0, 0)} \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right) = \lim_{r \rightarrow 0} \cos\left(\frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}\right) = \lim_{r \rightarrow 0} \cos\left[\frac{r(\cos^3 \theta - \sin^3 \theta)}{1}\right] = \cos 0 = 1$$

$$\begin{aligned} 57. \lim_{(x, y) \rightarrow (0, 0)} \frac{y^2}{x^2 + y^2} &= \lim_{r \rightarrow 0} \frac{r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} (\sin^2 \theta) = \sin^2 \theta; \text{ the limit does not exist since } \sin^2 \theta \text{ is between} \\ &0 \text{ and } 1 \text{ depending on } \theta \end{aligned}$$

$$58. \lim_{(x, y) \rightarrow (0, 0)} \frac{2x}{x^2 + x + y^2} = \lim_{r \rightarrow 0} \frac{2r \cos \theta}{r^2 + r \cos \theta} = \lim_{r \rightarrow 0} \frac{2 \cos \theta}{r + \cos \theta} = \frac{2 \cos \theta}{\cos \theta}; \text{ the limit does not exist for } \cos \theta = 0$$

$$59. \lim_{(x, y) \rightarrow (0, 0)} \tan^{-1} \left[\frac{|x| + |y|}{x^2 + y^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r \cos \theta| + |r \sin \theta|}{r^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right];$$

$$\text{if } r \rightarrow 0^+, \text{ then } \lim_{r \rightarrow 0^+} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right] = \lim_{r \rightarrow 0^+} \tan^{-1} \left[\frac{|\cos \theta| + |\sin \theta|}{r} \right] = \frac{\pi}{2}; \text{ if } r \rightarrow 0^-, \text{ then}$$

$$\lim_{r \rightarrow 0} \tan^{-1} \left[\frac{r(|\cos \theta| + |\sin \theta|)}{r^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left(\frac{|\cos \theta| + |\sin \theta|}{-r} \right) = \frac{\pi}{2} \Rightarrow \text{the limit is } \frac{\pi}{2}$$

60. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} (\cos^2 \theta - \sin^2 \theta) = \lim_{r \rightarrow 0} (\cos 2\theta)$ which ranges between -1 and 1 depending on $\theta \Rightarrow$ the limit does not exist

61. $\lim_{(x,y) \rightarrow (0,0)} \ln \left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2} \right) = \lim_{r \rightarrow 0} \ln \left(\frac{3r^2 \cos^2 \theta - r^4 \cos^2 \theta \sin^2 \theta + 3r^2 \sin^2 \theta}{r^2} \right)$
 $= \lim_{r \rightarrow 0} \ln(3 - r^2 \cos^2 \theta \sin^2 \theta) = \ln 3 \Rightarrow$ define $f(0,0) = \ln 3$

62. $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(2r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r \rightarrow 0} 2r \cos \theta \sin^2 \theta = 0 \Rightarrow$ define $f(0,0) = 0$

63. No, the limit depends only on the values $f(x,y)$ has when $(x,y) \neq (x_0,y_0)$

64. If f is continuous at (x_0,y_0) , then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$ must equal $f(x_0,y_0) = 3$. If f is not continuous at (x_0,y_0) , the limit could have any value different from 3 , and need not even exist.

65. (a) $f(x,y)|_{y=mx} = \frac{2m}{1+m^2} = \frac{2 \tan \theta}{1+\tan^2 \theta} = \sin 2\theta$. The value of $f(x,y) = \sin 2\theta$ varies with θ , which is the line's angle of inclination.

(b) Since $f(x,y)|_{y=mx} = \sin 2\theta$ and since $-1 \leq \sin 2\theta \leq 1$ for every θ , $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ varies from -1 to 1 along $y = mx$.

66. $|xy(x^2 - y^2)| = |xy||x^2 - y^2| \leq |x||y||x^2 + y^2| = \sqrt{x^2} \sqrt{y^2} |x^2 + y^2| \leq \sqrt{x^2 + y^2} \sqrt{x^2 + y^2} |x^2 + y^2|$
 $= (x^2 + y^2)^2 \Rightarrow \left| \frac{xy(x^2 - y^2)}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2 \Rightarrow -(x^2 + y^2) \leq \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq (x^2 + y^2)$
 $\Rightarrow \lim_{(x,y) \rightarrow (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0$ by the Sandwich Theorem, since $\lim_{(x,y) \rightarrow (0,0)} \pm(x^2 + y^2) = 0$; thus, define $f(0,0) = 0$

67. $\lim_{(x,y) \rightarrow (0,0)} \left(1 - \frac{x^2 y^2}{3} \right) = 1$ and $\lim_{(x,y) \rightarrow (0,0)} 1 = 1 \Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy} = 1$, by the Sandwich Theorem

68. If $xy > 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - \left(\frac{x^2 y^2}{6}\right)}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{2xy - \left(\frac{x^2 y^2}{6}\right)}{xy} = \lim_{(x,y) \rightarrow (0,0)} \left(2 - \frac{xy}{6} \right) = 2$ and

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} 2 = 2; \text{ if } xy < 0, \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - \left(\frac{x^2 y^2}{6}\right)}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{-2xy - \left(\frac{x^2 y^2}{6}\right)}{-xy}$$

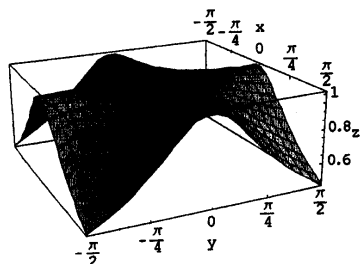
$$= \lim_{(x,y) \rightarrow (0,0)} \left(2 + \frac{xy}{6}\right) = 2 \text{ and } \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = 2 \Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} = 2, \text{ by the Sandwich Theorem}$$

Theorem

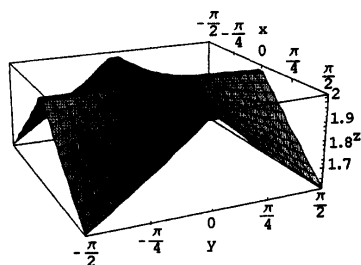
69. The limit is 0 since $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -y \leq y \sin\left(\frac{1}{x}\right) \leq y$ for $y \geq 0$, and $-y \geq y \sin\left(\frac{1}{x}\right) \geq y$ for $y \leq 0$. Thus as $(x, y) \rightarrow (0, 0)$, both $-y$ and y approach 0 $\Rightarrow y \sin\left(\frac{1}{x}\right) \rightarrow 0$, by the Sandwich Theorem.

70. The limit is 0 since $\left|\cos\left(\frac{1}{y}\right)\right| \leq 1 \Rightarrow -1 \leq \cos\left(\frac{1}{y}\right) \leq 1 \Rightarrow -x \leq x \cos\left(\frac{1}{y}\right) \leq x$ for $x \geq 0$, and $-x \geq x \cos\left(\frac{1}{y}\right) \geq x$ for $x \leq 0$. Thus as $(x, y) \rightarrow (0, 0)$, both $-x$ and x approach 0 $\Rightarrow x \cos\left(\frac{1}{y}\right) \rightarrow 0$, by the Sandwich Theorem.

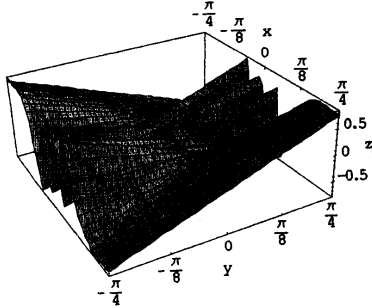
71. The graph for $f(x, y) = \frac{\tan^{-1} xy}{xy}$ in Exercise 67 supports that $\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy} = 1$.



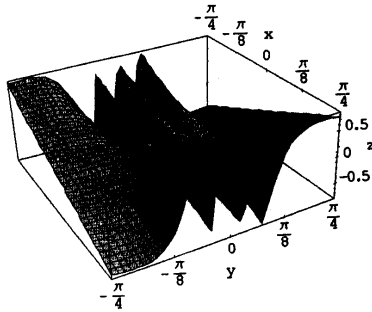
The graph $f(x, y) = \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|}$ in Exercise 68 supports that $\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} = 2$.



The graph for $f(x, y) = y \sin \frac{1}{x}$ in Exercise 69 supports that $\lim_{(x, y) \rightarrow (0, 0)} y \sin \frac{1}{x} = 0$.



The graph for $f(x, y) = x \sin \frac{1}{y}$ in Exercise 70 supports that $\lim_{(x, y) \rightarrow (0, 0)} x \sin \frac{1}{y} = 0$.



11.3 PARTIAL DERIVATIVES

1. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = -3$
2. $\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial y} = -x + 2y$
3. $\frac{\partial f}{\partial x} = 2x(y + 2)$, $\frac{\partial f}{\partial y} = x^2 - 1$
4. $\frac{\partial f}{\partial x} = 5y - 14x + 3$, $\frac{\partial f}{\partial y} = 5x - 2y - 6$
5. $\frac{\partial f}{\partial x} = 2y(xy - 1)$, $\frac{\partial f}{\partial y} = 2x(xy - 1)$
6. $\frac{\partial f}{\partial x} = 6(2x - 3y)^2$, $\frac{\partial f}{\partial y} = -9(2x - 3y)^2$
7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$
8. $\frac{\partial f}{\partial x} = \frac{2x^2}{3\sqrt{x^3 + (\frac{y}{2})}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt{x^3 + (\frac{y}{2})}}$

9. $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}$, $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$
10. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$
11. $\frac{\partial f}{\partial x} = \frac{(xy-1)(1) - (x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1) - (x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$
12. $\frac{\partial f}{\partial x} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{\partial}{\partial x}(\frac{y}{x}) = -\frac{y}{x^2[1+(\frac{y}{x})^2]} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{\partial}{\partial y}(\frac{y}{x}) = \frac{1}{x[1+(\frac{y}{x})^2]} = \frac{x}{x^2+y^2}$
13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x}(x+y+1) = e^{(x+y+1)}$, $\frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y}(x+y+1) = e^{(x+y+1)}$
14. $\frac{\partial f}{\partial x} = -e^{-x} \sin(x+y) + e^{-x} \cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x} \cos(x+y)$
15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$
16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \cdot \ln y = ye^{xy} \ln y$, $\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy} \ln y + \frac{e^{xy}}{y}$
17. $\frac{\partial f}{\partial x} = 2 \sin(x-3y) \cdot \frac{\partial}{\partial x} \sin(x-3y) = 2 \sin(x-3y) \cos(x-3y) \cdot \frac{\partial}{\partial x}(x-3y) = 2 \sin(x-3y) \cos(x-3y)$,
 $\frac{\partial f}{\partial y} = 2 \sin(x-3y) \cdot \frac{\partial}{\partial y} \sin(x-3y) = 2 \sin(x-3y) \cos(x-3y) \cdot \frac{\partial}{\partial y}(x-3y) = -6 \sin(x-3y) \cos(x-3y)$
18. $\frac{\partial f}{\partial x} = 2 \cos(3x-y^2) \cdot \frac{\partial}{\partial x} \cos(3x-y^2) = -2 \cos(3x-y^2) \sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$
 $= -6 \cos(3x-y^2) \sin(3x-y^2)$,
 $\frac{\partial f}{\partial y} = 2 \cos(3x-y^2) \cdot \frac{\partial}{\partial y} \cos(3x-y^2) = -2 \cos(3x-y^2) \sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$
 $= 4y \cos(3x-y^2) \sin(3x-y^2)$
19. $\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = x^y \ln x$
20. $f(x,y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x \ln y}$ and $\frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$
21. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$
22. $f(x,y) = \sum_{n=0}^{\infty} (xy)^n, |xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$ and
 $\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$
23. $f_x = 1 + y^2$, $f_y = 2xy$, $f_z = -4z$
24. $f_x = y + z$, $f_y = x + z$, $f_z = y + x$

25. $f_x = 1$, $f_y = -\frac{y}{\sqrt{y^2 + z^2}}$, $f_z = -\frac{z}{\sqrt{y^2 + z^2}}$
26. $f_x = -x(x^2 + y^2 + z^2)^{-3/2}$, $f_y = -y(x^2 + y^2 + z^2)^{-3/2}$, $f_z = -z(x^2 + y^2 + z^2)^{-3/2}$
27. $f_x = \frac{yz}{\sqrt{1 - x^2 y^2 z^2}}$, $f_y = \frac{xz}{\sqrt{1 - x^2 y^2 z^2}}$, $f_z = \frac{xy}{\sqrt{1 - x^2 y^2 z^2}}$
28. $f_x = \frac{1}{|x + yz|\sqrt{(x + yz)^2 - 1}}$, $f_y = \frac{z}{|x + yz|\sqrt{(x + yz)^2 - 1}}$, $f_z = \frac{y}{|x + yz|\sqrt{(x + yz)^2 - 1}}$
29. $f_x = \frac{1}{x + 2y + 3z}$, $f_y = \frac{2}{x + 2y + 3z}$, $f_z = \frac{3}{x + 2y + 3z}$
30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}$, $f_y = z \ln(xy) + yz \cdot \frac{\partial}{\partial y} \ln(xy) = z \ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z \ln(xy) + z$,
 $f_z = y \ln(xy) + yz \cdot \frac{\partial}{\partial z} \ln(xy) = y \ln(xy)$
31. $f_x = -2xe^{-(x^2 + y^2 + z^2)}$, $f_y = -2ye^{-(x^2 + y^2 + z^2)}$, $f_z = -2ze^{-(x^2 + y^2 + z^2)}$
32. $f_x = -yze^{-xyz}$, $f_y = -xze^{-xyz}$, $f_z = -xye^{-xyz}$
33. $f_x = \operatorname{sech}^2(x + 2y + 3z)$, $f_y = 2 \operatorname{sech}^2(x + 2y + 3z)$, $f_z = 3 \operatorname{sech}^2(x + 2y + 3z)$
34. $f_x = y \cosh(xy - z^2)$, $f_y = x \cosh(xy - z^2)$, $f_z = -2z \cosh(xy - z^2)$
35. $\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$
36. $\frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u} \left(\frac{2u}{v} \right) = 2ve^{(2u/v)}$, $\frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v} \left(\frac{2u}{v} \right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$
37. $\frac{\partial h}{\partial \rho} = \sin \phi \cos \theta$, $\frac{\partial h}{\partial \phi} = \rho \cos \phi \cos \theta$, $\frac{\partial h}{\partial \theta} = -\rho \sin \phi \sin \theta$
38. $\frac{\partial g}{\partial r} = 1 - \cos \theta$, $\frac{\partial g}{\partial \theta} = r \sin \theta$, $\frac{\partial g}{\partial z} = -1$
39. $W_P = V$, $W_V = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$
40. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$
41. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$
42. $\frac{\partial f}{\partial x} = y \cos xy$, $\frac{\partial f}{\partial y} = x \cos xy$, $\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$, $\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$

$$43. \frac{\partial g}{\partial x} = 2xy + y \cos x, \frac{\partial g}{\partial y} = x^2 - \sin y + \sin x, \frac{\partial^2 g}{\partial x^2} = 2y - y \sin x, \frac{\partial^2 g}{\partial y^2} = -\cos y, \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$$

$$44. \frac{\partial h}{\partial x} = e^y, \frac{\partial h}{\partial y} = xe^y + 1, \frac{\partial^2 h}{\partial x^2} = 0, \frac{\partial^2 h}{\partial y^2} = xe^y, \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$$

$$45. \frac{\partial r}{\partial x} = \frac{1}{x+y}, \frac{\partial r}{\partial y} = \frac{1}{x+y}, \frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}, \frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}, \frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$$

$$46. \frac{\partial s}{\partial x} = \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) = \left(-\frac{y}{x^2} \right) \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] = \frac{-y}{x^2 + y^2}, \frac{\partial s}{\partial y} = \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) = \left(\frac{1}{x} \right) \left[\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right] = \frac{x}{x^2 + y^2},$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}, \frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$47. \frac{\partial w}{\partial x} = \frac{2}{2x+3y}, \frac{\partial w}{\partial y} = \frac{3}{2x+3y}, \frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

$$48. \frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}, \frac{\partial w}{\partial y} = \frac{x}{y} + \ln x, \frac{\partial^2 w}{\partial y \partial x} = \frac{1}{y} + \frac{1}{x}, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$$

$$49. \frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4, \frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3, \frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$$

$$50. \frac{\partial w}{\partial x} = \sin y + y \cos x + y, \frac{\partial w}{\partial y} = x \cos y + \sin x + x, \frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$$

$$51. (a) \text{ x first} \quad (b) \text{ y first} \quad (c) \text{ x first} \quad (d) \text{ x first} \quad (e) \text{ y first} \quad (f) \text{ y first}$$

$$52. (a) \text{ y first three times} \quad (b) \text{ y first three times} \quad (c) \text{ y first twice} \quad (d) \text{ x first twice}$$

$$53. f_x(1,2) = \lim_{h \rightarrow 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim_{h \rightarrow 0} \frac{[1 - (1+h) + 2 - 6(1+h)^2] - (2-6)}{h} = \lim_{h \rightarrow 0} \frac{-h - 6(1+2h+h^2) + 6}{h} \\ = \lim_{h \rightarrow 0} \frac{-13h - 6h^2}{h} = \lim_{h \rightarrow 0} (-13 - 6h) = -13,$$

$$f_y(1,2) = \lim_{h \rightarrow 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim_{h \rightarrow 0} \frac{[1 - 1 + (2+h) - 3(2+h)] - (2-6)}{h} = \lim_{h \rightarrow 0} \frac{(2-6-2h) - (2-6)}{h} \\ = \lim_{h \rightarrow 0} (-2) = -2$$

$$54. f_x(-2,1) = \lim_{h \rightarrow 0} \frac{f(-2+h,1) - f(-2,1)}{h} = \lim_{h \rightarrow 0} \frac{[4 + 2(-2+h) - 3 - (-2+h)] - (-3+2)}{h} \\ = \lim_{h \rightarrow 0} \frac{(2h-1-h) + 1}{h} = \lim_{h \rightarrow 0} 1 = 1,$$

$$\begin{aligned} f_y(-2, 1) &= \lim_{h \rightarrow 0} \frac{f(-2, 1+h) - f(-2, 1)}{h} = \lim_{h \rightarrow 0} \frac{[4 - 4 - 3(1+h) + 2(1+h^2)] - (-3+2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-3 - 3h + 2 + 4h + 2h^2) + 1}{h} = \lim_{h \rightarrow 0} \frac{h + 2h^2}{h} = \lim_{h \rightarrow 0} (1 + 2h) = 1 \end{aligned}$$

$$\begin{aligned} 55. f_z(x_0, y_0, z_0) &= \lim_{h \rightarrow 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h}; \\ f_z(1, 2, 3) &= \lim_{h \rightarrow 0} \frac{f(1, 2, 3+h) - f(1, 2, 3)}{h} = \lim_{h \rightarrow 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim_{h \rightarrow 0} \frac{12h + 2h^2}{h} = \lim_{h \rightarrow 0} (12 + 2h) = 12 \end{aligned}$$

$$\begin{aligned} 56. f_y(x_0, y_0, z_0) &= \lim_{h \rightarrow 0} \frac{f(x_0, y_0+h, z_0) - f(x_0, y_0, z_0)}{h}; \\ f_y(-1, 0, 3) &= \lim_{h \rightarrow 0} \frac{f(-1, h, 3) - f(-1, 0, 3)}{h} = \lim_{h \rightarrow 0} \frac{(2h^2 + 9h) - 0}{h} = \lim_{h \rightarrow 0} (2h + 9) = 9 \end{aligned}$$

$$\begin{aligned} 57. y + \left(3z^2 \frac{\partial z}{\partial x}\right)x + z^3 - 2y \frac{\partial z}{\partial x} &= 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow \text{at } (1, 1, 1) \text{ we have } (3-2) \frac{\partial z}{\partial x} = -1-1 \text{ or} \\ \frac{\partial z}{\partial x} &= -2 \end{aligned}$$

$$\begin{aligned} 58. \left(\frac{\partial x}{\partial z}\right)z + x + \left(\frac{y}{x}\right)\frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} &= 0 \Rightarrow \left(z + \frac{y}{x} - 2x\right) \frac{\partial x}{\partial z} = -x \Rightarrow \text{at } (1, -1, -3) \text{ we have } (-3-1-2) \frac{\partial x}{\partial z} = -1 \text{ or} \\ \frac{\partial x}{\partial z} &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} 59. a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2a &= (2bc \sin A) \frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}; \text{ also } 0 = 2b - 2c \cos A + (2bc \sin A) \frac{\partial A}{\partial b} \\ \Rightarrow 2c \cos A - 2b &= (2bc \sin A) \frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A} \end{aligned}$$

$$\begin{aligned} 60. \frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A) \frac{\partial a}{\partial A} - a \cos A}{\sin^2 A} &= 0 \Rightarrow (\sin A) \frac{\partial a}{\partial A} - a \cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a \cos A}{\sin A}; \text{ also} \\ \left(\frac{1}{\sin A}\right) \frac{\partial a}{\partial B} &= b(-\csc B \cot B) \Rightarrow \frac{\partial a}{\partial B} = -b \csc B \cot B \sin A \end{aligned}$$

$$61. \text{Differentiating each equation implicitly gives } 1 = v_x \ln u + \left(\frac{v}{u}\right)u_x \text{ and } 0 = u_x \ln v + \left(\frac{u}{v}\right)v_x \text{ or}$$

$$\left. \begin{aligned} (\ln u) v_x + \left(\frac{v}{u}\right)u_x &= 1 \\ \left(\frac{u}{v}\right)v_x + (\ln v)u_x &= 0 \end{aligned} \right\} \Rightarrow v_x = \frac{\begin{vmatrix} 1 & \frac{v}{u} \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

$$62. \text{Differentiating each equation implicitly gives } 1 = (2x)x_u - (2y)y_u \text{ and } 0 = (2x)x_u - y_u \text{ or}$$

$$\left. \begin{aligned} (2x)x_u - (2y)y_u &= 1 \\ (2x)x_u - y_u &= 0 \end{aligned} \right\} \Rightarrow x_u = \frac{\begin{vmatrix} 1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = \frac{-1}{-2x+4xy} = \frac{1}{2x-4xy} \text{ and}$$

$$y_u = \frac{\begin{vmatrix} 2x & 1 \\ 2x & 0 \end{vmatrix}}{-2x+4xy} = \frac{-2x}{-2x+4xy} = \frac{2x}{2x-4xy} = \frac{1}{1-2y}; \text{ next } s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u}$$

$$= 2x \left(\frac{1}{2x-4xy} \right) + 2y \left(\frac{1}{1-2y} \right) = \frac{1}{1-2y} + \frac{2y}{1-2y} = \frac{1+2y}{1-2y}$$

$$63. \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -4z \Rightarrow \frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial y^2} = 2, \frac{\partial^2 f}{\partial z^2} = -4 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + (-4) = 0$$

$$64. \frac{\partial f}{\partial x} = -6xz, \frac{\partial f}{\partial y} = -6yz, \frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2), \frac{\partial^2 f}{\partial x^2} = -6z, \frac{\partial^2 f}{\partial y^2} = -6z, \frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= -6z - 6z + 12z = 0$$

$$65. \frac{\partial f}{\partial x} = -2e^{-2y} \sin 2x, \frac{\partial f}{\partial y} = -e^{-2y} \cos 2x, \frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos 2x, \frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos 2x \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= -4e^{-2y} \cos 2x + 4e^{-2y} \cos 2x = 0$$

$$66. \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2}, \frac{\partial^2 f}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$$

$$67. \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}, \frac{\partial f}{\partial y} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y)$$

$$= -y(x^2 + y^2 + z^2)^{-3/2}, \frac{\partial f}{\partial z} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z) = -z(x^2 + y^2 + z^2)^{-3/2};$$

$$\frac{\partial^2 f}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2}, \frac{\partial^2 f}{\partial y^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2},$$

$$\frac{\partial^2 f}{\partial z^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left[-(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2} \right] + \left[-(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2} \right]$$

$$+ \left[-(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2} \right] = -3(x^2 + y^2 + z^2)^{-3/2} + (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-5/2}$$

$$= 0$$

$$68. \frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z, \frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z, \frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z; \frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z, \frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z,$$

$$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$$

69. $\frac{\partial w}{\partial x} = \cos(x+ct)$, $\frac{\partial w}{\partial t} = c \cos(x+ct)$; $\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [-\sin(x+ct)]$
 $= c^2 \frac{\partial^2 w}{\partial x^2}$
70. $\frac{\partial w}{\partial x} = -2 \sin(2x+2ct)$, $\frac{\partial w}{\partial t} = -2c \sin(2x+2ct)$; $\frac{\partial^2 w}{\partial x^2} = -4 \cos(2x+2ct)$, $\frac{\partial^2 w}{\partial t^2} = -4c^2 \cos(2x+2ct)$
 $\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [-4 \cos(2x+2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
71. $\frac{\partial w}{\partial x} = \cos(x+ct) - 2 \sin(2x+2ct)$, $\frac{\partial w}{\partial t} = c \cos(x+ct) - 2c \sin(2x+2ct)$;
 $\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4 \cos(2x+2ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)$
 $\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [-\sin(x+ct) - 4 \cos(2x+2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
72. $\frac{\partial w}{\partial x} = \frac{1}{x+ct}$, $\frac{\partial w}{\partial t} = \frac{c}{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x+ct)^2}$, $\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 \left[\frac{-1}{(x+ct)^2} \right] = c^2 \frac{\partial^2 w}{\partial x^2}$
73. $\frac{\partial w}{\partial x} = 2 \sec^2(2x-2ct)$, $\frac{\partial w}{\partial t} = -2c \sec^2(2x-2ct)$; $\frac{\partial^2 w}{\partial x^2} = 8 \sec^2(2x-2ct) \tan(2x-2ct)$,
 $\frac{\partial^2 w}{\partial t^2} = 8c^2 \sec^2(2x-2ct) \tan(2x-2ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [8 \sec^2(2x-2ct) \tan(2x-2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$
74. $\frac{\partial w}{\partial x} = -15 \sin(3x+3ct) + e^{x+ct}$, $\frac{\partial w}{\partial t} = -15c \sin(3x+3ct) + ce^{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = -45 \cos(3x+3ct) + e^{x+ct}$,
 $\frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x+3ct) + c^2 e^{x+ct} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 [-45 \cos(3x+3ct) + e^{x+ct}] = c^2 \frac{\partial^2 w}{\partial x^2}$
75. $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u} (ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac) \left(\frac{\partial^2 f}{\partial u^2} \right) (ac) = a^2 c^2 \frac{\partial^2 f}{\partial u^2}$; $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a \frac{\partial^2 f}{\partial u^2} \right) \cdot a$
 $= a^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = a^2 c^2 \frac{\partial^2 f}{\partial u^2} = c^2 \left(a^2 \frac{\partial^2 f}{\partial u^2} \right) = c^2 \frac{\partial^2 w}{\partial x^2}$
76. If the first partial derivatives are continuous throughout an open region R , then by Eq. (3) in this section of the text, $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. Then as $(x, y) \rightarrow (x_0, y_0)$, $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0 \Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0) \Rightarrow f$ is continuous at every point (x_0, y_0) in R .
77. Yes, since f_{xx} , f_{yy} , f_{xy} , and f_{yx} are all continuous on R , use the same reasoning as in Exercise 76 with
 $f_x(x, y) = f_x(x_0, y_0) + f_{xx}(x_0, y_0) \Delta x + f_{xy}(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ and
 $f_y(x, y) = f_y(x_0, y_0) + f_{yx}(x_0, y_0) \Delta x + f_{yy}(x_0, y_0) \Delta y + \hat{\epsilon}_1 \Delta x + \hat{\epsilon}_2 \Delta y$. Then $\lim_{(x, y) \rightarrow (x_0, y_0)} f_x(x, y) = f_x(x_0, y_0)$
and $\lim_{(x, y) \rightarrow (x_0, y_0)} f_y(x, y) = f_y(x_0, y_0)$.

11.4 THE CHAIN RULE

1. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t = 0$; $w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$
 (b) $\frac{dw}{dt}(\pi) = 0$
2. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t + \cos t$, $\frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt} = (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) = 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = (2 \cos^2 t - 2 \sin^2 t) - (2 \cos^2 t - 2 \sin^2 t) = 0$; $w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2 \cos^2 t + 2 \sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$
 (b) $\frac{dw}{dt}(0) = 0$
3. (a) $\frac{\partial w}{\partial x} = \frac{1}{z}$, $\frac{\partial w}{\partial y} = \frac{1}{z}$, $\frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2}$, $\frac{dx}{dt} = -2 \cos t \sin t$, $\frac{dy}{dt} = 2 \sin t \cos t$, $\frac{dz}{dt} = -\frac{1}{t^2} \Rightarrow \frac{dw}{dt} = -\frac{2}{z} \cos t \sin t + \frac{2}{z} \sin t \cos t + \frac{x+y}{z^2 t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1$; $w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$
 (b) $\frac{dw}{dt}(3) = 1$
4. (a) $\frac{\partial w}{\partial x} = \frac{2x}{x^2 + y^2 + z^2}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$, $\frac{\partial w}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = 2t^{-1/2} \Rightarrow \frac{dw}{dt} = \frac{-2x \sin t}{x^2 + y^2 + z^2} + \frac{2y \cos t}{x^2 + y^2 + z^2} + \frac{4zt^{-1/2}}{x^2 + y^2 + z^2} = \frac{-2 \cos t \sin t + 2 \sin t \cos t + 4(4t^{-1/2})t^{-1/2}}{\cos^2 t + \sin^2 t + 16t} = \frac{16}{1 + 16t}$; $w = \ln(x^2 + y^2 + z^2) = \ln(\cos^2 t + \sin^2 t + 16t) = \ln(1 + 16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1 + 16t}$
 (b) $\frac{dw}{dt}(3) = \frac{16}{49}$
5. (a) $\frac{\partial w}{\partial x} = 2ye^x$, $\frac{\partial w}{\partial y} = 2e^x$, $\frac{\partial w}{\partial z} = -\frac{1}{z}$, $\frac{dx}{dt} = \frac{2t}{t^2 + 1}$, $\frac{dy}{dt} = \frac{1}{t^2 + 1}$, $\frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2 + 1} + \frac{2e^x}{t^2 + 1} - \frac{e^t}{z} = \frac{(4t)(\tan^{-1} t)(t^2 + 1)}{t^2 + 1} + \frac{2(t^2 + 1)}{t^2 + 1} - \frac{e^t}{e^t} = 4t \tan^{-1} t + 1$; $w = 2ye^x - \ln z = (2 \tan^{-1} t)(t^2 + 1) - t \Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2 + 1}\right)(t^2 + 1) + (2 \tan^{-1} t)(2t) - 1 = 4t \tan^{-1} t + 1$
 (b) $\frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$
6. (a) $\frac{\partial w}{\partial x} = -y \cos xy$, $\frac{\partial w}{\partial y} = -x \cos xy$, $\frac{\partial w}{\partial z} = 1$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{t}$, $\frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y \cos xy - \frac{x \cos xy}{t} + e^{t-1} = -(\ln t)[\cos(t \ln t)] - \frac{t \cos(t \ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t \ln t)] - \cos(t \ln t) + e^{t-1}$; $w = z - \sin xy = e^{t-1} - \sin(t \ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t \ln t)]\left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1 + \ln t) \cos(t \ln t)$
 (b) $\frac{dw}{dt}(1) = 1 - (1 + 0)(1) = 0$

7. (a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (4e^x \ln y) \left(\frac{\cos v}{u \cos v} \right) + \left(\frac{4e^x}{y} \right) (\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$
 $= \frac{4(u \cos v) \ln(u \sin v)}{u} + \frac{4(u \cos v)(\sin v)}{u \sin v} = (4 \cos v) \ln(u \sin v) + 4 \cos v;$
 $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (4e^x \ln y) \left(\frac{-u \sin v}{u \cos v} \right) + \left(\frac{4e^x}{y} \right) (u \cos v) = -(4e^x \ln y)(\tan v) + \frac{4e^x u \cos v}{y}$
 $= [-4(u \cos v) \ln(u \sin v)](\tan v) + \frac{4(u \cos v)(u \cos v)}{u \sin v} = (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v};$
 $z = 4e^x \ln y = 4(u \cos v) \ln(u \sin v) \Rightarrow \frac{\partial z}{\partial u} = (4 \cos v) \ln(u \sin v) + 4(u \cos v) \left(\frac{\sin v}{u \sin v} \right)$
 $= (4 \cos v) \ln(u \sin v) + 4 \cos v; \text{ also } \frac{\partial z}{\partial v} = (-4u \sin v) \ln(u \sin v) + 4(u \cos v) \left(\frac{u \cos v}{u \sin v} \right)$
 $= (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$
- (b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4 \cos \frac{\pi}{4} \ln \left(2 \sin \frac{\pi}{4}\right) + 4 \cos \frac{\pi}{4} = 2\sqrt{2} \ln \sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2);$
 $\frac{\partial z}{\partial v} = (-4)(2) \sin \frac{\pi}{4} \ln \left(2 \sin \frac{\pi}{4}\right) + \frac{(4)(2) \left(\cos^2 \frac{\pi}{4}\right)}{\left(\sin \frac{\pi}{4}\right)} = -4\sqrt{2} \ln \sqrt{2} + 4\sqrt{2} = -2\sqrt{2}(\ln 2 - 2)$
8. (a) $\frac{\partial z}{\partial u} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1} \right] \cos v + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1} \right] \sin v = \frac{y \cos v}{x^2 + y^2} - \frac{x \sin v}{x^2 + y^2} = \frac{(u \sin v)(\cos v) - (u \cos v)(\sin v)}{u^2} = 0;$
 $\frac{\partial z}{\partial v} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1} \right] (-u \sin v) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1} \right] u \cos v = -\frac{yu \sin v}{x^2 + y^2} - \frac{xu \cos v}{x^2 + y^2}$
 $= \frac{-(u \sin v)(u \sin v) - (u \cos v)(u \cos v)}{u^2} = -\sin^2 v - \cos^2 v = -1; z = \tan^{-1} \left(\frac{x}{y} \right) = \tan^{-1}(\cot v) \Rightarrow \frac{\partial z}{\partial u} = 0$
and $\frac{\partial z}{\partial v} = \left(\frac{1}{1 + \cot^2 v} \right) (-\csc^2 v) = \frac{-1}{\sin^2 v + \cos^2 v} = -1$
- (b) At $\left(1.3, \frac{\pi}{6}\right)$: $\frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = -1$
9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v) = x+y+2z+v(y+x)$
 $= (u+v) + (u-v) + 2uv + v(2u) = 2u + 4uv; \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$
 $= (y+z)(1) + (x+z)(-1) + (y+x)(u) = y-x + (y+x)u = -2v + (2u)u = -2v + 2u^2;$
 $w = xy + yz + xz = (u^2 - v^2) + (u^2 v - uv^2) + (u^2 v + uv^2) = u^2 - v^2 + 2u^2 v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv \text{ and } \frac{\partial w}{\partial v} = -2v + 2u^2$
- (b) At $\left(\frac{1}{2}, 1\right)$: $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -\frac{3}{2}$

10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2 + y^2 + z^2} \right) (e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2 + y^2 + z^2} \right) (e^v \cos u - ue^v \sin u) + \left(\frac{2z}{x^2 + y^2 + z^2} \right) (e^v)$

$$= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (e^v \sin u + ue^v \cos u)$$

$$+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (e^v \cos u - ue^v \sin u)$$

$$+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (e^v) = \frac{2}{u};$$

$$\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2 + y^2 + z^2} \right) (ue^v \sin u) + \left(\frac{2y}{x^2 + y^2 + z^2} \right) (ue^v \cos u) + \left(\frac{2z}{x^2 + y^2 + z^2} \right) (ue^v)$$

$$= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (ue^v \sin u)$$

$$+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (ue^v \cos u)$$

$$+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}} \right) (ue^v) = 2; w = \ln(u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}) = \ln(2u^2 e^{2v})$$

$$= \ln 2 + 2 \ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u} \text{ and } \frac{\partial w}{\partial v} = 2$$

(b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$

11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r+r-p+p-q}{(q-r)^2} = 0;$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} = \frac{1}{q-r} - \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r-r+p+p-q}{(q-r)^2} = \frac{2p-2r}{(q-r)^2}$$

$$= \frac{(2x+2y+2z)-(2x+2y-2z)}{(2z-2y)^2} = \frac{z}{(z-y)^2}; \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z}$$

$$= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} - \frac{p-q}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \frac{2q-2p}{(q-r)^2} = \frac{-4y}{(2z-2y)^2} = -\frac{y}{(z-y)^2};$$

$$u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2}, \text{ and } \frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$$

$$= -\frac{y}{(z-y)^2}$$

(b) At $(\sqrt{3}, 2, 1)$: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1$, and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$

12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}} (\cos x) + (re^{qr} \sin^{-1} p)(0) + (qe^{qr} \sin^{-1} p)(0) = \frac{e^{qr} \cos x}{\sqrt{1-p^2}} = \frac{e^{z \ln y} \cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;

$$\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}} (0) + (re^{qr} \sin^{-1} p) \left(\frac{z^2}{y} \right) + (qe^{qr} \sin^{-1} p)(0) = \frac{z^2 re^{qr} \sin^{-1} p}{y} = \frac{z^2 \left(\frac{1}{y} \right) y^z x}{y} = xzy^{z-1};$$

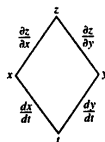
$$\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}} (0) + (re^{qr} \sin^{-1} p)(2z \ln y) + (qe^{qr} \sin^{-1} p) \left(-\frac{1}{z^2} \right) = (2zre^{qr} \sin^{-1} p)(\ln y) - \frac{qe^{qr} \sin^{-1} p}{z^2}$$

$$= (2z) \left(\frac{1}{z} \right) (y^z x \ln y) - \frac{(z^2 \ln y)(y^z)x}{z^2} = xy^z \ln y; u = e^{z \ln y} \sin^{-1}(\sin x) = xy^z \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z,$$

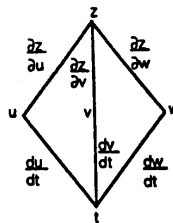
$$\frac{\partial u}{\partial y} = xzy^{z-1}, \text{ and } \frac{\partial u}{\partial z} = xy^z \ln y \text{ from direct calculations}$$

$$(b) \text{ At } \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2} \right): \frac{\partial u}{\partial x} = \left(\frac{1}{2} \right)^{-1/2} = \sqrt{2}, \frac{\partial u}{\partial y} = \left(\frac{\pi}{4} \right) \left(-\frac{1}{2} \right) \left(\frac{1}{2} \right)^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4}, \frac{\partial u}{\partial z} = \left(\frac{\pi}{4} \right) \left(\frac{1}{2} \right)^{-1/2} \ln \left(\frac{1}{2} \right) \\ = -\frac{\pi\sqrt{2} \ln 2}{4}$$

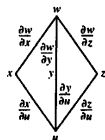
$$13. \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$



$$14. \frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$$



$$15. \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

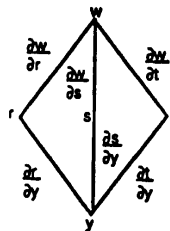
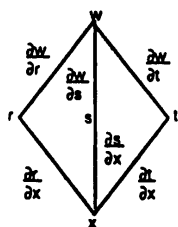


$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$

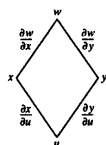


$$16. \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial x}$$

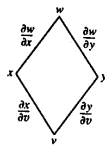
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial y}$$



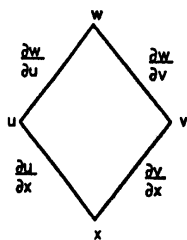
$$17. \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$$



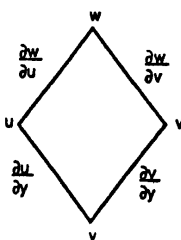
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}$$



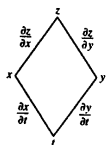
$$18. \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}$$



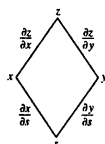
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y}$$



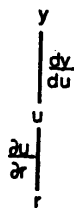
$$19. \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$



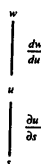
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$



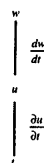
$$20. \frac{\partial y}{\partial r} = \frac{dy}{du} \frac{\partial u}{\partial r}$$



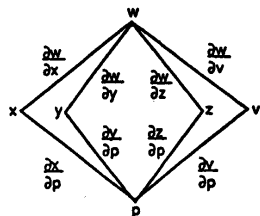
$$21. \frac{\partial w}{\partial s} = \frac{dw}{du} \frac{\partial u}{\partial s}$$



$$\frac{\partial w}{\partial t} = \frac{dw}{du} \frac{\partial u}{\partial t}$$

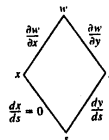
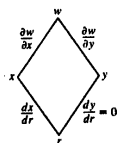


$$22. \frac{\partial w}{\partial p} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p}$$

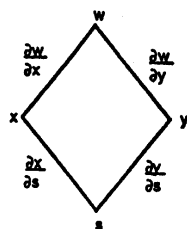


$$23. \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{dx}{dr} + \frac{\partial w}{\partial y} \frac{dy}{dr} = \frac{\partial w}{\partial x} \frac{dx}{dr} \text{ since } \frac{dy}{dr} = 0$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds} = \frac{\partial w}{\partial y} \frac{dy}{ds} \text{ since } \frac{dx}{ds} = 0$$



$$24. \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$



$$25. \text{ Let } F(x, y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x, y) = 3x^2 + y$$

$$\text{ and } F_y(x, y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 + y}{(-4y + x)}$$

$$\Rightarrow \frac{dy}{dx}(1, 1) = \frac{4}{3}$$

$$26. \text{ Let } F(x, y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x, y) = y - 3 \text{ and } F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y - 3}{x + 2y}$$

$$\Rightarrow \frac{dy}{dx}(-1, 1) = 2$$

$$27. \text{ Let } F(x, y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x, y) = 2x + y \text{ and } F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x + y}{x + 2y}$$

$$\Rightarrow \frac{dy}{dx}(1, 2) = -\frac{4}{5}$$

28. Let $F(x, y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x, y) = e^y + y \cos xy$ and $F_y(x, y) = xe^y + x \cos xy + 1$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y \cos xy}{xe^y + x \cos xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$$

29. Let $F(x, y, z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x, y, z) = -y$, $F_y(x, y, z) = -x + z + 3y^2$, $F_z(x, y, z) = 3z^2 + y$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2 + y} = \frac{y}{3z^2 + y} \Rightarrow \frac{\partial z}{\partial x}(1, 1, 1) = \frac{1}{4}; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x + z + 3y^2}{3z^2 + y} = \frac{x - z - 3y^2}{3z^2 + y}$$

$$\Rightarrow \frac{\partial z}{\partial y}(1, 1, 1) = -\frac{3}{4}$$

30. Let $F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x, y, z) = -\frac{1}{x^2}$, $F_y(x, y, z) = -\frac{1}{y^2}$, $F_z(x, y, z) = -\frac{1}{z^2}$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2, 3, 6) = -9; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2, 3, 6) = -4$$

31. Let $F(x, y, z) = \sin(x + y) + \sin(y + z) + \sin(z + x) = 0 \Rightarrow F_x(x, y, z) = \cos(x + y) + \cos(x + z)$,

$$F_y(x, y, z) = \cos(x + y) + \cos(y + z), F_z(x, y, z) = \cos(y + z) + \cos(x + z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{\cos(x + y) + \cos(x + z)}{\cos(y + z) + \cos(x + z)} \Rightarrow \frac{\partial z}{\partial x}(\pi, \pi, \pi) = -1; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x + y) + \cos(y + z)}{\cos(y + z) + \cos(x + z)} \Rightarrow \frac{\partial z}{\partial y}(\pi, \pi, \pi) = -1$$

32. Let $F(x, y, z) = xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2 = 0 \Rightarrow F_x(x, y, z) = e^y + \frac{2}{x}$, $F_y(x, y, z) = xe^y + e^z$, $F_z(x, y, z) = ye^z$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{4}{3 \ln 2}; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y + e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{5}{3 \ln 2}$$

33. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = 2(x + y + z)(1) + 2(x + y + z)[- \sin(r + s)] + 2(x + y + z)[\cos(r + s)]$

$$= 2(x + y + z)[1 - \sin(r + s) + \cos(r + s)] = 2[r - s + \cos(r + s) + \sin(r + s)][1 - \sin(r + s) + \cos(r + s)]$$

$$\Rightarrow \frac{\partial w}{\partial r} \Big|_{r=1, s=-1} = 2(3)(2) = 12$$

34. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \left(\frac{1}{z}\right)(0) = (u + v)\left(\frac{2v}{u}\right) + \frac{v^2}{u} \Rightarrow \frac{\partial w}{\partial v} \Big|_{u=-1, v=2} = (1)\left(\frac{4}{-1}\right) + \left(\frac{4}{-1}\right)$

$$= -8$$

35. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) = \left[2(u - 2v + 1) - \frac{2u + v - 2}{(u - 2v + 1)^2}\right](-2) + \frac{1}{u - 2v + 1}$

$$\Rightarrow \frac{\partial w}{\partial v} \Big|_{u=0, v=0} = -7$$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (y \cos xy + \sin y)(2u) + (x \cos xy + x \cos y)(v)$

$$= [uv \cos(u^3v + uv^3) + \sin uv](2u) + [(u^2 + v^2) \cos(u^3v + uv^3) + (u^2 + v^2) \cos uv](v)$$

$$\Rightarrow \frac{\partial z}{\partial u} \Big|_{u=0, v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$$

$$37. \frac{\partial z}{\partial u} = \frac{dz}{dx} \frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2} \right) e^u = \left[\frac{5}{1+(e^u + \ln v)^2} \right] e^u \Rightarrow \frac{\partial z}{\partial u} \Big|_{u=\ln 2, v=1} = \left[\frac{5}{1+(2)^2} \right] (2) = 2;$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx} \frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2} \right) \left(\frac{1}{v} \right) = \left[\frac{5}{1+(e^u + \ln v)^2} \right] \left(\frac{1}{v} \right) \Rightarrow \frac{\partial z}{\partial v} \Big|_{u=\ln 2, v=1} = \left[\frac{5}{1+(2)^2} \right] (1) = 1$$

$$38. \frac{\partial z}{\partial u} = \frac{dz}{dq} \frac{\partial q}{\partial u} = \left(\frac{1}{q} \right) \left(\frac{\sqrt{v+3}}{1+u^2} \right) = \left(\frac{1}{\sqrt{v+3} \tan^{-1} u} \right) \left(\frac{\sqrt{v+3}}{1+u^2} \right) = \frac{1}{(\tan^{-1} u)(1+u^2)}$$

$$\Rightarrow \frac{\partial z}{\partial u} \Big|_{u=1, v=-2} = \frac{1}{(\tan^{-1} 1)(1+1^2)} = \frac{2}{\pi}; \quad \frac{\partial z}{\partial v} = \frac{dz}{dq} \frac{\partial q}{\partial v} = \left(\frac{1}{q} \right) \left(\frac{\tan^{-1} u}{2\sqrt{v+3}} \right)$$

$$= \left(\frac{1}{\sqrt{v+3} \tan^{-1} u} \right) \left(\frac{\tan^{-1} u}{2\sqrt{v+3}} \right) = \frac{1}{2(v+3)} \Rightarrow \frac{\partial z}{\partial v} \Big|_{u=1, v=-2} = \frac{1}{2}$$

$$39. V = IR \Rightarrow \frac{\partial V}{\partial I} = R \text{ and } \frac{\partial V}{\partial R} = I; \quad \frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt} \Rightarrow -0.01 \text{ volts/sec}$$

$$= (600 \text{ ohms}) \frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005 \text{ amps/sec}$$

$$40. V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a} \frac{da}{dt} + \frac{\partial V}{\partial b} \frac{db}{dt} + \frac{\partial V}{\partial c} \frac{dc}{dt} = (bc) \frac{da}{dt} + (ac) \frac{db}{dt} + (ab) \frac{dc}{dt}$$

$$\Rightarrow \frac{dV}{dt} \Big|_{a=1, b=2, c=3} = (2 \text{ m})(3 \text{ m})(1 \text{ m/sec}) + (1 \text{ m})(3 \text{ m})(1 \text{ m/sec}) + (1 \text{ m})(2 \text{ m})(-3 \text{ m/sec}) = 3 \text{ m}^3/\text{sec}$$

$$\text{and the volume is increasing; } S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a} \frac{da}{dt} + \frac{\partial S}{\partial b} \frac{db}{dt} + \frac{\partial S}{\partial c} \frac{dc}{dt}$$

$$= 2(b+c) \frac{da}{dt} + 2(a+c) \frac{db}{dt} + 2(a+b) \frac{dc}{dt} \Rightarrow \frac{dS}{dt} \Big|_{a=1, b=2, c=3}$$

$$= 2(5 \text{ m})(1 \text{ m/sec}) + 2(4 \text{ m})(1 \text{ m/sec}) + 2(3 \text{ m})(-3 \text{ m/sec}) = 0 \text{ m}^2/\text{sec} \text{ and the surface area is not changing;}$$

$$D = \sqrt{a^2 + b^2 + c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a} \frac{da}{dt} + \frac{\partial D}{\partial b} \frac{db}{dt} + \frac{\partial D}{\partial c} \frac{dc}{dt} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left(a \frac{da}{dt} + b \frac{db}{dt} + c \frac{dc}{dt} \right) \Rightarrow \frac{dD}{dt} \Big|_{a=1, b=2, c=3}$$

$$= \left(\frac{1}{\sqrt{14} \text{ m}} \right) [(1 \text{ m})(1 \text{ m/sec}) + (2 \text{ m})(1 \text{ m/sec}) + (3 \text{ m})(-3 \text{ m/sec})] = -\frac{6}{\sqrt{14}} \text{ m/sec} < 0 \Rightarrow \text{the diagonals are}$$

decreasing in length

$$41. \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} (1) + \frac{\partial f}{\partial v} (0) + \frac{\partial f}{\partial w} (-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} (-1) + \frac{\partial f}{\partial v} (1) + \frac{\partial f}{\partial w} (0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}, \text{ and}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial z} = \frac{\partial f}{\partial u} (0) + \frac{\partial f}{\partial v} (-1) + \frac{\partial f}{\partial w} (1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

$$42. (a) \frac{\partial w}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = f_x \cos \theta + f_y \sin \theta \text{ and } \frac{\partial w}{\partial \theta} = f_x (-r \sin \theta) + f_y (r \cos \theta) \Rightarrow \frac{1}{r} \frac{\partial w}{\partial \theta} = -f_x \sin \theta + f_y \cos \theta$$

$$(b) \frac{\partial w}{\partial r} \sin \theta = f_x \sin \theta \cos \theta + f_y \sin^2 \theta \text{ and } \left(\frac{\cos \theta}{r} \right) \frac{\partial w}{\partial \theta} = -f_x \sin \theta \cos \theta + f_y \cos^2 \theta$$

$$\Rightarrow f_y = (\sin \theta) \frac{\partial w}{\partial r} + \left(\frac{\cos \theta}{r} \right) \frac{\partial w}{\partial \theta}; \text{ then } \frac{\partial w}{\partial r} = f_x \cos \theta + \left[(\sin \theta) \frac{\partial w}{\partial r} + \left(\frac{\cos \theta}{r} \right) \frac{\partial w}{\partial \theta} \right] (\sin \theta) \Rightarrow f_x \cos \theta$$

$$= \frac{\partial w}{\partial r} - (\sin^2 \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta \cos \theta}{r} \right) \frac{\partial w}{\partial \theta} = (1 - \sin^2 \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta \cos \theta}{r} \right) \frac{\partial w}{\partial \theta} \Rightarrow f_x = (\cos \theta) \frac{\partial w}{\partial r} - \left(\frac{\sin \theta}{r} \right) \frac{\partial w}{\partial \theta}$$

- (c) $(f_x)^2 = (\cos^2 \theta) \left(\frac{\partial w}{\partial r} \right)^2 - \left(\frac{2 \sin \theta \cos \theta}{r} \right) \left(\frac{\partial w}{\partial r} \frac{\partial w}{\partial \theta} \right) + \left(\frac{\sin^2 \theta}{r^2} \right) \left(\frac{\partial w}{\partial \theta} \right)^2$ and
- $$(f_y)^2 = (\sin^2 \theta) \left(\frac{\partial w}{\partial r} \right)^2 + \left(\frac{2 \sin \theta \cos \theta}{r} \right) \left(\frac{\partial w}{\partial r} \frac{\partial w}{\partial \theta} \right) + \left(\frac{\cos^2 \theta}{r^2} \right) \left(\frac{\partial w}{\partial \theta} \right)^2 \Rightarrow (f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2$$
43. $w_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v} \Rightarrow w_{xx} = \frac{\partial w}{\partial u} + x \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial u} \right) + y \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial v} \right)$
- $$= \frac{\partial w}{\partial u} + x \left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial x} \right) + y \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial x} \right) = \frac{\partial w}{\partial u} + x \left(x \frac{\partial^2 w}{\partial u^2} + y \frac{\partial^2 w}{\partial v \partial u} \right) + y \left(x \frac{\partial^2 w}{\partial u \partial v} + y \frac{\partial^2 w}{\partial v^2} \right)$$
- $$= \frac{\partial w}{\partial u} + x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial v \partial u} + y^2 \frac{\partial^2 w}{\partial v^2}; w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = -y \frac{\partial w}{\partial u} + x \frac{\partial w}{\partial v}$$
- $$\Rightarrow w_{yy} = -\frac{\partial w}{\partial u} - y \left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial y} \right) + x \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial y} \right)$$
- $$= -\frac{\partial w}{\partial u} - y \left(-y \frac{\partial^2 w}{\partial u^2} + x \frac{\partial^2 w}{\partial v \partial u} \right) + x \left(-y \frac{\partial^2 w}{\partial u \partial v} + x \frac{\partial^2 w}{\partial v^2} \right) = -\frac{\partial w}{\partial u} + y^2 \frac{\partial^2 w}{\partial u^2} - 2xy \frac{\partial^2 w}{\partial v \partial u} + x^2 \frac{\partial^2 w}{\partial v^2}; \text{ thus}$$
- $$w_{xx} + w_{yy} = (x^2 + y^2) \frac{\partial^2 w}{\partial u^2} + (x^2 + y^2) \frac{\partial^2 w}{\partial v^2} = (x^2 + y^2)(w_{uu} + w_{vv}) = 0, \text{ since } w_{uu} + w_{vv} = 0$$
44. $\frac{\partial w}{\partial x} = f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow w_{xx} = f''(u)(1) + g''(v)(1) = f''(u) + g''(v);$
- $$\frac{\partial w}{\partial y} = f'(u)(i) + g'(v)(-i) \Rightarrow w_{yy} = f''(u)(i^2) + g''(v)(i^2) = -f''(u) - g''(v) \Rightarrow w_{xx} + w_{yy} = 0$$
45. $f_x(x, y, z) = \cos t$, $f_y(x, y, z) = \sin t$, and $f_z(x, y, z) = t^2 + t - 2 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$
- $$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2 + t - 2)(1) = t^2 + t - 2; \frac{df}{dt} = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$$
- or $t = 1$; $t = -2 \Rightarrow x = \cos(-2)$, $y = \sin(-2)$, $z = -2$ for the point $(\cos(-2), \sin(-2), -2)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, $z = 1$ for the point $(\cos 1, \sin 1, 1)$
46. $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (2xe^{2y} \cos 3z)(-\sin t) + (2x^2e^{2y} \cos 3z) \left(\frac{1}{t+2} \right) + (-3x^2e^{2y} \sin 3z)(1)$
- $$= -2xe^{2y} \cos 3z \sin t + \frac{2x^2e^{2y} \cos 3z}{t+2} - 3x^2e^{2y} \sin 3z; \text{ at the point on the curve } z = 0 \Rightarrow t = z = 0$$
- $$\Rightarrow \left. \frac{dw}{dt} \right|_{(1, \ln 2, 0)} = 0 + \frac{2(1)^2(4)(1)}{2} - 0 = 4$$
47. (a) $\frac{\partial T}{\partial x} = 8x - 4y$ and $\frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = (8x - 4y)(-\sin t) + (8y - 4x)(\cos t)$
- $$= (8 \cos t - 4 \sin t)(-\sin t) + (8 \sin t - 4 \cos t)(\cos t) = 4 \sin^2 t - 4 \cos^2 t \Rightarrow \frac{d^2 T}{dt^2} = 16 \sin t \cos t;$$
- $$\frac{dT}{dt} = 0 \Rightarrow 4 \sin^2 t - 4 \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t \text{ or } \sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \text{ on the interval } 0 \leq t \leq 2\pi;$$
- $$\left. \frac{d^2 T}{dt^2} \right|_{t=\frac{\pi}{4}} = 16 \sin \frac{\pi}{4} \cos \frac{\pi}{4} > 0 \Rightarrow T \text{ has a minimum at } (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{3\pi}{4}} = 16 \sin \frac{3\pi}{4} \cos \frac{3\pi}{4} < 0 \Rightarrow T \text{ has a maximum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{5\pi}{4}} = 16 \sin \frac{5\pi}{4} \cos \frac{5\pi}{4} > 0 \Rightarrow T \text{ has a minimum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{7\pi}{4}} = 16 \sin \frac{7\pi}{4} \cos \frac{7\pi}{4} < 0 \Rightarrow T \text{ has a maximum at } (x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

- (b) $T = 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y$, and $\frac{\partial T}{\partial y} = 8y - 4x$ so the extreme values occur at the four points

$$\text{found in part (a): } T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 6, \text{ the maximum and}$$

$$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 2, \text{ the minimum}$$

48. (a) $\frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = y(-2\sqrt{2} \sin t) + x(\sqrt{2} \cos t)$
 $= (\sqrt{2} \sin t)(-2\sqrt{2} \sin t) + (2\sqrt{2} \cos t)(\sqrt{2} \cos t) = -4 \sin^2 t + 4 \cos^2 t = -4 \sin^2 t + 4(1 - \sin^2 t)$
 $= 4 - 8 \sin^2 t \Rightarrow \frac{d^2T}{dt^2} = -16 \sin t \cos t$; $\frac{dT}{dt} = 0 \Rightarrow 4 - 8 \sin^2 t = 0 \Rightarrow \sin^2 t = \frac{1}{2} \Rightarrow \sin t = \pm \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4},$

$$\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \text{ on the interval } 0 \leq t \leq 2\pi;$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{\pi}{4}} = -8 \sin 2\left(\frac{\pi}{4}\right) = -8 \Rightarrow T \text{ has a maximum at } (x, y) = (2, 1);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{3\pi}{4}} = -8 \sin 2\left(\frac{3\pi}{4}\right) = 8 \Rightarrow T \text{ has a minimum at } (x, y) = (-2, 1);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{5\pi}{4}} = -8 \sin 2\left(\frac{5\pi}{4}\right) = -8 \Rightarrow T \text{ has a maximum at } (x, y) = (-2, -1);$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{7\pi}{4}} = -8 \sin 2\left(\frac{7\pi}{4}\right) = 8 \Rightarrow T \text{ has a minimum at } (x, y) = (2, -1)$$

- (b) $T = xy - 2 \Rightarrow \frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x$ so the extreme values occur at the four points found in part (a):

$$T(2, 1) = T(-2, -1) = 0, \text{ the maximum and } T(-2, 1) = T(2, -1) = -4, \text{ the minimum}$$

49. $G(u, x) = \int_a^u g(t, x) dt$ where $u = f(x) \Rightarrow \frac{dG}{dx} = \frac{\partial G}{\partial u} \frac{du}{dx} + \frac{\partial G}{\partial x} \frac{dx}{dx} = g(u, x)f'(x) + \int_a^u g_x(t, x) dt$; thus

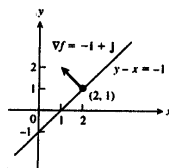
$$F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} dt \Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3} (2x) + \int_0^{x^2} \frac{\partial}{\partial x} \sqrt{t^4 + x^3} dt = 2x\sqrt{x^8 + x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}} dt$$

50. Using the result in Exercise 49, $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2} dt = - \int_1^{x^2} \sqrt{t^3 + x^2} dt \Rightarrow F'(x)$

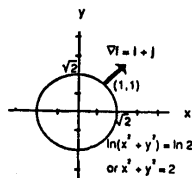
$$= -\sqrt{(x^2)^3 + x^2} (2x) - \int_1^{x^2} \frac{\partial}{\partial x} \sqrt{t^3 + x^2} dt = \int_{x^2}^1 \frac{x}{\sqrt{t^3 + x^2}} dt - 2x\sqrt{x^6 + x^2}$$

11.5 DIRECTIONAL DERIVATIVES, GRADIENT VECTORS, AND TANGENT PLANES

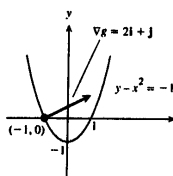
1. $\frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f = -\mathbf{i} + \mathbf{j}; f(2, 1) = -1$
 $\Rightarrow -1 = y - x$ is the level curve



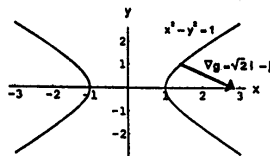
2. $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 1; \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$
 $\Rightarrow \frac{\partial f}{\partial y}(1, 1) = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j}; f(1, 1) = \ln 2 \Rightarrow \ln 2$
 $= \ln(x^2 + y^2) \Rightarrow 2 = x^2 + y^2$ is the level curve



3. $\frac{\partial g}{\partial x} = -2x \Rightarrow \frac{\partial g}{\partial x}(-1, 0) = 2; \frac{\partial g}{\partial y} = 1$
 $\Rightarrow \nabla g = 2\mathbf{i} + \mathbf{j}; g(-1, 0) = -1$
 $\Rightarrow -1 = y - x^2$ is the level curve



4. $\frac{\partial g}{\partial x} = x \Rightarrow \frac{\partial g}{\partial x}(\sqrt{2}, 1) = \sqrt{2}; \frac{\partial g}{\partial y} = -y$
 $\Rightarrow \frac{\partial g}{\partial y}(\sqrt{2}, 1) = -1 \Rightarrow \nabla g = \sqrt{2}\mathbf{i} - \mathbf{j}; g(\sqrt{2}, 1) = \frac{1}{2}$
 $\Rightarrow \frac{1}{2} = \frac{x^2}{2} - \frac{y^2}{2}$ or $1 = x^2 - y^2$ is the level curve

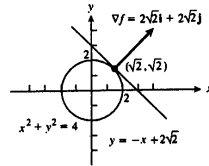


5. $\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \Rightarrow \frac{\partial f}{\partial x}(1, 1, 1) = 3$; $\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1, 1, 1) = 2$; $\frac{\partial f}{\partial z} = -4z + \ln x \Rightarrow \frac{\partial f}{\partial z}(1, 1, 1) = -4$;
thus $\nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
6. $\frac{\partial f}{\partial x} = -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\partial f}{\partial x}(1, 1, 1) = -\frac{11}{2}$; $\frac{\partial f}{\partial y} = -6yz \Rightarrow \frac{\partial f}{\partial y}(1, 1, 1) = -6$; $\frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2) + \frac{x}{x^2z^2+1}$
 $\Rightarrow \frac{\partial f}{\partial z}(1, 1, 1) = \frac{1}{2}$; thus $\nabla f = -\frac{11}{2}\mathbf{i} - 6\mathbf{j} + \frac{1}{2}\mathbf{k}$
7. $\frac{\partial f}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x}(-1, 2, -2) = -\frac{26}{27}$; $\frac{\partial f}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(-1, 2, -2) = \frac{23}{54}$;
 $\frac{\partial f}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\partial f}{\partial z}(-1, 2, -2) = -\frac{23}{54}$; thus $\nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}$
8. $\frac{\partial f}{\partial x} = e^{x+y} \cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\partial f}{\partial x}(0, 0, \frac{\pi}{6}) = \frac{\sqrt{3}}{2} + 1$; $\frac{\partial f}{\partial y} = e^{x+y} \cos z + \sin^{-1} x \Rightarrow \frac{\partial f}{\partial y}(0, 0, \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$;
 $\frac{\partial f}{\partial z} = -e^{x+y} \sin z \Rightarrow \frac{\partial f}{\partial z}(0, 0, \frac{\pi}{6}) = -\frac{1}{2}$; thus $\nabla f = \left(\frac{\sqrt{3}+2}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$
9. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{4\mathbf{i} + 3\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$; $f_x(x, y) = 2y \Rightarrow f_x(5, 5) = 10$; $f_y(x, y) = 2x - 6y \Rightarrow f_y(5, 5) = -20$
 $\Rightarrow \nabla f = 10\mathbf{i} - 20\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) = -4$
10. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$; $f_x(x, y) = 4x \Rightarrow f_x(-1, 1) = -4$; $f_y(x, y) = 2y \Rightarrow f_y(-1, 1) = 2$
 $\Rightarrow \nabla f = -4\mathbf{i} + 2\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4$
11. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{12\mathbf{i} + 5\mathbf{j}}{\sqrt{12^2 + 5^2}} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$; $g_x(x, y) = 1 + \frac{y^2}{x^2} + \frac{2y\sqrt{3}}{2xy\sqrt{4x^2y^2-1}} \Rightarrow g_x(1, 1) = 3$; $g_y(x, y)$
 $= -\frac{2y}{x} + \frac{2x\sqrt{3}}{2xy\sqrt{4x^2y^2-1}} \Rightarrow g_y(1, 1) = -1 \Rightarrow \nabla g = 3\mathbf{i} - \mathbf{j} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = \frac{36}{13} - \frac{5}{13} = \frac{31}{13}$
12. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 2\mathbf{j}}{\sqrt{3^2 + (-2)^2}} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$; $h_x(x, y) = \frac{\left(\frac{-y}{x^2}\right)}{\left(\frac{y}{x}\right)^2 + 1} + \frac{\left(\frac{y}{2}\right)\sqrt{3}}{\sqrt{1 - \left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_x(1, 1) = \frac{1}{2}$;
 $h_y(x, y) = \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1} + \frac{\left(\frac{x}{2}\right)\sqrt{3}}{\sqrt{1 - \left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_y(1, 1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}}$
 $= -\frac{3}{2\sqrt{13}}$
13. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x, y, z) = y + z \Rightarrow f_x(1, -1, 2) = 1$; $f_y(x, y, z) = x + z$

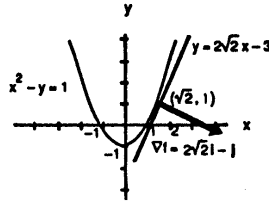
- $\Rightarrow f_y(1, -1, 2) = 3; f_z(x, y, z) = y + x \Rightarrow f_z(1, -1, 2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3$
14. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}; f_x(x, y, z) = 2x \Rightarrow f_x(1, 1, 1) = 2; f_y(x, y, z) = 4y$
 $\Rightarrow f_y(1, 1, 1) = 4; f_z(x, y, z) = -6z \Rightarrow f_z(1, 1, 1) = -6 \Rightarrow \nabla f = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$
 $= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) - 6\left(\frac{1}{\sqrt{3}}\right) = 0$
15. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}; g_x(x, y, z) = 3e^x \cos yz \Rightarrow g_x(0, 0, 0) = 3; g_y(x, y, z) = -3ze^x \sin yz$
 $\Rightarrow g_y(0, 0, 0) = 0; g_z(x, y, z) = -3ye^x \sin yz \Rightarrow g_z(0, 0, 0) = 0 \Rightarrow \nabla g = 3\mathbf{i} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = 2$
16. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}; h_x(x, y, z) = -y \sin xy + \frac{1}{x} \Rightarrow h_x\left(1, 0, \frac{1}{2}\right) = 1;$
 $h_y(x, y, z) = -x \sin xy + ze^{yz} \Rightarrow h_y\left(1, 0, \frac{1}{2}\right) = \frac{1}{2}; h_z(x, y, z) = ye^{yz} + \frac{1}{z} \Rightarrow h_z\left(1, 0, \frac{1}{2}\right) = 2 \Rightarrow \nabla h = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = 2$
17. $\nabla f = (2x + y)\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(-1, 1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{(-1)^2 + 1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}; f$ increases
 most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j};$
 $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$
18. $\nabla f = (2xy + ye^{xy} \sin y)\mathbf{i} + (x^2 + xe^{xy} \sin y + e^{xy} \cos y)\mathbf{j} \Rightarrow \nabla f(1, 0) = 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \mathbf{j}; f$ increases most
 rapidly in the direction $\mathbf{u} = \mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\mathbf{j}; (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f|$
 $= 2$ and $(D_{-\mathbf{u}}f)_{P_0} = -2$
19. $\nabla f = \frac{1}{y}\mathbf{i} - \left(\frac{x}{y^2} + z\right)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla f(4, 1, 1) = \mathbf{i} - 5\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{i} - 5\mathbf{j} - \mathbf{k}}{\sqrt{1^2 + (-5)^2 + (-1)^2}}$
 $= \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}; f$ increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$ and decreases
 most rapidly in the direction $-\mathbf{u} = -\frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}; (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 3\sqrt{3}$ and
 $(D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$
20. $\nabla g = e^y\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla g\left(1, \ln 2, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k};$
 g increases most rapidly in the direction $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and decreases most rapidly in the direction
 $-\mathbf{u} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}; (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = |\nabla g| = 3$ and $(D_{-\mathbf{u}}g)_{P_0} = -3$

21. $\nabla f = \left(\frac{1}{x} + \frac{1}{x}\right)\mathbf{i} + \left(\frac{1}{y} + \frac{1}{y}\right)\mathbf{j} + \left(\frac{1}{z} + \frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k};$
 f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}; (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$
22. $\nabla h = \left(\frac{2x}{x^2 + y^2 - 1}\right)\mathbf{i} + \left(\frac{2x}{x^2 + y^2 - 1}\right)\mathbf{j} + 6\mathbf{k} \Rightarrow \nabla h(1, 1, 0) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}}$
 $= \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}; h$ increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}; (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = |\nabla h| = 7$ and $(D_{-\mathbf{u}}h)_{P_0} = -7$
23. $\nabla f = \left(\frac{x}{x^2 + y^2 + z^2}\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2 + z^2}\right)\mathbf{j} + \left(\frac{z}{x^2 + y^2 + z^2}\right)\mathbf{k} \Rightarrow \nabla f(3, 4, 12) = \frac{3}{169}\mathbf{i} + \frac{4}{169}\mathbf{j} + \frac{12}{169}\mathbf{k};$
 $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{9}{1183}$ and $df = (\nabla f \cdot \mathbf{u}) ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$
24. $\nabla f = (e^x \cos yz)\mathbf{i} - (ze^x \sin yz)\mathbf{j} - (ye^x \sin yz)\mathbf{k} \Rightarrow \nabla f(0, 0, 0) = \mathbf{i}; \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + 2^2 + (-2)^2}}$
 $= \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and $df = (\nabla f \cdot \mathbf{u}) ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$
25. $\nabla g = (1 + \cos z)\mathbf{i} + (1 - \sin z)\mathbf{j} + (-x \sin z - y \cos z)\mathbf{k} \Rightarrow \nabla g(2, -1, 0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}; \mathbf{v} = P_0\vec{P}_1 = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
 $\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(-2)^2 + 2^2 + 2^2}} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and $dg = (\nabla g \cdot \mathbf{u}) ds = (0)(0.2) = 0$
26. $\nabla h = [-\pi y \sin(\pi xy) + z^2]\mathbf{i} - [\pi x \sin(\pi xy)]\mathbf{j} + 2xz\mathbf{k} \Rightarrow \nabla h(-1, -1, -1) = (\pi \sin \pi + 1)\mathbf{i} + (\pi \sin \pi)\mathbf{j} + 2\mathbf{k}$
 $= \mathbf{i} + 2\mathbf{k}; \mathbf{v} = P_0\vec{P}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0, 0, 0) \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
 $\Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $dh = (\nabla h \cdot \mathbf{u}) ds = \sqrt{3}(0.1) \approx 0.1732$
27. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane: $2(x-1) + 2(y-1) + 2(z-1) = 0$
 $\Rightarrow x + y + z = 3;$
 (b) Normal line: $x = 1 + 2t, y = 1 + 2t, z = 1 + 2t$
28. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(3, 5, -4) = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \Rightarrow$ Tangent plane: $6(x-3) + 10(y-5) + 8(z+4) = 0$
 $\Rightarrow 3x + 5y + 4z = 18;$
 (b) Normal line: $x = 3 + 6t, y = 5 + 10t, z = -4 + 8t$
29. (a) $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \Rightarrow \nabla f(2, 0, 2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane: $-4(x-2) + 2(z-2) = 0 \Rightarrow -4x + 2z + 4 = 0$
 $\Rightarrow -2x + z + 2 = 0;$
 (b) Normal line: $x = 2 - 4t, y = 0, z = 2 + 2t$
30. (a) $\nabla f = (2x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1, -1, 3) = 4\mathbf{j} + 6\mathbf{k} \Rightarrow$ Tangent plane: $4(y+1) + 6(z-3) = 0$
 $\Rightarrow 2y + 3z = 7;$
 (b) Normal line: $x = 1, y = -1 + 4t, z = 3 + 6t$

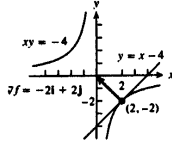
31. (a) $\nabla f = (-\pi \sin \pi x - 2xy + ze^{xz})\mathbf{i} + (-x^2 + z)\mathbf{j} + (xe^{xz} + y)\mathbf{k} \Rightarrow \nabla f(0, 1, 2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane:
 $2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0;$
 (b) Normal line: $x = 2t, y = 1 + 2t, z = 2 + t$
32. (a) $\nabla f = (2x - y)\mathbf{i} - (x + 2y)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, -1) = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
 $1(x-1) - 3(y-1) - 1(z+1) = 0 \Rightarrow x - 3y - z = -1;$
 (b) Normal line: $x = 1 + t, y = 1 - 3t, z = -1 - t$
33. (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0, 1, 0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0$
 $\Rightarrow x + y + z - 1 = 0;$
 (b) Normal line: $x = t, y = 1 + t, z = t$
34. (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2, -3, 18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
 $9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21;$
 (b) \Rightarrow Normal line: $x = 2 + 9t, y = -3 - 7t, z = 18 - t$
35. $z = f(x, y) = \ln(x^2 + y^2) \Rightarrow f_x(x, y) = \frac{2x}{x^2 + y^2}$ and $f_y(x, y) = \frac{2y}{x^2 + y^2} \Rightarrow f_x(1, 0) = 2$ and $f_y(1, 0) = 0 \Rightarrow$ from
 Eq. (9) the tangent plane at $(1, 0, 0)$ is $2(x-1) - z = 0$ or $2x - z - 2 = 0$
36. $z = f(x, y) = e^{-(x^2 + y^2)} \Rightarrow f_x(x, y) = -2xe^{-(x^2 + y^2)}$ and $f_y(x, y) = -2ye^{-(x^2 + y^2)} \Rightarrow f_x(0, 0) = 0$ and $f_y(0, 0) = 0$
 \Rightarrow from Eq. (9) the tangent plane at $(0, 0, 1)$ is $z - 1 = 0$ or $z = 1$
37. $z = f(x, y) = \sqrt{y-x} \Rightarrow f_x(x, y) = -\frac{1}{2}(y-x)^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(y-x)^{-1/2} \Rightarrow f_x(1, 2) = -\frac{1}{2}$ and $f_y(1, 2) = \frac{1}{2}$
 \Rightarrow from Eq. (9) the tangent plane at $(1, 2, 1)$ is $-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0 \Rightarrow x - y + 2z - 1 = 0$
38. $z = f(x, y) = 4x^2 + y^2 \Rightarrow f_x(x, y) = 8x$ and $f_y(x, y) = 2y \Rightarrow f_x(1, 1) = 8$ and $f_y(1, 1) = 2 \Rightarrow$ from Eq. (9) the
 tangent plane at $(1, 1, 5)$ is $8(x-1) + 2(y-1) - (z-5) = 0$ or $8x + 2y - z - 5 = 0$
39. $\nabla f = 2xi + 2yj \Rightarrow \nabla f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}$
 \Rightarrow Tangent line: $2\sqrt{2}(x - \sqrt{2}) + 2\sqrt{2}(y - \sqrt{2}) = 0$
 $\Rightarrow \sqrt{2}x + \sqrt{2}y = 4$



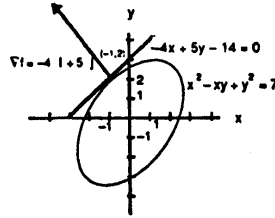
40. $\nabla f = 2xi - j \Rightarrow \nabla f(\sqrt{2}, 1) = 2\sqrt{2}\mathbf{i} - \mathbf{j}$
 \Rightarrow Tangent line: $2\sqrt{2}(x - \sqrt{2}) - (y - 1) = 0$
 $\Rightarrow y = 2\sqrt{2}x - 3$



41. $\nabla f = yi + xj \Rightarrow \nabla f(2, -2) = -2i + 2j$
 \Rightarrow Tangent line: $-2(x - 2) + 2(y + 2) = 0$
 $\Rightarrow y = x - 4$



42. $\nabla f = (2x - y)i + (2y - x)j \Rightarrow \nabla f(-1, 2) = -4i + 5j$
 \Rightarrow Tangent line: $-4(x + 1) + 5(y - 2) = 0$
 $\Rightarrow -4x + 5y - 14 = 0$



43. $\nabla f = i + 2yj + 2k \Rightarrow \nabla f(1, 1, 1) = i + 2j + 2k$ and $\nabla g = i$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1, y = 1 + 2t, z = 1 - 2t$$

44. $\nabla f = yzi + xzj + xyk \Rightarrow \nabla f(1, 1, 1) = i + j + k$; $\nabla g = 2xi + 4yj + 6zk \Rightarrow \nabla g(1, 1, 1) = 2i + 4j + 6k$;

$$\Rightarrow \mathbf{v} = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2i - 4j + 2k \Rightarrow \text{Tangent line: } x = 1 + 2t, y = 1 - 4t, z = 1 + 2t$$

45. $\nabla f = 2xi + 2j + 2k \Rightarrow \nabla f\left(1, 1, \frac{1}{2}\right) = 2i + 2j + 2k$ and $\nabla g = j$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2i + 2k \Rightarrow \text{Tangent line: } x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$$

46. $\nabla f = i + 2yj + k \Rightarrow \nabla f\left(\frac{1}{2}, 1, \frac{1}{2}\right) = i + 2j + k$ and $\nabla g = j$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -i + k \Rightarrow \text{Tangent line: } x = \frac{1}{2} - t, y = 1, z = \frac{1}{2} + t$$

$$47. \nabla f = (3x^2 + 6xy^2 + 4y)\mathbf{i} + (6x^2y + 3y^2 + 4x)\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 3) = 13\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}; \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\Rightarrow \nabla g(1, 1, 3) = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}; \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\mathbf{i} - 90\mathbf{j} \Rightarrow \text{Tangent line:}$$

$$x = 1 + 90t, y = 1 - 90t, z = 3$$

$$48. \nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}; \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g(\sqrt{2}, \sqrt{2}, 4)$$

$$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k}; \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} \Rightarrow \text{Tangent line:}$$

$$x = \sqrt{2} - 2\sqrt{2}t, y = \sqrt{2} + 2\sqrt{2}t, z = 4$$

$$49. \nabla f = y\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(3, 2) = 2\mathbf{i} + 7\mathbf{j}; \text{a vector orthogonal to } \nabla f \text{ is } \mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{7^2 + (-2)^2}}$$

$$= \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j} \text{ and } -\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j} \text{ are the directions where the derivative is zero}$$

$$50. \nabla f = \frac{4xy^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2}\mathbf{j} \Rightarrow \nabla f(1, 1) = \mathbf{i} - \mathbf{j}; \text{a vector orthogonal to } \nabla f \text{ is } \mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \text{ and } -\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} \text{ are the directions where the derivative is zero}$$

$$51. \nabla f = (2x - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j} \Rightarrow \nabla f(1, 2) = -4\mathbf{i} + 13\mathbf{j} \Rightarrow |\nabla f(1, 2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}; \text{no, the maximum rate of change is } \sqrt{185} < 14$$

$$52. \nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}; \text{no, the minimum rate of change is } -\sqrt{6} > -3$$

$$53. \nabla f = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j} \text{ and } \mathbf{u}_1 = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)(1, 2) = f_x(1, 2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1, 2)\left(\frac{1}{\sqrt{2}}\right) \\ = 2\sqrt{2} \Rightarrow f_x(1, 2) + f_y(1, 2) = 4; \mathbf{u}_2 = -\mathbf{j} \Rightarrow (D_{\mathbf{u}_2}f)(1, 2) = f_x(1, 2)(0) + f_y(1, 2)(-1) = -3 \Rightarrow -f_y(1, 2) = -3 \\ \Rightarrow f_y(1, 2) = 3; \text{then } f_x(1, 2) + 3 = 4 \Rightarrow f_x(1, 2) = 1; \text{thus } \nabla f(1, 2) = \mathbf{i} + 3\mathbf{j} \text{ and } \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{(-1)^2 + (-2)^2}} \\ = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

$$54. (a) (D_{\mathbf{u}}f)_P = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}; \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}; \text{thus } \mathbf{u} = \frac{\nabla f}{|\nabla f|}$$

$$\Rightarrow \nabla f = |\nabla f|\mathbf{u} \Rightarrow \nabla f = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

$$(b) \mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - 2(0) = 2\sqrt{2}$$

55. (a) The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;
- $$\nabla T = (\sin 2y)\mathbf{i} + (2x \cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (\sin \sqrt{3})\mathbf{i} + (\cos \sqrt{3})\mathbf{j} \Rightarrow D_{\mathbf{u}}T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u}$$
- $$= \frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3} \approx 0.935^\circ \text{ C/ft}$$
- (b) $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}(t) = (2 \cos 2t)\mathbf{i} - (2 \sin 2t)\mathbf{j}$ and $|\mathbf{v}| = 2$; $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}$
- $$= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) |\mathbf{v}| = (D_{\mathbf{u}}T) |\mathbf{v}|, \text{ where } \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}; \text{ at } \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ we have } \mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \text{ from part (a)}$$
- $$\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3} \right) \cdot 2 = \sqrt{3} \sin \sqrt{3} - \cos \sqrt{3} \approx 1.87^\circ \text{ C/sec}$$
56. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} = 2(\cos t + t \sin t)\mathbf{i} + 2(\sin t - t \cos t)\mathbf{j}$ and $\mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
- $$= \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{\sqrt{(t \cos t)^2 + (t \sin t)^2}} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \text{ since } t > 0 \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$$
- $$= 2(\cos t + t \sin t)(\cos t) + 2(\sin t - t \cos t)(\sin t) = 2$$
57. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
- $$= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{(\sin t)^2 + (\cos t)^2 + 1^2}} = \left(\frac{-\sin t}{\sqrt{2}} \right)\mathbf{i} + \left(\frac{\cos t}{\sqrt{2}} \right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$$
- $$= (2 \cos t) \left(\frac{-\sin t}{\sqrt{2}} \right) + (2 \sin t) \left(\frac{\cos t}{\sqrt{2}} \right) + (2t) \left(\frac{1}{\sqrt{2}} \right) = \frac{2t}{\sqrt{2}} \Rightarrow (D_{\mathbf{u}}f) \left(\frac{-\pi}{4} \right) = \frac{-\pi}{2\sqrt{2}}, (D_{\mathbf{u}}f)(0) = 0 \text{ and}$$
- $$(D_{\mathbf{u}}f) \left(\frac{\pi}{4} \right) = \frac{\pi}{2\sqrt{2}}$$
58. (a) $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8, 6, -4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}$; $\mathbf{r}(t) = 2t^2\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is at the point $P(8, 6, -4)$ when $t = 2$; $\mathbf{v}(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{v}(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
- $$= \frac{8}{\sqrt{89}}\mathbf{i} + \frac{3}{\sqrt{89}}\mathbf{j} - \frac{4}{\sqrt{89}}\mathbf{k} \Rightarrow D_{\mathbf{u}}T(8, 6, -4) = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{89}}[56 \cdot 8 + 32 \cdot 3 - 48 \cdot (-4)] = \frac{736}{\sqrt{89}}^\circ \text{ C/m}$$
- (b) $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = \nabla T \cdot \mathbf{v} = (\nabla T \cdot \mathbf{u}) |\mathbf{v}| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_{\mathbf{u}}T \Big|_{t=2} \cdot \mathbf{v}(2) = \left(\frac{736}{\sqrt{89}} \right) \sqrt{89} = 736^\circ \text{ C/sec}$
59. If (x, y) is a point on the line, then $\mathbf{T}(x, y) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}$ is a vector parallel to the line $\Rightarrow \mathbf{T} \cdot \mathbf{N} = 0 \Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.
60. (a) $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t+3)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = -1 \Rightarrow P_0 = (1, 1, -1)$
- and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $f(x, y, z) = x^2 + y^2 - z - 3 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$;
- therefore $\mathbf{v} = \frac{1}{4}(\nabla f) \Rightarrow$ the curve is normal to the surface
- (b) $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t-1)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} + 2\mathbf{k}$; $t = 1 \rightarrow x = 1, y = 1, z = 1 \Rightarrow P_0 = (1, 1, 1)$ and
- $$\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}; f(x, y, z) = x^2 + y^2 - z - 1 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k};$$

therefore $\mathbf{v} \cdot \nabla f = \frac{1}{2}(2) + \frac{1}{2}(2) + 2(-1) = 0 \Rightarrow$ the curve is tangent to the surface when $t = 1$

61. The directional derivative is the scalar component. With ∇f evaluated at P_0 , the scalar component of ∇f in the direction of \mathbf{u} is $\nabla f \cdot \mathbf{u} = (D_{\mathbf{u}}f)_{P_0}$.

62. $D_{\mathbf{i}}f = \nabla f \cdot \mathbf{i} = (f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}) \cdot \mathbf{i} = f_x$; similarly, $D_{\mathbf{j}}f = \nabla f \cdot \mathbf{j} = f_y$ and $D_{\mathbf{k}}f = \nabla f \cdot \mathbf{k} = f_z$

$$63. (a) \nabla(kf) = \frac{\partial(kf)}{\partial x} \mathbf{i} + \frac{\partial(kf)}{\partial y} \mathbf{j} + \frac{\partial(kf)}{\partial z} \mathbf{k} = k \left(\frac{\partial f}{\partial x} \right) \mathbf{i} + k \left(\frac{\partial f}{\partial y} \right) \mathbf{j} + k \left(\frac{\partial f}{\partial z} \right) \mathbf{k} = k \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) = k \nabla f$$

$$(b) \nabla(f+g) = \frac{\partial(f+g)}{\partial x} \mathbf{i} + \frac{\partial(f+g)}{\partial y} \mathbf{j} + \frac{\partial(f+g)}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right) \mathbf{k} \\ = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} + \frac{\partial g}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) + \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) = \nabla f + \nabla g$$

(c) $\nabla(f-g) = \nabla f - \nabla g$ (Substitute $-g$ for g in part (b) above)

$$(d) \nabla(fg) = \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} + \frac{\partial(fg)}{\partial z} \mathbf{k} = \left(\frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} g + \frac{\partial g}{\partial y} f \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} g + \frac{\partial g}{\partial z} f \right) \mathbf{k} \\ = \left(\frac{\partial f}{\partial x} g \right) \mathbf{i} + \left(\frac{\partial g}{\partial x} f \right) \mathbf{i} + \left(\frac{\partial f}{\partial y} g \right) \mathbf{j} + \left(\frac{\partial g}{\partial y} f \right) \mathbf{j} + \left(\frac{\partial f}{\partial z} g \right) \mathbf{k} + \left(\frac{\partial g}{\partial z} f \right) \mathbf{k} \\ = f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) + g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) = f \nabla g + g \nabla f$$

$$(e) \nabla \left(\frac{f}{g} \right) = \frac{\partial \left(\frac{f}{g} \right)}{\partial x} \mathbf{i} + \frac{\partial \left(\frac{f}{g} \right)}{\partial y} \mathbf{j} + \frac{\partial \left(\frac{f}{g} \right)}{\partial z} \mathbf{k} = \left(\frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} \right) \mathbf{i} + \left(\frac{g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}}{g^2} \right) \mathbf{j} + \left(\frac{g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z}}{g^2} \right) \mathbf{k} \\ = \left(\frac{g \frac{\partial f}{\partial x} \mathbf{i} + g \frac{\partial f}{\partial y} \mathbf{j} + g \frac{\partial f}{\partial z} \mathbf{k}}{g^2} \right) - \left(\frac{f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k}}{g^2} \right) = \frac{g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right)}{g^2} - \frac{f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right)}{g^2} \\ = \frac{g \nabla f}{g^2} - \frac{f \nabla g}{g^2} = \frac{g \nabla f - f \nabla g}{g^2}$$

11.6 LINEARIZATION AND DIFFERENTIALS

1. (a) $f(0,0) = 1$, $f_x(x,y) = 2x \Rightarrow f_x(0,0) = 0$, $f_y(x,y) = 2y \Rightarrow f_y(0,0) = 0 \Rightarrow L(x,y) = 1 + 0(x-0) + 0(y-0) = 1$

(b) $f(1,1) = 3$, $f_x(1,1) = 2$, $f_y(1,1) = 2 \Rightarrow L(x,y) = 3 + 2(x-1) + 2(y-1) = 2x + 2y - 1$

2. (a) $f(0,0) = 4$, $f_x(x,y) = 2(x+y+2) \Rightarrow f_x(0,0) = 4$, $f_y(x,y) = 2(x+y+2) \Rightarrow f_y(0,0) = 4$

$\Rightarrow L(x,y) = 4 + 4(x-0) + 4(y-0) = 4x + 4y + 4$

(b) $f(1,2) = 25$, $f_x(1,2) = 10$, $f_y(1,2) = 10 \Rightarrow L(x,y) = 25 + 10(x-1) + 10(y-2) = 10x + 10y - 5$

3. (a) $f(0,0) = 5$, $f_x(x,y) = 3$ for all (x,y) , $f_y(x,y) = -4$ for all $(x,y) \Rightarrow L(x,y) = 5 + 3(x-0) - 4(y-0)$

$= 3x - 4y + 5$

(b) $f(1,1) = 4$, $f_x(1,1) = 3$, $f_y(1,1) = -4 \Rightarrow L(x,y) = 4 + 3(x-1) - 4(y-1) = 3x - 4y + 5$

4. (a) $f(1,1) = 1$, $f_x(x,y) = 3x^2y^4 \Rightarrow f_x(1,1) = 3$, $f_y(x,y) = 4x^3y^3 \Rightarrow f_y(1,1) = 4$
 $\Rightarrow L(x,y) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$
 (b) $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0 \Rightarrow L(x,y) = 0$
5. (a) $f(0,0) = 1$, $f_x(x,y) = e^x \cos y \Rightarrow f_x(0,0) = 1$, $f_y(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$
 $\Rightarrow L(x,y) = 1 + 1(x-0) + 0(y-0) = x + 1$
 (b) $f(0, \frac{\pi}{2}) = 0$, $f_x(0, \frac{\pi}{2}) = 0$, $f_y(0, \frac{\pi}{2}) = -1 \Rightarrow L(x,y) = 0 + 0(x-0) - 1(y - \frac{\pi}{2}) = -y + \frac{\pi}{2}$
6. (a) $f(0,0) = 1$, $f_x(x,y) = -e^{2y-x} \Rightarrow f_x(0,0) = -1$, $f_y(x,y) = 2e^{2y-x} \Rightarrow f_y(0,0) = 2$
 $\Rightarrow L(x,y) = 1 - 1(x-0) + 2(y-0) = -x + 2y + 1$
 (b) $f(1,2) = e^3$, $f_x(1,2) = -e^3$, $f_y(1,2) = 2e^3 \Rightarrow L(x,y) = e^3 - e^3(x-1) + 2e^3(y-2)$
 $= -e^3x + 2e^3y - 2e^3$
7. $f(2,1) = 3$, $f_x(x,y) = 2x - 3y \Rightarrow f_x(2,1) = 1$, $f_y(x,y) = -3x \Rightarrow f_y(2,1) = -6$
 $\Rightarrow L(x,y) = 3 + 1(x-2) - 6(y-1) = 7 + x - 6y$; $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 0$, $f_{xy}(x,y) = -3$
 $\Rightarrow M = 3$; thus $|E(x,y)| \leq (\frac{1}{2})(3)(|x-2| + |y-1|)^2 \leq (\frac{3}{2})(0.1 + 0.1)^2 = 0.06$
8. $f(2,2) = 11$, $f_x(x,y) = x + y + 3 \Rightarrow f_x(2,2) = 7$, $f_y(x,y) = x + \frac{y}{2} - 3 \Rightarrow f_y(2,2) = 0$
 $\Rightarrow L(x,y) = 11 + 7(x-2) + 0(y-2) = 7x - 3$; $f_{xx}(x,y) = 1$, $f_{yy}(x,y) = \frac{1}{2}$, $f_{xy}(x,y) = 1$
 $\Rightarrow M = 1$; thus $|E(x,y)| \leq (\frac{1}{2})(1)(|x-2| + |y-2|)^2 \leq (\frac{1}{2})(0.1 + 0.1)^2 = 0.02$
9. $f(0,0) = 1$, $f_x(x,y) = \cos y \Rightarrow f_x(0,0) = 1$, $f_y(x,y) = 1 - x \sin y \Rightarrow f_y(0,0) = 1$
 $\Rightarrow L(x,y) = 1 + 1(x-0) + 1(y-0) = x + y + 1$; $f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -x \cos y$, $f_{xy}(x,y) = -\sin y \Rightarrow M = 1$;
 thus $|E(x,y)| \leq (\frac{1}{2})(1)(|x| + |y|)^2 \leq (\frac{1}{2})(0.2 + 0.2)^2 = 0.08$
10. $f(1,2) = 6$, $f_x(x,y) = y^2 - y \sin(x-1) \Rightarrow f_x(1,2) = 4$, $f_y(x,y) = 2xy + \cos(x-1) \Rightarrow f_y(1,2) = 5$
 $\Rightarrow L(x,y) = 6 + 4(x-1) + 5(y-2) = 4x + 5y - 8$; $f_{xx}(x,y) = -y \cos(x-1)$, $f_{yy}(x,y) = 2x$,
 $f_{xy}(x,y) = 2y - \sin(x-1)$; $|x-1| \leq 0.1 \Rightarrow 0.9 \leq x \leq 1.1$ and $|y-2| \leq 0.1 \Rightarrow 1.9 \leq y \leq 2.1$; thus the max of
 $|f_{xx}(x,y)|$ on R is 2.1, the max of $|f_{yy}(x,y)|$ on R is 2.2, and the max of $|f_{xy}(x,y)|$ on R is $2(2.1) - \sin(0.9-1)$
 $\leq 4.3 \Rightarrow M = 4.3$; thus $|E(x,y)| \leq (\frac{1}{2})(4.3)(|x-1| + |y-2|)^2 \leq (2.15)(0.1 + 0.1)^2 = 0.086$
11. $f(0,0) = 1$, $f_x(x,y) = e^x \cos y \Rightarrow f_x(0,0) = 1$, $f_y(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$
 $\Rightarrow L(x,y) = 1 + 1(x-0) + 0(y-0) = 1 + x$; $f_{xx}(x,y) = e^x \cos y$, $f_{yy}(x,y) = -e^x \cos y$, $f_{xy}(x,y) = -e^x \sin y$;
 $|x| \leq 0.1 \Rightarrow -0.1 \leq x \leq 0.1$ and $|y| \leq 0.1 \Rightarrow -0.1 \leq y \leq 0.1$; thus the max of $|f_{xx}(x,y)|$ on R is $e^{0.1} \cos(0.1)$
 ≤ 1.11 , the max of $|f_{yy}(x,y)|$ on R is $e^{0.1} \cos(0.1) \leq 1.11$, and the max of $|f_{xy}(x,y)|$ on R is $e^{0.1} \sin(0.1)$
 $\leq 0.002 \Rightarrow M = 1.11$; thus $|E(x,y)| \leq (\frac{1}{2})(1.11)(|x| + |y|)^2 \leq (0.555)(0.1 + 0.1)^2 = 0.0222$

12. $f(1,1) = 0$, $f_x(x,y) = \frac{1}{x} \Rightarrow f_x(1,1) = 1$, $f_y(x,y) = \frac{1}{y} \Rightarrow f_y(1,1) = 1 \Rightarrow L(x,y) = 0 + 1(x-1) + 1(y-1)$
 $= x + y - 2$; $f_{xx}(x,y) = -\frac{1}{x^2}$, $f_{yy}(x,y) = -\frac{1}{y^2}$, $f_{xy}(x,y) = 0$; $|x-1| \leq 0.2 \Rightarrow 0.98 \leq x \leq 1.2$ so the max of
 $|f_{xx}(x,y)|$ on R is $\frac{1}{(0.98)^2} \leq 1.04$; $|y-1| \leq 0.2 \Rightarrow 0.98 \leq y \leq 1.2$ so the max of $|f_{yy}(x,y)|$ on R is
 $\frac{1}{(0.98)^2} \leq 1.04 \Rightarrow M = 1.04$; thus $|E(x,y)| \leq \left(\frac{1}{2}\right)(1.04)(|x-1| + |y-1|)^2 \leq (0.52)(0.2 + 0.2)^2 = 0.0832$
13. $A = xy \Rightarrow dA = x dy + y dx$; if $x > y$ then a 1-unit change in y gives a greater change in dA than a 1-unit change in x . Thus, pay more attention to y which is the smaller of the two dimensions.
14. (a) $f_x(x,y) = 2x(y+1) \Rightarrow f_x(1,0) = 2$ and $f_y(x,y) = x^2 \Rightarrow f_y(1,0) = 1 \Rightarrow df = 2 dx + 1 dy \Rightarrow df$ is more sensitive to changes in x
 (b) $df = 0 \Rightarrow 2 dx + dy = 0 \Rightarrow 2 \frac{dx}{dy} + 1 = 0 \Rightarrow \frac{dx}{dy} = -\frac{1}{2}$
15. $T_x(x,y) = e^y + e^{-y}$ and $T_y(x,y) = x(e^y - e^{-y}) \Rightarrow dT = T_x(x,y) dx + T_y(x,y) dy$
 $= (e^y + e^{-y})dx + x(e^y - e^{-y}) dy \Rightarrow dT|_{(2, \ln 2)} = 2.5 dx + 3.0 dy$. If $|dx| \leq 0.1$ and $|dy| \leq 0.02$, then the maximum possible error in the computed value of T is $(2.5)(0.1) + (3.0)(0.02) = 0.31$ in magnitude.
16. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r dr + V_h dh \Rightarrow \frac{dV}{V} = \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h} = \frac{2}{r} dr + \frac{1}{h} dh$; now $\left|\frac{dr}{r} \cdot 100\right| \leq 1$ and
 $\left|\frac{dh}{h} \cdot 100\right| \leq 1 \Rightarrow \left|\frac{dV}{V} \cdot 100\right| \leq \left|2 \frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \leq 2 \left|\frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \leq 2(1) + 1 = 3 \Rightarrow 3\%$
17. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r dr + V_h dh \Rightarrow dV = 2\pi rh dr + \pi r^2 dh \Rightarrow dV|_{(5,12)} = 120\pi dr + 25\pi dh$;
 $|dr| \leq 0.1$ cm and $|dh| \leq 0.1$ cm $\Rightarrow dV \leq (120\pi)(0.1) + (25\pi)(0.1) = 14.5\pi$ cm³; $V(5,12) = 300\pi$ cm³
 \Rightarrow maximum percentage error is $\pm \frac{14.5\pi}{300\pi} \times 100 = \pm 4.83\%$
18. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r dr + V_h dh \Rightarrow dV = 2\pi rh dr + \pi r^2 dh$; assuming $dr = dh$
 $\Rightarrow dV = 2\pi rh dr + \pi r^2 dr = (2\pi rh + \pi r^2) dr$; $dV \leq 0.1$ m³ when $r = 2$ m and $h = 3$ m $\Rightarrow [2\pi(2)(3) + \pi(2)^2] dr$
 $\leq 0.1 \Rightarrow dr \leq \frac{0.1}{16\pi} \approx 0.001$ m (rounded down). Thus, the absolute value of the error in measuring r and h should be less than or equal to 0.002 m.
19. $df = f_x(x,y) dx + f_y(x,y) dy = 3x^2y^4 dx + 4x^3y^3 dy \Rightarrow df|_{(1,1)} = 3 dx + 4 dy$; for a square, $dx = dy$
 $\Rightarrow df = 7 dx$ so that $|df| \leq 0.1 \Rightarrow 7|dx| \leq 0.1 \Rightarrow |dx| \leq \frac{0.1}{7} \approx 0.014 \Rightarrow$ for the square, $|x-1| \leq 0.014$ and
 $|y-1| \leq 0.014$
20. (a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow -\frac{1}{R^2} dR = -\frac{1}{R_1^2} dR_1 - \frac{1}{R_2^2} dR_2 \Rightarrow dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$
 (b) $dR = R^2 \left[\left(\frac{1}{R_1^2}\right) dR_1 + \left(\frac{1}{R_2^2}\right) dR_2 \right] \Rightarrow dR|_{(100,400)} = R^2 \left[\frac{1}{(100)^2} dR_1 + \frac{1}{(400)^2} dR_2 \right] \Rightarrow R$ will be more sensitive to a variation in R_1 since $\frac{1}{(100)^2} > \frac{1}{(400)^2}$

21. From Exercise 20, $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$ so that R_1 changing from 20 to 20.1 ohms $\Rightarrow dR_1 = 0.1$ ohm and R_2 changing from 25 to 24.9 ohms $\Rightarrow dR_2 = -0.1$ ohms; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = \frac{100}{9}$ ohms

$$\begin{aligned} \Rightarrow dR|_{(20,25)} &= \frac{\left(\frac{100}{9}\right)^2}{(20)^2} (0.1) + \frac{\left(\frac{100}{9}\right)^2}{(25)^2} (-0.1) \approx 0.011 \text{ ohms} \Rightarrow \text{percentage change is } \frac{dR}{R}|_{(20,25)} \times 100 \\ &= \frac{0.011}{\left(\frac{100}{9}\right)} \times 100 \approx 0.1\% \end{aligned}$$

22. (a) $r^2 = x^2 + y^2 \Rightarrow 2r dr = 2x dx + 2y dy \Rightarrow dr = \frac{x}{r} dx + \frac{y}{r} dy \Rightarrow dr|_{(3,4)} = \left(\frac{3}{5}\right)(\pm 0.01) + \left(\frac{4}{5}\right)(\pm 0.01)$

$$= \pm \frac{0.07}{5} = \pm 0.014 \Rightarrow \left| \frac{dr}{r} \times 100 \right| = \left| \pm \frac{0.014}{5} \times 100 \right| = 0.28\%; d\theta = \frac{\left(-\frac{y}{x^2}\right) dx + \left(\frac{1}{x}\right) dy}{\left(\frac{y}{x}\right)^2 + 1}$$

$$= \frac{-y}{y^2 + x^2} dx + \frac{x}{y^2 + x^2} dy \Rightarrow d\theta|_{(3,4)} = \left(\frac{-4}{25}\right)(\pm 0.01) + \left(\frac{3}{25}\right)(\pm 0.01) = \frac{\mp 0.04}{25} + \frac{\pm 0.03}{25}$$

\Rightarrow maximum change in $d\theta$ occurs when dx and dy have opposite signs ($dx = 0.01$ and $dy = -0.01$ or vice versa)

$$\Rightarrow d\theta = \frac{\pm 0.07}{25} \approx \pm 0.0028; \theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927255218 \Rightarrow \left| \frac{d\theta}{\theta} \times 100 \right| = \left| \frac{\pm 0.0028}{0.927255218} \times 100 \right| \approx 0.30\%$$

(b) the radius r is more sensitive to changes in y , and the angle θ is more sensitive to changes in x .

23. (a) $f(1, 1, 1) = 3$, $f_x(1, 1, 1) = y + z|_{(1,1,1)} = 2$, $f_y(1, 1, 1) = x + z|_{(1,1,1)} = 2$, $f_z(1, 1, 1) = y + x|_{(1,1,1)} = 2$
 $\Rightarrow L(x, y, z) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$

- (b) $f(1, 0, 0) = 0$, $f_x(1, 0, 0) = 0$, $f_y(1, 0, 0) = 1$, $f_z(1, 0, 0) = 1 \Rightarrow L(x, y, z) = 0 + 0(x-1) + (y-0) + (z-0) = y + z$

- (c) $f(0, 0, 0) = 0$, $f_x(0, 0, 0) = 0$, $f_y(0, 0, 0) = 0$, $f_z(0, 0, 0) = 0 \Rightarrow L(x, y, z) = 0$

24. (a) $f(1, 1, 1) = 3$, $f_x(1, 1, 1) = 2x|_{(1,1,1)} = 2$, $f_y(1, 1, 1) = 2y|_{(1,1,1)} = 2$, $f_z(1, 1, 1) = 2z|_{(1,1,1)} = 2$
 $\Rightarrow L(x, y, z) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$

- (b) $f(0, 1, 0) = 1$, $f_x(0, 1, 0) = 0$, $f_y(0, 1, 0) = 2$, $f_z(0, 1, 0) = 0 \Rightarrow L(x, y, z) = 1 + 0(x-0) + 2(y-1) + 0(z-0) = 2y - 1$

- (c) $f(1, 0, 0) = 1$, $f_x(1, 0, 0) = 2$, $f_y(1, 0, 0) = 0$, $f_z(1, 0, 0) = 0 \Rightarrow L(x, y, z) = 1 + 2(x-1) + 0(y-0) + 0(z-0) = 2x - 1$

25. (a) $f(1, 0, 0) = 1$, $f_x(1, 0, 0) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}|_{(1,0,0)} = 1$, $f_y(1, 0, 0) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}|_{(1,0,0)} = 0$,
 $f_z(1, 0, 0) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}|_{(1,0,0)} = 0 \Rightarrow L(x, y, z) = 1 + 1(x-1) + 0(y-0) + 0(z-0) = x$

- (b) $f(1, 1, 0) = \sqrt{2}$, $f_x(1, 1, 0) = \frac{1}{\sqrt{2}}$, $f_y(1, 1, 0) = \frac{1}{\sqrt{2}}$, $f_z(1, 1, 0) = 0$

$$\Rightarrow L(x, y, z) = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1) + 0(z-0) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

- (c) $f(1, 2, 2) = 3$, $f_x(1, 2, 2) = \frac{1}{3}$, $f_y(1, 2, 2) = \frac{2}{3}$, $f_z(1, 2, 2) = \frac{2}{3} \Rightarrow L(x, y, z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$
 $= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$
26. (a) $f\left(\frac{\pi}{2}, 1, 1\right) = 1$, $f_x\left(\frac{\pi}{2}, 1, 1\right) = \frac{y \cos xy}{z}\bigg|_{\left(\frac{\pi}{2}, 1, 1\right)} = 0$, $f_y\left(\frac{\pi}{2}, 1, 1\right) = \frac{x \cos xy}{z}\bigg|_{\left(\frac{\pi}{2}, 1, 1\right)} = 0$,
 $f_z\left(\frac{\pi}{2}, 1, 1\right) = \frac{-\sin xy}{z^2}\bigg|_{\left(\frac{\pi}{2}, 1, 1\right)} = -1 \Rightarrow L(x, y, z) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y-1) - 1(z-1) = 2 - z$
 (b) $f(2, 0, 1) = 0$, $f_x(2, 0, 1) = 0$, $f_y(2, 0, 1) = 2$, $f_z(2, 0, 1) = 0 \Rightarrow L(x, y, z) = 0 + 0(x-2) + 2(y-0) + 0(z-1) = 2y$
27. (a) $f(0, 0, 0) = 2$, $f_x(0, 0, 0) = e^x|_{(0,0,0)} = 1$, $f_y(0, 0, 0) = -\sin(y+z)|_{(0,0,0)} = 0$,
 $f_z(0, 0, 0) = -\sin(y+z)|_{(0,0,0)} = 0 \Rightarrow L(x, y, z) = 2 + 1(x-0) + 0(y-0) + 0(z-0) = 2 + x$
 (b) $f\left(0, \frac{\pi}{2}, 0\right) = 1$, $f_x\left(0, \frac{\pi}{2}, 0\right) = 1$, $f_y\left(0, \frac{\pi}{2}, 0\right) = -1$, $f_z\left(0, \frac{\pi}{2}, 0\right) = -1 \Rightarrow L(x, y, z)$
 $= 1 + 1(x-0) - 1\left(y - \frac{\pi}{2}\right) - 1(z-0) = x - y - z + \frac{\pi}{2} + 1$
 (c) $f\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = 1$, $f_x\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = 1$, $f_y\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = -1$, $f_z\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = -1 \Rightarrow L(x, y, z)$
 $= 1 + 1(x-0) - 1\left(y - \frac{\pi}{4}\right) - 1\left(z - \frac{\pi}{4}\right) = x - y - z + \frac{\pi}{2} + 1$
28. (a) $f(1, 0, 0) = 0$, $f_x(1, 0, 0) = \frac{yz}{(xyz)^2 + 1}\bigg|_{(1,0,0)} = 0$, $f_y(1, 0, 0) = \frac{xz}{(xyz)^2 + 1}\bigg|_{(1,0,0)} = 0$,
 $f_z(1, 0, 0) = \frac{xy}{(xyz)^2 + 1}\bigg|_{(1,0,0)} = 0 \Rightarrow L(x, y, z) = 0$
 (b) $f(1, 1, 0) = 0$, $f_x(1, 1, 0) = 0$, $f_y(1, 1, 0) = 0$, $f_z(1, 1, 0) = 1 \Rightarrow L(x, y, z) = 0 + 0(x-1) + 0(y-1) + 1(z-0) = z$
 (c) $f(1, 1, 1) = \frac{\pi}{4}$, $f_x(1, 1, 1) = \frac{1}{2}$, $f_y(1, 1, 1) = \frac{1}{2}$, $f_z(1, 1, 1) = \frac{1}{2} \Rightarrow L(x, y, z) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}(z-1)$
 $= \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z + \frac{\pi}{4} - \frac{3}{2}$
29. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = -2$; $f_x = z$, $f_y = -3z$, $f_z = x - 3y \Rightarrow L(x, y, z)$
 $= -2 + 2(x-1) - 6(y-1) - 2(z-2) = 2x - 6y - 2z + 6$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 0$, $f_{yz} = -3$
 $\Rightarrow M = 3$; thus, $|E(x, y, z)| \leq \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$
30. $f(x, y, z) = x^2 + xy + yz + \frac{1}{4}z^2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = 5$; $f_x = 2x + y$, $f_y = x + z$, $f_z = y + \frac{1}{2}z$
 $\Rightarrow L(x, y, z) = 5 + 3(x-1) + 3(y-1) + 2(z-2) = 3x + 3y + 2z - 5$; $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = \frac{1}{2}$, $f_{xy} = 1$, $f_{xz} = 0$,
 $f_{yz} = 1 \Rightarrow M = 2$; thus $|E(x, y, z)| \leq \left(\frac{1}{2}\right)(2)(0.01 + 0.01 + 0.08)^2 = 0.01$
31. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0) \Rightarrow f(1, 1, 0) = 1$; $f_x = y - 3z$, $f_y = x + 2z$, $f_z = 2y - 3x$
 $\Rightarrow L(x, y, z) = 1 + (x-1) + (y-1) - (z-0) = x + y - z - 1$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 1$, $f_{xz} = -3$,
 $f_{yz} = 2 \Rightarrow M = 3$; thus $|E(x, y, z)| \leq \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.01)^2 = 0.00135$

32. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $P_0(x, y, \frac{\pi}{4}) \Rightarrow f(0, 0, \frac{\pi}{4}) = 1$; $f_x = -\sqrt{2} \sin x \sin(y + z)$,
 $f_y = \sqrt{2} \cos x \cos(y + z)$, $f_z = \sqrt{2} \cos x \cos(y + z) \Rightarrow L(x, y, z) = 1 - 0(x - 0) + (y - 0) + (z - \frac{\pi}{4})$
 $= y + z - \frac{\pi}{4} + 1$; $f_{xx} = -\sqrt{2} \cos x \sin(y + z)$, $f_{yy} = -\sqrt{2} \cos x \sin(y + z)$, $f_{zz} = -\sqrt{2} \cos x \sin(y + z)$,
 $f_{xy} = -\sqrt{2} \sin x \cos(y + z)$, $f_{xz} = -\sqrt{2} \sin x \cos(y + z)$, $f_{yz} = -\sqrt{2} \cos x \sin(y + z)$. The absolute value of
each of these second partial derivatives is bounded above by $\sqrt{2} \Rightarrow M = \sqrt{2}$; thus $|E(x, y, z)|$
 $\leq (\frac{1}{2})(\sqrt{2})(0.01 + 0.01 + 0.01)^2 = 0.000636$.
33. (a) $dS = S_p dp + S_x dx + S_w dw + S_h dh = C\left(\frac{x^4}{wh^3} dp + \frac{4px^3}{wh^3} dx - \frac{px^4}{w^2h^3} dw - \frac{3px^4}{wh^4} dh\right)$
 $= C\left(\frac{px^4}{wh^3}\right)\left(\frac{1}{p} dp + \frac{4}{x} dx - \frac{1}{w} dw - \frac{3}{h} dh\right) = S_0\left(\frac{1}{p_0} dp + \frac{4}{x_0} dx - \frac{1}{w_0} dw - \frac{3}{h_0} dh\right)$
 $= S_0\left(\frac{1}{100} dp + dx - 5 dw - 30 dh\right)$, where $p_0 = 100$ N/m, $x_0 = 4$ m, $w_0 = 0.2$ m, $h_0 = 0.1$ m
(b) More sensitive to a change in height
34. (a) $V = \pi r^2 h \Rightarrow dV = 2\pi r h dr + \pi r^2 dh \Rightarrow$ at $r = 1$ and $h = 5$ we have $dV = 10\pi dr + \pi dh \Rightarrow$ the volume is
about 10 times more sensitive to a change in r
(b) $dV = 0 \Rightarrow 0 = 2\pi r h dr + \pi r^2 dh = 2h dr + r dh = 10 dr + dh \Rightarrow dr = -\frac{1}{10} dh$; choose $dh = 1.5$
 $\Rightarrow dr = -0.15 \Rightarrow h = 6.5$ in. and $r = 0.85$ in. is one solution for $\Delta V \approx dV = 0$
35. $f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \Rightarrow f_a = d, f_b = -c, f_c = -b, f_d = a \Rightarrow df = d da - c db - b dc + a dd$; since
 $|a|$ is much greater than $|b|, |c|$, and $|d|$, the function f is most sensitive to a change in d .
36. $p(a, b, c) = abc \Rightarrow p_a = bc, p_b = ac, p_c = ab \Rightarrow dp = bc da + ac db + ab dc \Rightarrow \frac{dp}{p} = \frac{bc da + ac db + ab dc}{abc}$
 $= \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$. Now $\left|\frac{da}{a} \cdot 100\right| = 2$, $\left|\frac{db}{b} \cdot 100\right| = 2$, and $\left|\frac{dc}{c} \cdot 100\right| = 2 \Rightarrow \left|\frac{dp}{p} \cdot 100\right|$
 $= \left|\frac{da}{a} \cdot 100 + \frac{db}{b} \cdot 100 + \frac{dc}{c} \cdot 100\right| \leq \left|\frac{da}{a} \cdot 100\right| + \left|\frac{db}{b} \cdot 100\right| + \left|\frac{dc}{c} \cdot 100\right| = 2 + 2 + 2 = 6$ or 6%
37. $V = lwh \Rightarrow V_l = wh, V_w = lh, V_h = lw \Rightarrow dV = wh dl + lh dw + lw dh \Rightarrow dV|_{(5, 3, 2)} = 6 dl + 10 dw + 15 dh$;
 $dl = 1$ in. $= \frac{1}{12}$ ft, $dw = 1$ in. $= \frac{1}{12}$ ft, $dh = \frac{1}{2}$ in. $= \frac{1}{24}$ ft $\Rightarrow dV = 6\left(\frac{1}{12}\right) + 10\left(\frac{1}{12}\right) + 15\left(\frac{1}{24}\right) = \frac{47}{24}$ ft³
38. $A = \frac{1}{2}ab \sin C \Rightarrow A_a = \frac{1}{2}b \sin C, A_b = \frac{1}{2}a \sin C, A_c = \frac{1}{2}ab \cos C$
 $\Rightarrow dA = \left(\frac{1}{2}b \sin C\right) da + \left(\frac{1}{2}a \sin C\right) db + \left(\frac{1}{2}ab \cos C\right) dC$; $dC = |2^\circ| = |0.0349|$ radians, $da = |0.5|$ ft,
 $db = |0.5|$ ft; at $a = 150$ ft, $b = 200$ ft, and $C = 60^\circ$, we see that the change is approximately
 $dA = \frac{1}{2}(200)(\sin 60^\circ)|0.5| + \frac{1}{2}(150)(\sin 60^\circ)|0.5| + \frac{1}{2}(200)(150)(\cos 60^\circ)|0.0349| = \pm 338$ ft²

39. $u_x = e^y$, $u_y = xe^y + \sin z$, $u_z = y \cos z \Rightarrow du = e^y dx + (xe^y + \sin z) dy + (y \cos z) dz$
 $\Rightarrow du|_{(2, \ln 3, \frac{\pi}{2})} = 3 dx + 7 dy + 0 dz = 3 dx + 7 dy \Rightarrow \text{magnitude of the maximum possible error}$
 $\leq 3(0.2) + 7(0.6) = 4.8$

40. $Q_K = \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{2M}{h} \right)$, $Q_M = \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{2K}{h} \right)$, and $Q_h = \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{-2KM}{h^2} \right)$
 $\Rightarrow dQ = \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{2M}{h} \right) dK + \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{2K}{h} \right) dM + \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left(\frac{-2KM}{h^2} \right) dh$
 $= \frac{1}{2} \left(\frac{2KM}{h} \right)^{-1/2} \left[\frac{2M}{h} dK + \frac{2K}{h} dM - \frac{2KM}{h^2} dh \right] \Rightarrow dQ|_{(2, 20, 0.05)}$
 $= \frac{1}{2} \left[\frac{(2)(2)(20)}{0.05} \right]^{-1/2} \left[\frac{(2)(20)}{0.05} dK + \frac{(2)(2)}{0.05} dM - \frac{(2)(2)(20)}{(0.05)^2} dh \right] = (0.0125)(800 dK + 80 dM - 32,000 dh)$
 $\Rightarrow Q$ is most sensitive to changes in h

41. $z = f(x, y) \Rightarrow g(x, y, z) = f(x, y) - z = 0 \Rightarrow g_x(x, y, z) = f_x(x, y)$, $g_y(x, y, z) = f_y(x, y)$ and $g_z(x, y, z) = -1$
 $\Rightarrow g_x(x_0, y_0, f(x_0, y_0)) = f_x(x_0, y_0)$, $g_y(x_0, y_0, f(x_0, y_0)) = f_y(x_0, y_0)$ and $g_z(x_0, y_0, f(x_0, y_0)) = -1 \Rightarrow$ the tangent
plane at the point P_0 is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - [z - f(x_0, y_0)] = 0$ or
 $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$

11.7 EXTREME VALUES AND SADDLE POINTS

- $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2y - 3 = 0 \Rightarrow x = -3$ and $y = 3 \Rightarrow$ critical point is $(-3, 3)$;
 $f_{xx}(-3, 3) = 2$, $f_{yy}(-3, 3) = 2$, $f_{xy}(-3, 3) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
 $f(-3, 3) = -5$
- $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$;
 $f_{xx}\left(\frac{2}{3}, \frac{4}{3}\right) = -10$, $f_{yy}\left(\frac{2}{3}, \frac{4}{3}\right) = -4$, $f_{xy}\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of
 $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$
- $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2, 1)$;
 $f_{xx}(-2, 1) = 2$, $f_{yy}(-2, 1) = 0$, $f_{xy}(-2, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point
- $f_x(x, y) = 5y - 14x + 3 = 0$ and $f_y(x, y) = 5x - 6 = 0 \Rightarrow x = \frac{6}{5}$ and $y = \frac{69}{25} \Rightarrow$ critical point is $\left(\frac{6}{5}, \frac{69}{25}\right)$;
 $f_{xx}\left(\frac{6}{5}, \frac{69}{25}\right) = -14$, $f_{yy}\left(\frac{6}{5}, \frac{69}{25}\right) = 0$, $f_{xy}\left(\frac{6}{5}, \frac{69}{25}\right) = 5 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -25 < 0 \Rightarrow$ saddle point
- $f_x(x, y) = 6x + 6y - 2 = 0$ and $f_y(x, y) = 6x + 14y + 4 = 0 \Rightarrow x = \frac{13}{12}$ and $y = -\frac{3}{4} \Rightarrow$ critical point is $\left(\frac{13}{12}, -\frac{3}{4}\right)$;
 $f_{xx}\left(\frac{13}{12}, -\frac{3}{4}\right) = 6$, $f_{yy}\left(\frac{13}{12}, -\frac{3}{4}\right) = 14$, $f_{xy}\left(\frac{13}{12}, -\frac{3}{4}\right) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 48 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of

$$f\left(\frac{13}{12}, -\frac{3}{4}\right) = -\frac{31}{12}$$

6. $f_x(x, y) = 4x + 3y - 5 = 0$ and $f_y(x, y) = 3x + 8y + 2 = 0 \Rightarrow x = 2$ and $y = -1 \Rightarrow$ critical point is $(2, -1)$;
 $f_{xx}(2, -1) = 4$, $f_{yy}(2, -1) = 8$, $f_{xy}(2, -1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 23 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
 $f(2, -1) = -6$
7. $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1, 2)$; $f_{xx}(1, 2) = 2$,
 $f_{yy}(1, 2) = -2$, $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point
8. $f_x(x, y) = 2x - 2y - 2 = 0$ and $f_y(x, y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1, 0)$;
 $f_{xx}(1, 0) = 2$, $f_{yy}(1, 0) = 4$, $f_{xy}(1, 0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
 $f(1, 0) = 0$
9. $f_x(x, y) = 2 - 4x - 2y = 0$ and $f_y(x, y) = 2 - 2x - 2y = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow$ critical point is $(0, 1)$;
 $f_{xx}(0, 1) = -4$, $f_{yy}(0, 1) = -2$, $f_{xy}(0, 1) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, 1) = 4$
10. $f_x(x, y) = 3x^2 - 2y = 0$ and $f_y(x, y) = -3y^2 - 2x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -\frac{2}{3}$ and $y = \frac{2}{3} \Rightarrow$ critical points
are $(0, 0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = -6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -2$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point; for $\left(-\frac{2}{3}, \frac{2}{3}\right)$: $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{yy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{xy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -2$
 $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27}$
11. $f_x(x, y) = 3x^2 + 3y = 0$ and $f_y(x, y) = 3x + 3y^2 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -1$ and $y = -1 \Rightarrow$ critical points
are $(0, 0)$ and $(-1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= -9 < 0 \Rightarrow$ saddle point; for $(-1, -1)$: $f_{xx}(-1, -1) = -6$, $f_{yy}(-1, -1) = -6$, $f_{xy}(-1, -1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1, -1) = 1$
12. $f_x(x, y) = 12x - 6x^2 + 6y = 0$ and $f_y(x, y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical
points are $(0, 0)$ and $(1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0, 0) = 6$, $f_{xy}(0, 0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 0) = 0$; for $(1, -1)$: $f_{xx}(1, -1) = 0$, $f_{yy}(1, -1) = 6$,
 $f_{xy}(1, -1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point
13. $f_x(x, y) = 27x^2 - 4y = 0$ and $f_y(x, y) = y^2 - 4x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{4}{9}$ and $y = \frac{4}{3} \Rightarrow$ critical points are
 $(0, 0)$ and $\left(\frac{4}{9}, \frac{4}{3}\right)$; for $(0, 0)$: $f_{xx}(0, 0) = 54x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 2y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$
 $= -16 < 0 \Rightarrow$ saddle point; for $\left(\frac{4}{9}, \frac{4}{3}\right)$: $f_{xx}\left(\frac{4}{9}, \frac{4}{3}\right) = 24$, $f_{yy}\left(\frac{4}{9}, \frac{4}{3}\right) = \frac{8}{3}$, $f_{xy}\left(\frac{4}{9}, \frac{4}{3}\right) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 48 > 0$
and $f_{xx} > 0 \Rightarrow$ local minimum of $f\left(\frac{4}{9}, \frac{4}{3}\right) = -\frac{64}{81}$

14. $f_x(x, y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 2) = -12$; for $(-2, 0)$: $f_{xx}(-2, 0) = -6$, $f_{yy}(-2, 0) = -6$, $f_{xy}(-2, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = -4$; for $(-2, 2)$: $f_{xx}(-2, 2) = -6$, $f_{yy}(-2, 2) = 6$, $f_{xy}(-2, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point
15. $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = y \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = -12x^2|_{(0,0)} = 0$, $f_{yy}(0, 0) = -12y^2|_{(0,0)} = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, 1)$: $f_{xx}(1, 1) = -12$, $f_{yy}(1, 1) = -12$, $f_{xy}(1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(1, 1) = 2$; for $(-1, -1)$: $f_{xx}(-1, -1) = -12$, $f_{yy}(-1, -1) = -12$, $f_{xy}(-1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1, -1) = 2$
16. $f_x(x, y) = 4x^3 + 4y = 0$ and $f_y(x, y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0, 0)$, $(1, -1)$, and $(-1, 1)$; $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$, and $f_{xy}(x, y) = 4$; for $(0, 0)$: $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, -1)$: $f_{xx}(1, -1) = 12$, $f_{yy}(1, -1) = 12$, $f_{xy}(1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1, -1) = -2$; for $(-1, 1)$: $f_{xx}(-1, 1) = 12$, $f_{yy}(-1, 1) = 12$, $f_{xy}(-1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-1, 1) = -2$
17. $f_x(x, y) = \frac{-2x}{(x^2 + y^2 - 1)^2} = 0$ and $f_y(x, y) = \frac{-2y}{(x^2 + y^2 - 1)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0, 0)$; $f_{xx} = \frac{4x^2 - 2y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{yy} = \frac{-2x^2 + 4y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{xy} = \frac{6xy}{(x^2 + y^2 - 1)^3}$; $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = -2$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, 0) = -1$
18. $f_x(x, y) = -\frac{1}{x^2} + y = 0$ and $f_y(x, y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow$ the critical point is $(1, 1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1, 1) = 2$, $f_{yy}(1, 1) = 2$, $f_{xy}(1, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2 \Rightarrow$ local minimum of $f(1, 1) = 3$
19. $f_x(x, y) = y \cos x = 0$ and $f_y(x, y) = \sin x = 0 \Rightarrow x = n\pi$, n an integer, and $y = 0 \Rightarrow$ the critical points are $(n\pi, 0)$, n an integer (Note: $\cos x$ and $\sin x$ cannot both be 0 for the same x , so $\sin x$ must be 0 and $y = 0$); $f_{xx} = -y \sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$; $f_{xx}(n\pi, 0) = 0$, $f_{yy}(n\pi, 0) = 0$, $f_{xy}(n\pi, 0) = 1$ if n is even and $f_{xy}(n\pi, 0) = -1$ if n is odd $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point; $f(n\pi, 0) = 0$ for every n
20. $f_x(x, y) = 2e^{2x} \cos y = 0$ and $f_y(x, y) = -e^{2x} \sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions $\cos y$ and $\sin y$ cannot equal 0 for the same $y \Rightarrow$ no critical points \Rightarrow no extrema and no saddle points

21. (i) On OA, $f(x, y) = f(0, y) = y^2 - 4y + 1$ on $0 \leq y \leq 2$;
 $f'(0, y) = 2y - 4 = 0 \Rightarrow y = 2$;
 $f(0, 0) = 1$ and $f(0, 2) = -3$
- (ii) On AB, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ on $0 \leq x \leq 1$;
 $f'(x, 2) = 4x - 4 = 0 \Rightarrow x = 1$;
 $f(0, 2) = -3$ and $f(1, 2) = -5$
- (iii) On OB, $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \leq x \leq 1$;
 endpoint values have been found above; $f'(x, 2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of OB
- (iv) For interior points of the triangular region,
 $f_x(x, y) = 4x - 4 = 0$ and $f_y(x, y) = 2y - 4 = 0$
 $\Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0, 0)$ and the absolute minimum is -5 at $(1, 2)$.
22. (i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \leq y \leq 2$; $f'(0, y) = 2y = 0$
 $\Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and $f(0, 2) = 4$
- (ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \leq x \leq 1$; $f'(x, 0) = 2x = 0$
 $\Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and $f(1, 0) = 1$
- (iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on $0 \leq x \leq 1$;
 $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$; $f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5}$ and $f(0, 2) = 4$
- (iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0$
 $\Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.
23. (i) On OC, $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \leq x \leq 5$;
 $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$,
 $T(0, 0) = 2$, and $T(5, 0) = -3$
- (ii) On CB, $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \leq y \leq 0$;
 $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T(5, -\frac{5}{2}) = -\frac{37}{4}$ and $T(5, -3) = -9$
- (iii) On AB, $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \leq x \leq 5$;
 $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T(\frac{9}{2}, -3) = -\frac{37}{4}$ and $T(0, -3) = 11$
- (iv) On AO, $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \leq y \leq 0$;
 $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO
- (v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.

