CHAPTER 10 VECTORS AND MOTION IN SPACE

10.1 CARTESIAN (RECTANGULAR) COORDINATES AND VECTORS IN SPACE

1. The line through the point $(2,3,0)$ parallel to the z-axis
2. The line through the point $(-1,0,0)$ parallel to the y-axis
3. The x-axis
4. The line through the point $(1,0,0)$ parallel to the z-axis
5. The circle $x^2+y^2=4$ in the plane $z=-2$
6. The circle $x^2+z^2=4$ in the xz-plane
7. The circle $y^2+z^2=1$ in the yz-plane
8. The circle $x^2+z^2=9$ in the plane $y=-4$
9. The circle $x^2+y^2=16$ in the xy-plane
10. The circle $x^2+z^2=3$ in the xz-plane
11. (a) The first quadrant of the xy-plane (b) The fourth quadrant of the xy-plane
12. (a) The slab bounded by the planes $x=0$ and $x=1$
    (b) The square column bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$
    (c) The unit cube in the first octant having one vertex at the origin
13. (a) The ball of radius 1 centered at the origin
    (b) All points at distance greater than 1 unit from the origin
14. (a) The circumference and interior of the circle $x^2+y^2=1$ in the xy-plane
    (b) The circumference and interior of the circle $x^2+y^2=1$ in the plane $z=3$
    (c) A solid cylindrical column of radius 1 whose axis is the z-axis
15. (a) The upper hemisphere of radius 1 centered at the origin
    (b) The solid upper hemisphere of radius 1 centered at the origin
16. (a) The line $y=x$ in the xy-plane
    (b) The plane $y=x$ consisting of all points of the form (x,x,z)
17. (a) $x=3$ (b) $y=-1$ (c) $z=-2$
18. (a) $x=3$ (b) $y=-1$ (c) $z=2$
19. (a) $z=1$ (b) $x=3$ (c) $y=-1$
20. (a) $x^2+y^2=4,\ z=0$ (b) $y^2+z^2=4,\ x=0$ (c) $x^2+z^2=4,\ y=0$
21. (a) $x^2+(y-2)^2=4,\ z=0$ (b) $(y-2)^2+z^2=4,\ x=0$ (c) $x^2+z^2=4,\ y=2$
22. (a) $(x+3)^2+(y-4)^2=1,\ z=1$ (b) $(y-4)^2+(z-1)^2=1,\ x=-3$ (c) $(x+3)^2+(z-1)^2=1,\ y=4$
23. (a) $y=3,\ z=-1$ (b) $x=1,\ z=-1$ (c) $x=1,\ y=3$
24. $\sqrt{x^2+y^2+z^2}=\sqrt{x^2+(y-2)^2+z^2} \Rightarrow x^2+y^2+z^2=x^2+(y-2)^2+z^2 \Rightarrow y^2=y^2-4y+4 \Rightarrow y=1$
25. $x^2+y^2+z^2=25,\ z=3$
26. $x^2+y^2+(z-1)^2=4$ and $x^2+y^2+(z+1)^2=4 \Rightarrow x^2+y^2+(z-1)^2=x^2+y^2+(z+1)^2 \Rightarrow z=0,\ x^2+y^2=3$
27. $0\le z\le 1$
28. $0\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 2$

29. $z \le 0$

30. $z = \sqrt{1 - x^2 - y^2}$

31. (a) $(x-1)^2 + (y-1)^2 + (z-1)^2 < 1$ (b) $(x-1)^2 + (y-1)^2 + (z-1)^2 > 1$

32. $1 \le x^2 + y^2 + z^2 \le 4$

33. length $= |2\mathbf{i} + \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

34. length $= |9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{81 + 4 + 36} = 11$, the direction is $\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k} \Rightarrow 9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
$= 11\left(\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}\right)$

35. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \Rightarrow 5\mathbf{k} = 5(\mathbf{k})$

36. length $= \left|\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, the direction is $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} = 1\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right)$

37. length $= \left|\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow \frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$

38. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$

39. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j} + \frac{2}{5}\mathbf{k}$ (d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

40. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\mathbf{i} - \frac{4\sqrt{2}}{5}\mathbf{k}$ (c) $\frac{1}{4}\mathbf{i} - \frac{1}{3}\mathbf{j} - \mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\mathbf{i} + \frac{a}{\sqrt{3}}\mathbf{j} - \frac{a}{\sqrt{6}}\mathbf{k}$

41. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

42. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$
$= -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

43. (a) the distance = the length $= \left|\overrightarrow{P_1P_2}\right| = |3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

(b) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$

(c) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

44. (a) the distance = the length $= \left|\overrightarrow{P_1P_2}\right| = |3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}| = \sqrt{9+36+4} = 7$

(b) $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$

(c) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$

45. (a) the distance = the length $= \left|\overrightarrow{P_1P_2}\right| = |-\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{3}$

(b) $-\mathbf{i} - \mathbf{j} - \mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$

(c) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

46. (a) the distance = the length $= \left|\overrightarrow{P_1P_2}\right| = |2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}| = \sqrt{3 \cdot 2^2} = 2\sqrt{3}$

(b) $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$

(c) the midpoint is $(1, -1, -1)$

47. $\overrightarrow{AB} = (5-a)\mathbf{i} + (1-b)\mathbf{j} + (3-c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow 5 - a = 1$, $1 - b = 4$, and $3 - c = -2 \Rightarrow a = 4$, $b = -3$, and $c = 5 \Rightarrow$ A is the point $(4, -3, 5)$

48. $\overrightarrow{AB} = (a+2)\mathbf{i} + (b+3)\mathbf{j} + (c-6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} \Rightarrow a + 2 = -7$, $b + 3 = 3$, and $c - 6 = 8 \Rightarrow a = -9$, $b = 0$, and $c = 14 \Rightarrow$ B is the point $(-9, 0, 14)$

49. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$

50. $x^2 + (y+1)^2 + (z-5)^2 = 4$

51. center $(-2, 0, 2)$, radius $2\sqrt{2}$

52. center $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{21}}{2}$

53. $x^2 + y^2 + z^2 + 4x - 4z = 0 \Rightarrow \left(x^2 + 4x + 4\right) + y^2 + \left(z^2 - 4z + 4\right) = 4 + 4 \Rightarrow (x+2)^2 + (y-0)^2 + (z-2)^2 = \left(\sqrt{8}\right)^2$
$\Rightarrow$ the center is at $(-2, 0, 2)$ and the radius is $2\sqrt{2}$

54. $x^2 + y^2 + z^2 - 6y + 8z = 0 \Rightarrow x^2 + \left(y^2 - 6y + 9\right) + \left(z^2 + 8z + 16\right) = 9 + 16 \Rightarrow (x-0)^2 + (y-3)^2 + (z+4)^2 = 5^2$
$\Rightarrow$ the center is at $(0, 3, -4)$ and the radius is 5

55. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}$
$\Rightarrow \left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) + \left(z^2 + \frac{1}{2}z + \frac{1}{16}\right) = \frac{9}{2} + \frac{3}{16} = \frac{75}{16} \Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \left(\frac{5\sqrt{3}}{4}\right)^2$
$\Rightarrow$ the center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

56. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \Rightarrow x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3 \Rightarrow x^2 + \left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) + \left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) = 3 + \frac{2}{9}$
$\Rightarrow (x-0)^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

57. (a) the distance between (x,y,z) and $(x,0,0)$ is $\sqrt{y^2+z^2}$

(b) the distance between (x,y,z) and $(0,y,0)$ is $\sqrt{x^2+z^2}$

(c) the distance between (x,y,z) and $(0,0,z)$ is $\sqrt{x^2+y^2}$

58. (a) the distance between (x,y,z) and $(x,y,0)$ is $|z|$
(b) the distance between (x,y,z) and $(0,y,z)$ is $|x|$
(c) the distance between (x,y,z) and $(x,0,z)$ is $|y|$

59. (a) the midpoint of AB is $M\left(\frac{5}{2},\frac{5}{2},0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2}-1\right)\mathbf{i}+\left(\frac{5}{2}-1\right)\mathbf{j}+(0-3)\mathbf{k} = \frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}$

(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-3\mathbf{k}\right) = \mathbf{i}+\mathbf{j}-2\mathbf{k}$

(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i}+\mathbf{j}+3\mathbf{k})+(\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is the point $(2,2,1)$, which is the location of the center of mass

60. The midpoint of AB is $M\left(\frac{3}{2},0,\frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left[\left(\frac{3}{2}+1\right)\mathbf{i}+(0-2)\mathbf{j}+\left(\frac{5}{2}+1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i}-2\mathbf{j}+\frac{7}{2}\mathbf{k}\right)$ $= \frac{5}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}+\frac{7}{3}\mathbf{k}$. The terminal point of $\left(\frac{5}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}+\frac{7}{3}\mathbf{k}\right)+\overrightarrow{OC} = \left(\frac{5}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}+\frac{7}{3}\mathbf{k}\right)+(-\mathbf{i}+2\mathbf{j}-\mathbf{k})$ $= \frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{4}{3}\mathbf{k}$ is the point $\left(\frac{2}{3},\frac{2}{3},\frac{4}{3}\right)$ which is the location of the intersection of the medians.

61. Without loss of generality we identify the vertices of the quadrilateral such that $A(0,0,0)$, $B(x_b,0,0)$, $C(x_c,y_c,0)$ and $D(x_d,y_d,z_d)$ $\Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2},0,0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b+x_c}{2},\frac{y_c}{2},0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c+x_d}{2},\frac{y_c+y_d}{2},\frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2},\frac{y_d}{2},\frac{z_d}{2}\right)$ $\Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2}+\frac{x_c+x_d}{2}}{2},\frac{y_c+y_d}{4},\frac{z_d}{4}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b+x_c}{2}+\frac{x_d}{2}}{2},\frac{y_c+y_d}{4},\frac{z_d}{4}\right)$.

62. Let V_1, V_2, V_3, …, V_n be the vertices of a regular n-sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \ldots, n$. If $\mathbf{S} = \sum_{i=1}^{n} \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \ldots, n$, then $\mathbf{S}$ would remain the same. Since $\mathbf{S}$ does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.

63. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0,0)$, $B(b,0)$ and $C(x_c,y_c)$ $\Rightarrow$ a is located at $\left(\frac{b+x_c}{2},\frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2},\frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2},0\right)$. Therefore, $\overrightarrow{Aa} = \left(\frac{b}{2}+\frac{x_c}{2}\right)\mathbf{i}+\left(\frac{y_c}{2}\right)\mathbf{j}$, $\overrightarrow{Bb} = \left(\frac{x_c}{2}-b\right)\mathbf{i}+\left(\frac{y_c}{2}\right)\mathbf{j}$, and $\overrightarrow{Cc} = \left(\frac{b}{2}-x_c\right)\mathbf{i}+(-y_c)\mathbf{j} \Rightarrow \overrightarrow{Aa}+\overrightarrow{Bb}+\overrightarrow{Cc} = \mathbf{0}$.

10.2 DOT AND CROSS PRODUCTS

NOTE: In Exercises 1-6 below we calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i}+4\mathbf{j}-\sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i}+11\mathbf{j}-2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i}+10\mathbf{j}-11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j}-3\mathbf{k})$
6.	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$

7. $\mathbf{u}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)+\left(\mathbf{u}-\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)=\frac{3}{2}(\mathbf{i}+\mathbf{j})+\left[(3\mathbf{j}+4\mathbf{k})-\frac{3}{2}(\mathbf{i}+\mathbf{j})\right]=\left(\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}\right)+\left(-\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}+4\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u}=3$ and $\mathbf{v}\cdot\mathbf{v}=2$

8. $\mathbf{u}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)+\left(\mathbf{u}-\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)=\frac{1}{2}\mathbf{v}+\left(\mathbf{u}-\frac{1}{2}\mathbf{v}\right)=\frac{1}{2}(\mathbf{i}+\mathbf{j})+\left[(\mathbf{j}+\mathbf{k})-\frac{1}{2}(\mathbf{i}+\mathbf{j})\right]=\left(\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)+\left(-\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u}=1$ and $\mathbf{v}\cdot\mathbf{v}=2$

9. $\mathbf{u}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)+\left(\mathbf{u}-\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)=\frac{14}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k})+\left[(8\mathbf{i}+4\mathbf{j}-12\mathbf{k})-\left(\frac{14}{3}\mathbf{i}+\frac{28}{3}\mathbf{j}-\frac{14}{3}\mathbf{k}\right)\right]$
$=\left(\frac{14}{3}\mathbf{i}+\frac{28}{3}\mathbf{j}-\frac{14}{3}\mathbf{k}\right)+\left(\frac{10}{3}\mathbf{i}-\frac{16}{3}\mathbf{j}-\frac{22}{3}\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u}=28$ and $\mathbf{v}\cdot\mathbf{v}=6$

10. $\mathbf{u}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)+\left(\mathbf{u}-\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right)=\frac{1}{1}(\mathbf{v})+\left[(\mathbf{i}+\mathbf{j}+\mathbf{k})-\left(\frac{1}{1}\right)\mathbf{v}\right]=(\mathbf{i})+(\mathbf{j}+\mathbf{k})$, where $\mathbf{v}\cdot\mathbf{u}=1$ and $\mathbf{v}\cdot\mathbf{v}=1$; yes

11. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{(2)(1)+(1)(2)+(0)(-1)}{\sqrt{2^2+1^2+0^2}\,\sqrt{1^2+2^2+(-1)^2}}\right)=\cos^{-1}\left(\frac{4}{\sqrt{5}\,\sqrt{6}}\right)=\cos^{-1}\left(\frac{4}{\sqrt{30}}\right)\approx 0.75$ rad

12. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{(2)(3)+(-2)(0)+(1)(4)}{\sqrt{2^2+(-2)^2+1^2}\,\sqrt{3^2+0^2+4^2}}\right)=\cos^{-1}\left(\frac{10}{\sqrt{9}\,\sqrt{25}}\right)=\cos^{-1}\left(\frac{2}{3}\right)\approx 0.84$ rad

13. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)=\cos^{-1}\left(\frac{(\sqrt{3})(\sqrt{3})+(-7)(1)+(0)(-2)}{\sqrt{(\sqrt{3})^2+(-7)^2+0^2}\,\sqrt{(\sqrt{3})^2+(1)^2+(-2)^2}}\right)=\cos^{-1}\left(\frac{3-7}{\sqrt{52}\,\sqrt{8}}\right)$
$=\cos^{-1}\left(\frac{-1}{\sqrt{26}}\right)\approx 1.77$ rad

14. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1)+(\sqrt{2})(1)+(-\sqrt{2})(1)}{\sqrt{(1)^2+(\sqrt{2})^2+(-\sqrt{2})^2}\sqrt{(-1)^2+(1)^2+(1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right)$

$= \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83$ rad

15. (a) $\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{i}||\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{\mathbf{j}\cdot\mathbf{v}}{|\mathbf{j}||\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{\mathbf{k}\cdot\mathbf{v}}{|\mathbf{k}||\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$ and

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 = \frac{a^2+b^2+c^2}{|\mathbf{v}||\mathbf{v}|} = \frac{|\mathbf{v}||\mathbf{v}|}{|\mathbf{v}||\mathbf{v}|} = 1$

(b) $|\mathbf{v}| = 1 \Rightarrow \cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$ and $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\mathbf{v}$

16. $\mathbf{u} = 10\mathbf{i} + 2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{j} + \mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\sqrt{101}}\right) \approx 1.55$ rad $\approx 88.88°$.

17. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

18. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $-\mathbf{k}$

19. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

20. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

21. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $-\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $\mathbf{k}$

22. $\mathbf{u}\times\mathbf{v} = (\mathbf{i}\times\mathbf{j})\times(\mathbf{j}\times\mathbf{k}) = \mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow$ length $= 1$ and the direction is $\mathbf{j}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{b}) = -\mathbf{j} \Rightarrow$ length $= 1$ and the direction is $-\mathbf{j}$

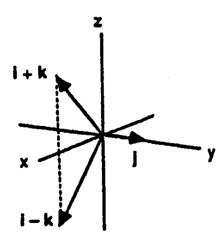
23. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

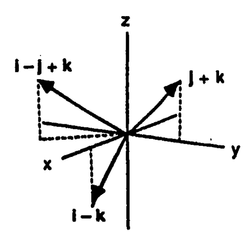
24. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ length $= 2\sqrt{3}$ and the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \Rightarrow$ length $= 2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$

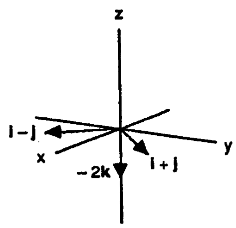
25. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{k}$



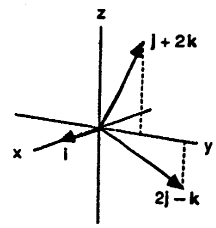
26. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$



27. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{k}$



28. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j}-\mathbf{k}$



29. (a) $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i}+4\mathbf{j}+4\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{64+16+16} = 2\sqrt{6}$

(b) $\mathbf{u} = \pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm\frac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$

30. (a) $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i}+4\mathbf{j}-2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{16+16+4} = 3$

(b) $\mathbf{u} = \pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm\frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

31. (a) $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i}+\mathbf{j} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$

(b) $\mathbf{u} = \pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = \pm\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$

32. (a) $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+9+1} = \frac{\sqrt{14}}{2}$

(b) $\mathbf{u} = \pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm\frac{1}{\sqrt{14}}(2\mathbf{i}+3\mathbf{j}+\mathbf{k})$

33. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which all have the same value, since the interchanging of two pair of rows in a determinant does not change its value $\Rightarrow$ the volume is

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

34. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$ (for details about verification, see Exercise 33)

35. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$ (for details about verification, see Exercise 33)

36. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8$ (for details about verification, see Exercise 33)

37. (a) $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = -81$, $\mathbf{v} \cdot \mathbf{w} = 18 \Rightarrow$ none

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$

$\Rightarrow$ **u** and **w** are parallel

38. (a) $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{u} \cdot \mathbf{r} = -3\pi$, $\mathbf{v} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{r} = 0$, $\mathbf{w} \cdot \mathbf{r} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{r}$ and $\mathbf{w} \perp \mathbf{r}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} = \mathbf{0}$

$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\1&0&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{v}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0},\ \mathbf{w}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0}$$

$\Rightarrow$ **u** and **r** are parallel

39. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right||\mathbf{F}|\sin(60^\circ)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{3}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{3}\text{ ft}\cdot\text{lb}$

40. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right||\mathbf{F}|\sin(135^\circ)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{2}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{2}\text{ ft}\cdot\text{lb}$

41. (a) true, $|\mathbf{u}|=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{\mathbf{u}\cdot\mathbf{u}}$

(b) not always true, $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

(c) true, $\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\0&0&0\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$

(d) true, $\mathbf{u}\times(-\mathbf{u})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\-a_1&-a_2&-a_3\end{vmatrix}=(-a_2a_3+a_2a_3)\mathbf{i}+(-a_1a_3+a_1a_3)\mathbf{j}+(-a_1a_2+a_1a_2)\mathbf{k}=\mathbf{0}$

(e) not always true, $\mathbf{i}\times\mathbf{j}=\mathbf{k}\neq-\mathbf{k}=\mathbf{j}\times\mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=\mathbf{u}\cdot\mathbf{0}=0$

(h) true, the volume of a parallelepiped with **u**, **v**, and **w** along the three edges is $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$ $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, since the dot product is commutative.

42. (a) true, $\mathbf{u}\cdot\mathbf{v}=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\mathbf{v}\cdot\mathbf{u}$

(b) true, $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\b_1&b_2&b_3\\a_1&a_2&a_3\end{vmatrix}=-(\mathbf{v}\times\mathbf{u})$

(c) true, $(-\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a_1&-a_2&-a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-(\mathbf{u}\times\mathbf{v})$

(d) true, $(c\mathbf{u})\cdot\mathbf{v}=(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=a_1(cb_1)+a_2(cb_2)+a_3(cb_3)=\mathbf{u}\cdot(c\mathbf{v})=c(a_1b_1+a_2b_2+a_3b_3)$ $=c(\mathbf{u}\cdot\mathbf{v})$

(e) true, $c(\mathbf{u}\times\mathbf{v})=c\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ca_1&ca_2&ca_3\\b_1&b_2&b_3\end{vmatrix}=(c\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\cb_1&cb_2&cb_3\end{vmatrix}=\mathbf{u}\times(c\mathbf{v})$

(f) true, $\mathbf{u}\cdot\mathbf{u} = a_1^2 + a_2^2 + a_3^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = |\mathbf{u}|^2$

(g) true, $(\mathbf{u}\times\mathbf{u})\cdot\mathbf{u} = \mathbf{0}\cdot\mathbf{u} = 0$

(h) true, $\mathbf{u}\times\mathbf{v} \perp \mathbf{u}$ and $\mathbf{u}\times\mathbf{v} \perp \mathbf{v} \Rightarrow (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$

43. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ (b) $\pm(\mathbf{u}\times\mathbf{v})$ (c) $\pm(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ (d) $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$

44. (a) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})$

(b) $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = (\mathbf{u}+\mathbf{v})\times\mathbf{u} - (\mathbf{u}+\mathbf{v})\times\mathbf{v} = \mathbf{u}\times\mathbf{u} + \mathbf{v}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} - \mathbf{v}\times\mathbf{v}$
$= \mathbf{0} + \mathbf{v}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} - \mathbf{0} = 2(\mathbf{v}\times\mathbf{u})$, or simply $\mathbf{u}\times\mathbf{v}$

(c) $|\mathbf{u}|\frac{\mathbf{v}}{|\mathbf{v}|}$ (d) $|\mathbf{u}\times\mathbf{w}|$

45. (a) yes, $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}$ are both vectors (b) no, $\mathbf{u}$ is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar
(c) yes, $\mathbf{u}$ and $\mathbf{u}\times\mathbf{w}$ are both vectors (d) no, $\mathbf{u}$ is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar

46. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is perpendicular to $\mathbf{u}\times\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \Rightarrow (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.

47. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i}+\mathbf{j} \neq -\mathbf{i}+\mathbf{j}$, but $\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{i}\times\mathbf{i} + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$ and $\mathbf{i}\times(-\mathbf{i}+\mathbf{j}) = -\mathbf{i}\times\mathbf{i} + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$.

48. Yes. If $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ and $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ and $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = 0$. Suppose now that $\mathbf{v} \neq \mathbf{w}$. Then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v}-\mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot(k\mathbf{u}) = k|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, it cannot be true that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

49. $\overrightarrow{AB} = -\mathbf{i}+\mathbf{j}$ and $\overrightarrow{AD} = -\mathbf{i}-\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 2$

50. $\overrightarrow{AB} = 7\mathbf{i}+3\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i}+5\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 29$

51. $\overrightarrow{AB} = 3\mathbf{i}-2\mathbf{j}$ and $\overrightarrow{AD} = 5\mathbf{i}+\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 13$

52. $\overrightarrow{AB} = 7\mathbf{i}-4\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i}+5\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 43$

53. $\overrightarrow{AB} = -2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{11}{2}$

54. $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2$

55. $\overrightarrow{AB} = 6\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 11\mathbf{i} - 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{25}{2}$

56. $\overrightarrow{AB} = 16\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i} + 4\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 42$

57. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$ and the triangle's area is

$\frac{1}{2}|\mathbf{u} \times \mathbf{v}| = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$. The applicable sign is (+) if the acute angle from $\mathbf{u}$ to $\mathbf{v}$ runs counterclockwise in the xy-plane, and (−) if it runs clockwise, because the area must be a nonnegative number.

58. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$. Now,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix}\mathbf{k} \Rightarrow \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\left|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)\right| = \frac{1}{2}\left|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)\right|$$

$= \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}$. The applicable sign ensures the area formula gives a nonnegative number.

10.3 LINES AND PLANES IN SPACE

1. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (3\mathbf{i} - 4\mathbf{j} - \mathbf{k}) + t(\mathbf{i} + \mathbf{j} + \mathbf{k}) = (3 + t)\mathbf{i} + (-4 + t)\mathbf{j} + (-1 + t)\mathbf{k}$
   Parametric form: $x = 3 + t$, $y = -4 + t$, $z = -1 + t$

2. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = (1 - 2t)\mathbf{i} + (2 - 2t)\mathbf{j} + (-1 + 2t)\mathbf{k}$
   Parametric form: $x = 1 - 2t$, $y = 2 - 2t$, $z = -1 + 2t$

3. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (-2\mathbf{i} + 0\mathbf{j} + 3\mathbf{k}) + t(5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}) = (-2 + 5t)\mathbf{i} + (5t)\mathbf{j} + (3 - 5t)\mathbf{k}$
   Parametric form: $x = -2 + 5t$, $y = 5t$, $z = 3 - 5t$

4. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{j} + \mathbf{k}) = 0\mathbf{i} + (2t)\mathbf{j} + (t)\mathbf{k}$
   Parametric form: $x = 0$, $y = 2t$, $z = t$

5. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = (3 + 2t)\mathbf{i} + (-2 - t)\mathbf{j} + (1 + 3t)\mathbf{k}$
   Parametric form: $x = (3 + 2t)$, $y = -2 - t$, $z = 1 + 3t$

6. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(0\mathbf{i} + 0\mathbf{j} + \mathbf{k}) = \mathbf{i} + \mathbf{j} + (1 + t)\mathbf{k}$
   Parametric form: $x = 1$, $y = 1$, $z = 1 + t$

7. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) + t(3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}) = (2 + 3t)\mathbf{i} + (4 + 7t)\mathbf{j} + (5 - 5t)\mathbf{k}$
   Parametric form: $x = 2 + 3t$, $y = 4 + 7t$, $z = 5 - 5t$

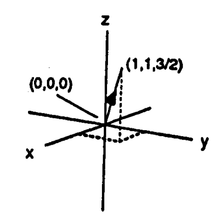
8. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} - 7\mathbf{j} + 0\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = (t)\mathbf{i} + (-7 + 2t)\mathbf{j} + (2t)\mathbf{k}$
   Parametric form: $x = t$, $y = -7 + 2t$, $z = 2t$

9. The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

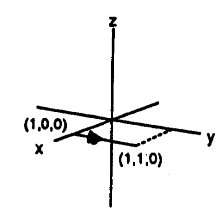
   Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{u} \times \mathbf{v}) = (2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) + t(-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = (2 - 2t)\mathbf{i} + (3 + 4t)\mathbf{j} + (-2t)\mathbf{k}$
   Parametric form: $x = 2 - 2t$, $y = 3 + 4t$, $z = -2t$

10. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) = t\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = t\mathbf{i}$
    Parametric form: $x = t$, $y = 0$, $z = 0$

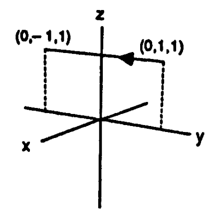
11. The direction $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0,0,0) \Rightarrow x = t$, $y = t$, $z = \frac{3}{2}t$,
    where $0 \le t \le 1$



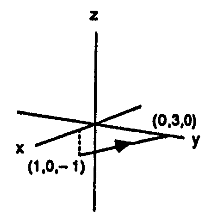
12. The direction $\overrightarrow{PQ} = \mathbf{j}$ and $P(1,1,0) \Rightarrow x = 1$, $y = 1 + t$, $z = 0$,
    where $-1 \le t \le 0$



13. The direction $\overrightarrow{PQ} = -2\mathbf{j}$ and $P(0,1,1) \Rightarrow x = 0,\ y = 1 - 2t,\ z = 1,$ where $0 \le t \le 1$



14. The direction $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1,0,-1) \Rightarrow x = 1 - t,$ $y = 3t,\ z = -1 + t,$ where $0 \le t \le 1$



15. $3(x-0) + (-2)(y-2) + (-1)(z+1) = 0 \Rightarrow 3x - 2y - z = -3$

16. $3(x-1) + (1)(y+1) + (1)(z-3) = 0 \Rightarrow 3x + y + z = 5$

17. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k},\ \overrightarrow{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane

$\Rightarrow 7(x-2) + (-5)(y-0) + (-4)(z-2) = 0 \Rightarrow 7x - 5y - 4z = 6$

18. $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \overrightarrow{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane

$\Rightarrow (-1)(x-1) + (-3)(y-5) + (1)(z-7) = 0 \Rightarrow x + 3y - z = 9$

19. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k},\ P(2,4,5) = (1)(x-2) + (3)(y-4) + (4)(z-5) = 0 \Rightarrow x + 3y + 4z = 34$

20. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},\ P(1,-2,1) = (1)(x-1) + (-2)(y+2) + (1)(z-1) = 0 \Rightarrow x - 2y + z = 6$

21. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0$ and $s = -1$; then $z = 4t + 3 = -4s - 1$

$\Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines do intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1,\ y = 2,$ and $z = 3$ or $P(1,2,3)$. A vector normal to the plane determined by these lines is

$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by$(-20)(x-1)+(12)(y-2)+(1)(z-3)=0 \Rightarrow -20x+12y+z=7$.

22. $\begin{cases} x = t \quad = 2s+2 \\ y = -t+2 = s+3 \end{cases} \Rightarrow \begin{cases} t-2s=2 \\ -t-s=1 \end{cases} \Rightarrow s=-1$ and $t=0$; then $z=t+1=5s+6 \Rightarrow 0+1=5(-1)+6$ is satisfied $\Rightarrow$ the lines do intersect when $s=-1$ and $t=0 \Rightarrow$ the point of intersection is $x=0$, $y=2$ and $z=1$ or $P(0,2,1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix}$ $= -6\mathbf{i}-3\mathbf{j}+3\mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x-0)+(-3)(y-2)+(3)(z-1)=0 \Rightarrow 6x+3y-3z=3$.

23. The cross product of $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $-4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ has the same direction as the normal to the plane $\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j}+6\mathbf{k}$. Select a point on either line, such as $P(-1,2,1)$. Since the lines are given to intersect, the desired plane is $0(x+1)+6(y-2)+6(z-1)=0 \Rightarrow 6y+6z=18 \Rightarrow y+z=3$.

24. The cross product of $\mathbf{i}-3\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ has the same direction as the normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$. Select a point on either line, such as $P(0,3,-2)$. Since the lines are given to intersect, the desired plane is $(-2)(x-0)+(-2)(y-3)+(4)(z+2)=0 \Rightarrow -2x-2y+4z=-14$ $\Rightarrow x+y-2z=7$.

25. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x-2)+(-3)(y-1)+3(z+1)=0 \Rightarrow 3x-3y+3z=0 \Rightarrow x-y+z=0$ is the desired plane containing $P_0(2,1,-1)$

26. A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i}-12\mathbf{j}-2\mathbf{k}$; choosing $P_1(1,2,3)$ as a point on the plane $\Rightarrow (-2)(x-1)+(-12)(y-2)+(-2)(z-3)=0 \Rightarrow -2x-12y-2z=-32 \Rightarrow x+6y+z=16$

is the desired plane

27. (a) If $\theta = 0$ or $\theta = \pi$, then S is on the line through P in the direction of $\mathbf{v}$. In these cases, the distance from the point S to the line is 0 and $|\overrightarrow{PS}| \sin\theta = 0$. If $0 < \theta < \frac{\pi}{2}$, then by the trigonometry of a right triangle, the distance from the point S to the line is $|\overrightarrow{PS}| \sin\theta$. If $\theta = \frac{\pi}{2}$ then $|\overrightarrow{PS}| \sin\theta = |\overrightarrow{PS}|$, which is the distance from S to the line. If $\frac{\pi}{2} < \theta < \pi$ then the distance from S to the line is $|\overrightarrow{PS}| \sin(\pi - \theta) = |\overrightarrow{PS}| \sin\theta$. Therefore, for all $0 \le \theta \le \pi$, the distance from the point S to the line through P and in the direction of $\mathbf{v}$ is given by $|\overrightarrow{PS}| \sin\theta$.

    (b) $|\overrightarrow{PS} \times \mathbf{v}| = |\overrightarrow{PS}||\mathbf{v}| \sin\theta \Rightarrow \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = |\overrightarrow{PS}| \sin\theta = d$

28. $S(0,0,0)$, $P(5,5,-3)$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} + 16\mathbf{j} - 5\mathbf{k}$

    $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169 + 256 + 25}}{\sqrt{9 + 16 + 25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3$ is the distance from S to the line

29. $S(2,1,3)$, $P(2,1,3)$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0$ is the distance from S to the line (i.e., the point S lies on the line)

30. $S(3,-1,4)$, $P(4,3,-5)$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$

    $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900 + 36 + 36}}{\sqrt{1 + 4 + 9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \frac{\sqrt{81 \cdot 6}}{\sqrt{7}} = \frac{9\sqrt{42}}{7}$ is the distance from S to the line

31. (a) P is any point with coordinates (x_0, y_0, z_0) that satisfies the equation $Ax_0 + By_0 + Cz_0 = D$.

    (b) If S has coordinates (x, y, z), then $\overrightarrow{PS} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$

    (c) If the angle between $\overrightarrow{PS}$ and the normal to the plane is θ, then $d = |\overrightarrow{PS}||\cos\theta|$. We know, however, that $\overrightarrow{PS} \cdot \overline{\mathbf{n}} = |\overrightarrow{PS}||\overline{\mathbf{n}}| \cos\theta$, where $\overline{\mathbf{n}} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane. This implies that $|\overrightarrow{PS}| \cos\theta = \frac{\overrightarrow{PS} \cdot \overline{\mathbf{n}}}{|\overline{\mathbf{n}}|} = \overrightarrow{PS} \cdot \frac{\overline{\mathbf{n}}}{|\overline{\mathbf{n}}|}$ and $\left||\overrightarrow{PS}| \cos\theta\right| = |\overrightarrow{PS}||\cos\theta| = \left|\overrightarrow{PS} \cdot \frac{\overline{\mathbf{n}}}{|\overline{\mathbf{n}}|}\right|$, and that $d = \left|\overrightarrow{PS} \cdot \frac{\overline{\mathbf{n}}}{|\overline{\mathbf{n}}|}\right|$.

32. $S(2,-3,4)$, $x + 2y + 2z = 13$ and $P(13,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = -11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

    $\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-11 - 6 + 8}{\sqrt{1 + 4 + 4}}\right| = \left|\frac{-9}{\sqrt{9}}\right| = 3$

33. $S(0,1,1)$, $4y+3z=-12$ and $P(0,-3,0)$ is on the plane $\Rightarrow \overrightarrow{PS}=4\mathbf{j}+\mathbf{k}$ and $\mathbf{n}=4\mathbf{j}+3\mathbf{k}$

$\Rightarrow d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{16+3}{\sqrt{16+9}}\right|=\frac{19}{5}$

34. $S(0,-1,0)$, $2x+y+2z=4$ and $P(2,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS}=-2\mathbf{i}-\mathbf{j}$ and $\mathbf{n}=2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$\Rightarrow d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{-4-1+0}{\sqrt{4+1+4}}\right|=\frac{5}{3}$

35. The point $P(1,0,0)$ is on the first plane and $S(10,0,0)$ is a point on the second plane $\Rightarrow \overrightarrow{PS}=9\mathbf{i}$, and $\mathbf{n}=\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$ $=\left|\frac{9}{\sqrt{1+4+36}}\right|=\frac{9}{\sqrt{41}}$, which is also the distance between the planes.

36. The line is parallel to the plane since $\mathbf{v}\cdot\mathbf{n}=\left(\mathbf{i}+\mathbf{j}-\frac{1}{2}\mathbf{k}\right)\cdot(\mathbf{i}+2\mathbf{j}+6\mathbf{k})=1+2-3=0$. Also the point $S(1,0,0)$ when $t=-1$ lies on the line, and the point $P(10,0,0)$ lies on the plane $\Rightarrow \overrightarrow{PS}=-9\mathbf{i}$. The distance from S to the plane is $d=\left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{-9}{\sqrt{1+4+36}}\right|=\frac{9}{\sqrt{41}}$, which is also the distance from the line to the plane.

37. (a) If two planes intersect, then the angle between the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is equal to the angle between the planes. We know, however, that $\mathbf{n}_1\cdot\mathbf{n}_2=|\mathbf{n}_1||\mathbf{n}_2|\cos\theta$, where θ is the angle between the normal vectors and between the planes. Therefore, $\theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$ with $0\le\theta\le\pi$.

(b) $\mathbf{n}_1=3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$ and $\mathbf{n}_2=2\mathbf{i}+\mathbf{j}-2\mathbf{k} \Rightarrow \theta=\cos^{-1}\left(\frac{6-6+4}{\sqrt{3^2+6^2+2^2}\sqrt{2^2+1^2+2^2}}\right)=\cos^{-1}\left(\frac{4}{21}\right)$

≈ 1.38 radians.

38. $\mathbf{n}_1=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{n}_2=\mathbf{k} \Rightarrow \theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)=\cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{1}}\right)\approx 0.96$ rad

39. $\mathbf{n}_1=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}+2\mathbf{j}+\mathbf{k} \Rightarrow \theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)=\cos^{-1}\left(\frac{2+4-1}{\sqrt{9}\sqrt{6}}\right)=\cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)\approx 0.82$ rad

40. $\mathbf{n}_1=4\mathbf{j}+3\mathbf{k}$ and $\mathbf{n}_2=3\mathbf{i}+2\mathbf{j}+6\mathbf{k} \Rightarrow \theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)=\cos^{-1}\left(\frac{8+18}{\sqrt{25}\sqrt{49}}\right)=\cos^{-1}\left(\frac{26}{35}\right)\approx 0.73$ rad

41. $2x-y+3z=6 \Rightarrow 2(1-t)-(3t)+3(1+t)=6 \Rightarrow -2t+5=6 \Rightarrow t=-\frac{1}{2} \Rightarrow x=\frac{3}{2}$, $y=-\frac{3}{2}$ and $z=\frac{1}{2}$

$\Rightarrow \left(\frac{3}{2},-\frac{3}{2},\frac{1}{2}\right)$ is the point

42. $6x+3y-4z=-12 \Rightarrow 6(2)+3(3+2t)-4(-2-2t)=-12 \Rightarrow 14t+29=-12 \Rightarrow t=-\frac{41}{14} \Rightarrow x=2$, $y=3-\frac{41}{7}$, and $z=-2+\frac{41}{7} \Rightarrow \left(2,-\frac{20}{7},\frac{27}{7}\right)$ is the point

43. $x+y+z=2 \Rightarrow (1+2t)+(1+5t)+(3t)=2 \Rightarrow 10t+2=2 \Rightarrow t=0 \Rightarrow x=1,\ y=1$ and $z=0$ $\Rightarrow (1,1,0)$ is the point

44. $2x-3z=7 \Rightarrow 2(-1+3t)-3(5t)=7 \Rightarrow -9t-2=7 \Rightarrow t=-1 \Rightarrow x=-1-3,\ y=-2$ and $z=-5$ $\Rightarrow (-4,-2,-5)$ is the point

45. $\mathbf{n}_1=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=-\mathbf{i}+\mathbf{j}$, the direction of the desired line; $(1,1,-1)$ is on both planes $\Rightarrow$ the desired line is $x=1-t,\ y=1+t,\ z=-1$

46. $\mathbf{n}_1=3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$ and $\mathbf{n}_2=2\mathbf{i}+\mathbf{j}-2\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -6 & -2\\ 2 & 1 & -2\end{vmatrix}=14\mathbf{i}+2\mathbf{j}+15\mathbf{k}$, the direction of the desired line; $(1,0,0)$ is on both planes $\Rightarrow$ the desired line is $x=1+14t,\ y=2t,\ z=15t$

47. $\mathbf{n}_1=\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}+\mathbf{j}-2\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 4\\ 1 & 1 & -2\end{vmatrix}=6\mathbf{j}+3\mathbf{k}$, the direction of the desired line; $(4,3,1)$ is on both planes $\Rightarrow$ the desired line is $x=4,\ y=3+6t,\ z=1+3t$

48. $\mathbf{n}_1=5\mathbf{i}-2\mathbf{j}$ and $\mathbf{n}_2=4\mathbf{j}-5\mathbf{k} \Rightarrow \mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & -2 & 0\\ 0 & 4 & -5\end{vmatrix}=10\mathbf{i}+25\mathbf{j}+20\mathbf{k}$, the direction of the desired line; $(1,-3,1)$ is on both planes $\Rightarrow$ the desired line is $x=1+10t,\ y=-3+25t,\ z=1+20t$

49. <u>L1 & L2</u>: $x=3+2t=1+4s$ and $y=-1+4t=1+2s \Rightarrow \begin{cases}2t-4s=-2\\ 4t-2s=2\end{cases} \Rightarrow \begin{cases}2t-4s=-2\\ 2t-s=1\end{cases}$
$\Rightarrow -3s=-3 \Rightarrow s=1$ and $t=1 \Rightarrow$ on L1, $z=1$ and on L2, $z=1 \Rightarrow$ L1 and L2 intersect at $(5,3,1)$.

<u>L2 & L3</u>: The direction of L2 is $\frac{1}{6}(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})=\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ of L3; hence L2 and L3 are parallel.

<u>L1 & L3</u>: $x=3+2t=3+2r$ and $y=-1+4t=2+r \Rightarrow \begin{cases}2t-2r=0\\ 4t-r=3\end{cases} \Rightarrow \begin{cases}t-r=0\\ 4t-r=3\end{cases} \Rightarrow 3t=3$
$\Rightarrow t=1$ and $r=1 \Rightarrow$ on L1, $z=2$ while on L3, $z=0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i}+4\mathbf{j}-\mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and neither is a multiple of the other; hence L1 and L3 are skew.

50. <u>L1 & L2</u>: $x = 1 + 2t = 2 - s$ and $y = -1 - t = 3s \Rightarrow \begin{cases} 2t + s = 1 \\ -t - 3s = 1 \end{cases} \Rightarrow -5s = 3 \Rightarrow s = -\frac{3}{5}$ and $t = \frac{4}{5} \Rightarrow$ on L1, $z = \frac{12}{5}$ while on L2, $z = 1 - \frac{3}{5} = \frac{2}{5} \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ while the direction of L2 is $\frac{1}{\sqrt{11}}(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and neither is a multiple of the other; hence, L1 and L2 are skew.

<u>L2 & L3</u>: $x = 2 - s = 5 + 2r$ and $y = 3s = 1 - r \Rightarrow \begin{cases} -s - 2r = 3 \\ 3s + r = 1 \end{cases} \Rightarrow 5s = 5 \Rightarrow s = 1$ and $r = -2 \Rightarrow$ on L2, $z = 2$ and on L3, $z = 2 \Rightarrow$ L2 and L3 intersect at $(1, 3, 2)$.

<u>L1 & L3</u>: L1 and L3 have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$; hence L1 and L3 are parallel.

51. $x = 2 + 2t,\ y = -4 - t,\ z = 7 + 3t;\ x = -2 - t,\ y = -2 + \frac{1}{2}t,\ z = 1 - \frac{3}{2}t$

52. $1(x - 4) - 2(y - 1) + 1(z - 5) = 0 \Rightarrow x - 4 - 2y + 2 + z - 5 = 0 \Rightarrow x - 2y + z = 7;$
$-\sqrt{2}(x - 3) + 2\sqrt{2}(y + 2) - \sqrt{2}(z - 0) = 0 \Rightarrow -\sqrt{2}x + 2\sqrt{2}y - \sqrt{2}z = -7\sqrt{2}$

53. $x = 0 \Rightarrow t = -\frac{1}{2},\ y = -\frac{1}{2},\ z = -\frac{3}{2} \Rightarrow \left(0, -\frac{1}{2}, -\frac{3}{2}\right);\ y = 0 \Rightarrow t = -1,\ x = -1,\ z = -3 \Rightarrow (-1, 0, -3);\ z = 0$
$\Rightarrow t = 0,\ x = 1,\ y = -1 \Rightarrow (1, -1, 0)$

54. The line contains $(0, 0, 3)$ and $(\sqrt{3}, 1, 3)$ because the projection of the line onto the xy-plane contains the origin and intersects the positive x-axis at a 30° angle. The direction of the line is $\sqrt{3}\mathbf{i} + \mathbf{j} + 0\mathbf{k} \Rightarrow$ the line in question is $x = \sqrt{3}t,\ y = t,\ z = 3$.

55. With substitution of the line into the plane we have $2(1 - 2t) + (2 + 5t) - (-3t) = 8 \Rightarrow 2 - 4t + 2 + 5t + 3t = 8$
$\Rightarrow 4t + 4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

56. The planes are parallel when either vector $A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ or $A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ is a multiple of the other or when $|(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \times (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}| = 0$. The planes are perpendicular when their normals are perpendicular, or$(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \cdot (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0$.

57. There are many possible answers. One is found as follows: eliminate t to get $t = x - 1 = 2 - y = \frac{z - 3}{2}$
$\Rightarrow x - 1 = 2 - y$ and $2 - y = \frac{z - 3}{2} \Rightarrow x + y = 3$ and $2y + z = 7$ are two such planes.

58. Since the plane passes through the origin, its general equation is of the form $Ax + By + Cz = 0$. Since it meets the plane M at a right angle, their normal vectors are perpendicular $\Rightarrow 2A + 3B + C = 0$. One choice satisfying this equation is $A = 1$, $B = -1$ and $C = 1 \Rightarrow x - y + z = 0$. Any plane $Ax + By + Cz = 0$ with $2A + 3B + C = 0$ will pass through the origin and be perpendicular to M.

59. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the x, y, and z intercepts of the plane. Since a, b, and c are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes <u>except</u> those through the origin or parallel to a coordinate axis.

60. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.

61. (a) $\overrightarrow{EP} = c\overrightarrow{EP}_1 \Rightarrow -x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c[(x_1 - x_0)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}] \Rightarrow -x_0 = c(x_1 - x_0)$, $y = cy_1$ and $z = cz_1$, where c is a positive real number

(b) At $x_1 = 0 \Rightarrow c = 1 \Rightarrow y = y_1$ and $z = z_1$; at $x_1 = x_0 \Rightarrow x_0 = 0$, $y = 0$, $z = 0$; $\lim\limits_{x_0 \to \infty} c = \lim\limits_{x_0 \to \infty} \frac{-x_0}{x_1 - x_0}$ $= \lim\limits_{x_0 \to \infty} \frac{-1}{-1} = 1 \Rightarrow c \to 1$ so that $y \to y_1$ and $z \to z_1$

62. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t$, $y = 2t$, $z = 2t$. The plane and the line intersect at $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. The visible section of the line segment is $\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow \frac{2}{3}$ of the line segment is hidden from view.

10.4 CYLINDERS AND QUADRIC SURFACES

1. d, ellipsoid
2. i, hyperboloid
3. a, cylinder
4. g, cone
5. l, hyperbolic paraboloid
6. e, paraboloid
7. b, cylinder
8. j, hyperboloid
9. k, hyperbolic paraboloid
10. f, paraboloid
11. h, cone
12. c, ellipsoid

13. (a) If $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $z = c$, then $x^2 + \frac{y^2}{4} = \frac{9 - c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9 - c^2}{9}\right)} + \frac{y^2}{\left[\frac{4(9 - c^2)}{9}\right]} = 1 \Rightarrow A = ab\pi$ $= \pi\left(\frac{\sqrt{9 - c^2}}{3}\right)\left(\frac{2\sqrt{9 - c^2}}{3}\right) = \frac{2\pi(9 - c^2)}{9}$

(b) From part (a), each slice has the area $\frac{2\pi(9 - z^2)}{9}$, where $-3 \le z \le 3$. Thus $V = 2\int_0^3 \frac{2\pi}{9}(9 - z^2)\,dz$ $= \frac{4\pi}{9}\int_0^3 (9 - z^2)\,dz = \frac{4\pi}{9}\left[9z - \frac{z^3}{3}\right]_0^3 = \frac{4\pi}{9}(27 - 9) = 8\pi$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{\left[\frac{a^2(c^2 - z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2 - z^2)}{c^2}\right]} = 1 \Rightarrow A = \pi\left(\frac{a\sqrt{c^2 - z^2}}{c}\right)\left(\frac{b\sqrt{c^2 - z^2}}{c}\right)$ $\Rightarrow V = 2\int_0^c \frac{\pi ab}{c^2}(c^2 - z^2)\,dz = \frac{2\pi ab}{c^2}\left[c^2 z - \frac{z^3}{3}\right]_0^c = \frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right) = \frac{4\pi abc}{3}$. Note that if $r = a = b = c$, then $V = \frac{4\pi r^3}{3}$, which is the volume of a sphere.

14. The ellipsoid has the form $\frac{x^2}{R^2}+\frac{y^2}{R^2}+\frac{z^2}{c^2}=1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface of the barrel. Thus, $\frac{r^2}{R^2}+\frac{h^2}{c^2}=1 \Rightarrow c^2=\frac{h^2R^2}{R^2-r^2}$. We calculate the volume by the disk method:

$V=\pi\int_{-h}^{h} y^2\,dz$. Now, $\frac{y^2}{R^2}+\frac{z^2}{c^2}=1 \Rightarrow y^2=R^2\left(1-\frac{z^2}{c^2}\right)=R^2\left[1-\frac{z^2(R^2-r^2)}{h^2R^2}\right]=R^2-\left(\frac{R^2-r^2}{h^2}\right)z^2$

$\Rightarrow V=\pi\int_{-h}^{h}\left[R^2-\left(\frac{R^2-r^2}{h^2}\right)z^2\right]dz=\pi\left[R^2z-\frac{1}{3}\left(\frac{R^2-r^2}{h^2}\right)z^3\right]_{-h}^{h}=2\pi\left[R^2h-\frac{1}{3}(R^2-r^2)h\right]=2\pi\left(\frac{2R^2h}{3}+\frac{r^2h}{3}\right)$

$=\frac{4}{3}\pi R^2h+\frac{2}{3}\pi r^2h$, the volume of the barrel. If $r=R$, then $V=2\pi R^2h$ which is the volume of a cylinder of radius R and height 2h. If $r=0$ and $h=R$, then $V=\frac{4}{3}\pi R^3$ which is the volume of a sphere.

15. We calculate the volume by the slicing method, taking slices parallel to the xy-plane. For fixed z, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)}+\frac{y^2}{\left(\frac{zb^2}{c}\right)}=1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right)=\frac{\pi abz}{c}$ (see Exercise 13a). Hence the volume is given by $V=\int_0^h \frac{\pi abz}{c}\,dz=\left[\frac{\pi abz^2}{2c}\right]_0^h=\frac{\pi abh^2}{2c}$. Now the area of the elliptic base when $z=h$ is $A=\frac{\pi abh}{c}$, as determined previously. Thus, $V=\frac{\pi abh^2}{2c}=\frac{1}{2}\left(\frac{\pi abh}{c}\right)h=\frac{1}{2}$(base)(altitude), as claimed.

16. (a) For each fixed value of z, the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ results in a cross-sectional ellipse

$\frac{x^2}{\left[\frac{a^2(c^2+z^2)}{c^2}\right]}+\frac{y^2}{\left[\frac{b^2(c^2+z^2)}{c^2}\right]}=1$. The area of the cross-sectional ellipse (see Exercise 13a) is

$A(z)=\pi\left(\frac{a}{c}\sqrt{c^2+z^2}\right)\left(\frac{b}{c}\sqrt{c^2+z^2}\right)=\frac{\pi ab}{c^2}(c^2+z^2)$. The volume of the solid by the method of slices is

$V=\int_0^h A(z)\,dz=\int_0^h \frac{\pi ab}{c^2}(c^2+z^2)\,dz=\frac{\pi ab}{c^2}\left[c^2z+\frac{1}{3}z^3\right]_0^h=\frac{\pi ab}{c^2}\left(c^2h+\frac{1}{3}h^3\right)=\frac{\pi abh}{3c^2}(3c^2+h^2)$

(b) $A_0=A(0)=\pi ab$ and $A_h=A(h)=\frac{\pi ab}{c^2}(c^2+h^2)$, from part (a) $\Rightarrow V=\frac{\pi abh}{3c^2}(3c^2+h^2)$

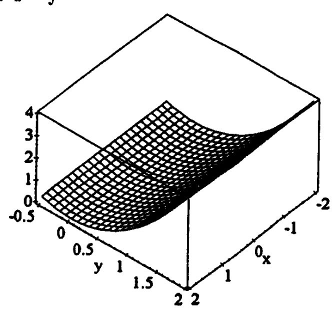
$=\frac{\pi abh}{3}\left(2+1+\frac{h^2}{c^2}\right)=\frac{\pi abh}{3}\left(2+\frac{c^2+h^2}{c^2}\right)=\frac{h}{3}\left[2\pi ab+\frac{\pi ab}{c^2}(c^2+h^2)\right]=\frac{h}{3}(2A_0+A_h)$

(c) $A_m=A\left(\frac{h}{2}\right)=\frac{\pi ab}{c^2}\left(c^2+\frac{h^2}{4}\right)=\frac{\pi ab}{4c^2}(4c^2+h^2) \Rightarrow \frac{h}{6}(A_0+4A_m+A_h)$

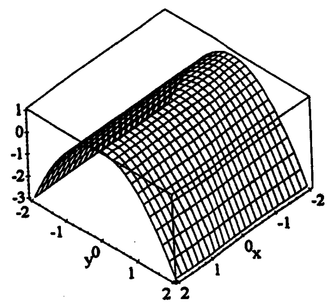
$=\frac{h}{6}\left[\pi ab+\frac{\pi ab}{c^2}(4c^2+h^2)+\frac{\pi ab}{c^2}(c^2+h^2)\right]=\frac{\pi abh}{6c^2}(c^2+4c^2+h^2+c^2+h^2)=\frac{\pi abh}{6c^2}(6c^2+2h^2)$

$=\frac{\pi abh}{3c^2}(3c^2+h^2)=V$ from part (a)

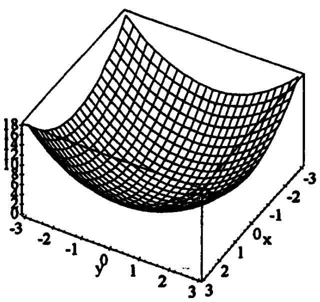
17. $z = y^2$



18. $z = 1 - y^2$

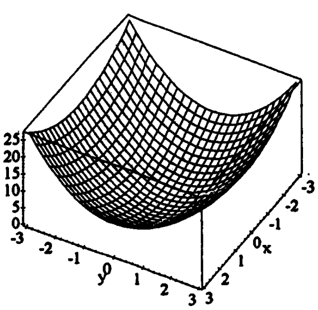


19. $z = x^2 + y^2$

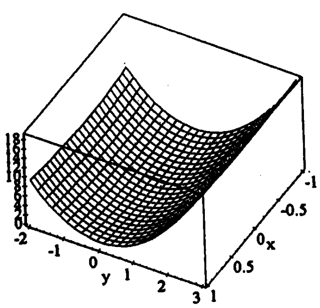


20. $z = x^2 + 2y^2$

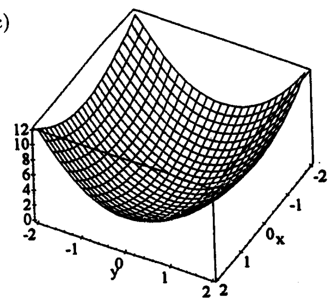
(a)



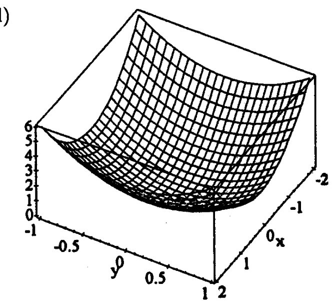
(b)



(c)



(d)



21-26. Example CAS commands:

Maple:
```
with(plots):
eq1:= x^2/9 - y^2/16 -z^2/2 = 1;
implicitplot3d(eq1, x = -15..15, y = -9..9, z = -7..7, title = `Hyperboloid of Two Sheets`);
```

Mathematica:
```
ContourPlot3D[ x^2/9 - y^2/16 - z^2/2 - 1,
{x,-9,9}, {y,-12,12}, {z,-5,5},
PlotLabel -> "Elliptic Hyperboloid of Two Sheets" ]
```

10.5 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

1. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$;

Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + 2(1)\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

2. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)|$

$= \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1)$

$= 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$

3. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$;

Speed: $\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(3\cos\frac{\pi}{2}\right)^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|}$

$$= \left(-\frac{2}{2\sqrt{5}}\sin\frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}}\cos\frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}\right)$$

4. $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j} + \frac{4}{3}\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$

$= (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2\sec^2 t \tan t)\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right| = \sqrt{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)^2 + \left(\sec^2\frac{\pi}{6}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$;

Direction: $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|} = \frac{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)\mathbf{i} + \left(\sec^2\frac{\pi}{6}\right)\mathbf{j} + \frac{4}{3}\mathbf{k}}{2} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

5. $\mathbf{r} = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;

Speed: $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}}$

$= \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$

6. $\mathbf{r} = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\mathbf{i} - (6\sin 3t)\mathbf{j} + (6\cos 3t)\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$

$= (e^{-t})\mathbf{i} - (18\cos 3t)\mathbf{j} - (18\sin 3t)\mathbf{k}$; Speed: $|\mathbf{v}(0)| = \sqrt{(-e^0)^2 + [-6\sin 3(0)]^2 + [6\cos 3(0)]^2} = \sqrt{37}$;

Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\mathbf{i} - 6\sin 3(0)\mathbf{j} + 6\cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$

7. $\mathbf{v} = 3\mathbf{i} + \sqrt{3}\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{k} \Rightarrow |\mathbf{v}(0)| = \sqrt{3^2 + (\sqrt{3})^2 + 0^2} = \sqrt{12}$ and $|\mathbf{a}(0)| = \sqrt{2^2} = 2$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

8. $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\mathbf{j}$ and $\mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}(0) = -32\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$

$= 1$ and $|\mathbf{a}(0)| = \sqrt{(-32)^2} = 32$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \left(\frac{\sqrt{2}}{2}\right)(-32) = -16\sqrt{2} \Rightarrow \cos\theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}$

9. $\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-1/2}\mathbf{k}$ and $\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \Rightarrow |\mathbf{v}(0)| = 1$ and $|\mathbf{a}(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

10. $\mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and

$\mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \frac{2}{9} - \frac{2}{9}$

$= 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

11. $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{a} = (\sin t)(1 - \cos t) + (\sin t)(\cos t) = \sin t$. Thus, $\mathbf{v}\cdot\mathbf{a} = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0,\ \pi$, or 2π

12. $\mathbf{v} = (\cos t)\mathbf{i} + \mathbf{j} - (\sin t)\mathbf{k}$ and $\mathbf{a} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{k} \Rightarrow \mathbf{v}\cdot\mathbf{a} = -\sin t\cos t + \sin t\cos t = 0$ for all $t \ge 0$

13. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt = \left[\frac{t^4}{4}\right]_0^1\mathbf{i} + [7t]_0^1\mathbf{j} + \left[\frac{t^2}{2} + t\right]_0^1\mathbf{k} = \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}$

14. $\int_1^2 \left[(6 - 6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right]dt = \left[6t - 3t^2\right]_1^2\mathbf{i} + \left[2t^{3/2}\right]_1^2\mathbf{j} + \left[-4t^{-1}\right]_1^2\mathbf{k} = -3\mathbf{i} + \left(4\sqrt{2} - 2\right)\mathbf{j} + 2\mathbf{k}$

15. $\int_{-\pi/4}^{\pi/4} \left[(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + \left(\sec^2 t\right)\mathbf{k}\right]dt = [-\cos t]_{-\pi/4}^{\pi/4}\mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4}\mathbf{j} + [\tan t]_{-\pi/4}^{\pi/4}\mathbf{k}$

$= \left(\frac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + 2\mathbf{k}$

16. $\int_0^{\pi/3} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\,\mathbf{k}]\,dt = \int_0^{\pi/3} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}]\,dt$

$= [\sec t]_0^{\pi/3}\mathbf{i} + [-\ln(\cos t)]_0^{\pi/3}\mathbf{j} + \left[-\frac{1}{2}\cos 2t\right]_0^{\pi/3}\mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4}\mathbf{k}$

17. $\int_1^4 \left(\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right)dt = = [\ln t]_1^4\mathbf{i} + [-\ln(5-t)]_1^4\mathbf{j} + \left[\frac{1}{2}\ln t\right]_1^4\mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$

18. $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\mathbf{j} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right)dt = \left[2\sin^{-1} t\right]_0^1\mathbf{i} + \left[\sqrt{3}\tan^{-1} t\right]_0^1\mathbf{k} = \pi\mathbf{i} + \frac{\pi\sqrt{3}}{4}\mathbf{k}$

19. $\mathbf{r} = \int (-t\mathbf{i} - t\mathbf{j} - t\mathbf{k})\,dt = -\frac{t^2}{2}\mathbf{i} - \frac{t^2}{2}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1\right)\mathbf{i} + \left(-\frac{t^2}{2} + 2\right)\mathbf{j} + \left(-\frac{t^2}{2} + 3\right)\mathbf{k}$

20. $\mathbf{r} = \int \left[(180t)\mathbf{i} + \left(180t - 16t^2\right)\mathbf{j}\right]dt = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3\right)\mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 90(0)^2\mathbf{i} + \left[90(0)^2 - \frac{16}{3}(0)^3\right]\mathbf{j} + \mathbf{C}$

$= 100\mathbf{j} \Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\mathbf{j}$

21. $\mathbf{r} = \int \left[\left(\frac{3}{2}(t+1)^{1/2}\right)\mathbf{i} + e^{-t}\mathbf{j} + \left(\frac{1}{t+1}\right)\mathbf{k}\right] dt = (t+1)^{3/2}\mathbf{i} - e^{-t}\mathbf{j} + \ln(t+1)\mathbf{k} + \mathbf{C}$;

$\mathbf{r}(0) = (0+1)^{3/2}\mathbf{i} - e^{-0}\mathbf{j} + \ln(0+1)\mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{r} = \left[(t+1)^{3/2} - 1\right]\mathbf{i} + \left(1 - e^{-t}\right)\mathbf{j} + [1 + \ln(t+1)]\mathbf{k}$

22. $\mathbf{r} = \int [(t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}]\, dt = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{2t^3}{3}\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = \left[\frac{0^4}{4} + 2(0)^2\right]\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{2(0)^3}{3}\mathbf{k} + \mathbf{C}$

$= \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j} + \frac{2t^3}{3}\mathbf{k}$

23. $\frac{d\mathbf{r}}{dt} = \int (-32\mathbf{k})\, dt = -32t\mathbf{k} + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$

$\Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}$; $\mathbf{r} = \int (8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k})\, dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = 100\mathbf{k}$

$\Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2\mathbf{k} + \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + \left(100 - 16t^2\right)\mathbf{k}$

24. $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})$; $\mathbf{r} = \int -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\, dt = -\left(\frac{t^2}{2}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}\right) + \mathbf{C}_2$; $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$

$\Rightarrow -\left(\frac{0^2}{2}\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{0^2}{2}\mathbf{k}\right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$

$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$

25. $\mathbf{r}(t) = (\sin t)\mathbf{i} + \left(t^2 - \cos t\right)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}$; $t_0 = 0 \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(0) = P_0 = (0, -1, 1) \Rightarrow x = 0 + t = t$, $y = -1$, and $z = 1 + t$ are parametric equations of the tangent line

26. $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v}(t) = (2 \cos t)\mathbf{i} - (2 \sin t)\mathbf{j} + 5\mathbf{k}$; $t_0 = 4\pi \Rightarrow \mathbf{v}(0) = 2\mathbf{i} + 5\mathbf{k}$ and $\mathbf{r}(0) = P_0 = (0, 2, 20\pi) \Rightarrow x = 0 + 2t = 2t$, $y = 2$, and $z = 20\pi + 5t$ are parametric equations of the tangent line

27. $\mathbf{r}(t) = (a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v}(t) = (a \cos t)\mathbf{i} - (a \sin t)\mathbf{j} + b\mathbf{k}$; $t_0 = 2\pi \Rightarrow \mathbf{v}(0) = a\mathbf{i} + b\mathbf{k}$ and $\mathbf{r}(0) = P_0 = (0, a, 2b\pi) \Rightarrow x = 0 + at = at$, $y = a$, and $z = 2\pi b + bt$ are parametric equations of the tangent line

28. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2 \cos 2t)\mathbf{k}$; $t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(0) = -\mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}(0) = P_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t$, $y = 1$, and $z = 0 - 2t = -2t$ are parametric equations of the tangent line

29. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed $2 \Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}$

$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{22}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$

$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k}$

$= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

30. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2 $\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$

$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$

$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

31. $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1-\cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 4 \Rightarrow \max |\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 0 \Rightarrow \min |\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max |\mathbf{a}| = \min |\mathbf{a}| = \sqrt{1} = 1$

32. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2,2,1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $(2,2,1)$ the vector $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ traces out a circle of radius 1 and center $(2,2,1)$ in the plane $x + y - 2z = 2$.

33. $\mathbf{v} = (-3\sin t)\mathbf{j} + (2\cos t)\mathbf{k}$ and $\mathbf{a} = (-3\cos t)\mathbf{j} - (2\sin t)\mathbf{k}$; $|\mathbf{v}|^2 = 9\sin^2 t + 4\cos^2 t \Rightarrow \frac{d}{dt}(|\mathbf{v}|^2)$ $= 18 \sin t \cos t - 8 \cos t \sin t = 10 \sin t \cos t$; $\frac{d}{dt}(|\mathbf{v}|^2) = 0 \Rightarrow 10 \sin t \cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$ $\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{v}|^2 = 4 \Rightarrow |\mathbf{v}| = \sqrt{4} = 2$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{v}| = \sqrt{9} = 3$. Therefore $\max |\mathbf{v}|$ is 3 when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, and $\min |\mathbf{v}| = 2$ when $t = 0, \pi$. Next, $|\mathbf{a}|^2 = 9\cos^2 t + 4\sin^2 t$ $\Rightarrow \frac{d}{dt}(|\mathbf{a}|^2) = -18 \cos t \sin t + 8 \sin t \cos t = -10 \sin t \cos t$; $\frac{d}{dt}(|\mathbf{a}|^2) = 0 \Rightarrow -10 \sin t \cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0 \Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{a}|^2 = 9 \Rightarrow |\mathbf{a}| = 3$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{a}|^2 = 4 \Rightarrow |\mathbf{a}| = 2$. Therefore, $\max |\mathbf{a}| = 3$ when $t = 0, \pi$, and $\min |\mathbf{a}| = 2$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

34. $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2 \cdot 0 = 0 \Rightarrow \mathbf{v} \cdot \mathbf{v}$ is a constant $\Rightarrow |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is constant

35. $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

36. (a) $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c\,\frac{df}{dt}\mathbf{i} + c\,\frac{dg}{dt}\mathbf{j} + c\,\frac{dh}{dt}\mathbf{k}$
$= c\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = c\,\frac{d\mathbf{u}}{dt}$

(b) $f\mathbf{u} = ff(t)\mathbf{i} + fg(t)\mathbf{j} + fh(t)\mathbf{k} \Rightarrow \frac{d}{dt}(f\mathbf{u}) = \left[\frac{df}{dt}f(t) + f\,\frac{df}{dt}\right]\mathbf{i} + \left[\frac{df}{dt}g(t) + f\,\frac{dg}{dt}\right]\mathbf{j} + \left[\frac{df}{dt}h(t) + f\,\frac{dh}{dt}\right]\mathbf{k}$
$= \frac{df}{dt}[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] + f\left[\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right] = \frac{df}{dt}\mathbf{u} + f\,\frac{d\mathbf{u}}{dt}$

37. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$. Then
$\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f_1'(t) + g_1'(t)]\mathbf{i} + [f_2'(t) + g_2'(t)]\mathbf{j} + [f_3'(t) + g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] + [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$;
$\mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f_1'(t) - g_1'(t)]\mathbf{i} + [f_2'(t) - g_2'(t)]\mathbf{j} + [f_3'(t) - g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] - [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

38. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim\limits_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim\limits_{t\to t_0} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
$= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim\limits_{t\to t_0} f(t) = f(t_0),\ \lim\limits_{t\to t_0} g(t) = g(t_0)$, and $\lim\limits_{t\to t_0} h(t) = h(t_0) \Leftrightarrow$ f, g, and h are continuous at $t = t_0$.

39. $$\lim_{t\to t_0} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \lim_{t\to t_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lim\limits_{t\to t_0} f_1(t) & \lim\limits_{t\to t_0} f_2(t) & \lim\limits_{t\to t_0} f_3(t) \\ \lim\limits_{t\to t_0} g_1(t) & \lim\limits_{t\to t_0} g_2(t) & \lim\limits_{t\to t_0} g_3(t) \end{vmatrix}$$
$= \lim\limits_{t\to t_0} \mathbf{r}_1(t) \times \lim\limits_{t\to t_0} \mathbf{r}_2(t) = \mathbf{u} \times \mathbf{v}$

40. $\mathbf{r}'(t_0)$ exists $\Rightarrow f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow f'(t_0),\ g'(t_0),\ h'(t_0)$ all exist $\Rightarrow$ f, g, and h are continuous at $t = t_0 \Rightarrow \mathbf{r}(t)$ is continuous at $t = t_0$

41. (a) $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{v}\times\frac{d\mathbf{w}}{dt}\right)$
$= \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}$

(b) Each of the determinants is equivalent to each expression in Eq. (7) in part (a) because of the determinant formula for the Triple Scalar Product in Section 10.2.

42. $\frac{d}{dt}\left[\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right] = \frac{d\mathbf{r}}{dt}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$, since $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ and $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = 0$ for any vectors $\mathbf{u}$ and $\mathbf{v}$

43. (a) $\int_a^b k\mathbf{r}(t)\,dt = \int_a^b [kf(t)\mathbf{i} + kg(t)\mathbf{j} + kh(t)\mathbf{k}]\,dt = \int_a^b [kf(t)]\,dt\,\mathbf{i} + \int_a^b [kg(t)]\,dt\,\mathbf{j} + \int_a^b [kh(t)]\,dt\,\mathbf{k}$

$= k\left(\int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}\right) = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)]\,dt = \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}])\,dt$

$= \int_a^b ([f_1(t) \pm f_2(t)]\mathbf{i} + [g_1(t) \pm g_2(t)]\mathbf{j} + [h_1(t) \pm h_2(t)]\mathbf{k})\,dt$

$= \int_a^b [f_1(t) \pm f_2(t)]\,dt\,\mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)]\,dt\,\mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)]\,dt\,\mathbf{k}$

$= \left[\int_a^b f_1(t)\,dt\,\mathbf{i} \pm \int_a^b f_2(t)\,dt\,\mathbf{i}\right] + \left[\int_a^b g_1(t)\,dt\,\mathbf{j} \pm \int_a^b g_2(t)\,dt\,\mathbf{j}\right] + \left[\int_a^b h_1(t)\,dt\,\mathbf{k} \pm \int_a^b h_2(t)\,dt\,\mathbf{k}\right]$

$= \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$

(c) Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)]\,dt$

$= c_1\int_a^b f(t)\,dt + c_2\int_a^b g(t)\,dt + c_3\int_a^b h(t)\,dt = \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt;$

$\int_a^b \mathbf{C}\times\mathbf{r}(t)\,dt = \int_a^b [c_2h(t) - c_3g(t)]\mathbf{i} + [c_3f(t) - c_1h(t)]\mathbf{j} + [c_1g(t) - c_2f(t)]\mathbf{k}\,dt$

$= \left[c_2\int_a^b h(t)\,dt - c_3\int_a^b g(t)\,dt\right]\mathbf{i} + \left[c_3\int_a^b f(t)\,dt - c_1\int_a^b h(t)\,dt\right]\mathbf{j} + \left[c_1\int_a^b g(t)\,dt - c_2\int_a^b f(t)\,dt\right]\mathbf{k}$

$= \mathbf{C}\times\int_a^b \mathbf{r}(t)\,dt$

44. (a) Let u and $\mathbf{r}$ be continuous on $[a,b]$. Then $\lim_{t\to t_0} u(t)\mathbf{r}(t) = \lim_{t\to t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r}$ is continuous for every t_0 in $[a,b]$.

(b) Let u and $\mathbf{r}$ be differentiable. Then $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt}\right)\mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt}\right)\mathbf{k}$

$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\frac{du}{dt} + u(t)\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = \mathbf{r}\frac{du}{dt} + u\frac{d\mathbf{r}}{dt}$

45. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on $\mathbf{i}$, then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$ $= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt}, \frac{dg_1}{dt} = \frac{dg_2}{dt}, \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1, g_1(t) = g_2(t) + c_2, h_1(t) = h_2(t) + c_3$ $\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$, where $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on $\mathbf{i}$. Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on $\mathbf{i}$, then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on $\mathbf{i} \Rightarrow \mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

46. $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \frac{d}{dt}\int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}]\,d\tau = \frac{d}{dt}\int_a^t f(\tau)\,d\tau\,\mathbf{i} + \frac{d}{dt}\int_a^t g(\tau)\,d\tau\,\mathbf{j} + \frac{d}{dt}\int_a^t h(\tau)\,d\tau\,\mathbf{k}$

$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t)$. Since $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$, we have that $\int_a^t \mathbf{r}(\tau)\,d\tau$ is an antiderivative of $\mathbf{r}$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ by Exercise 45(b). Then $\mathbf{R}(a) = \int_a^a \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ $= \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau)\,d\tau = \mathbf{R}(b) - \mathbf{R}(a)$.

47-50. Example CAS commands:

Maple:
```
with(plots):
x:= t -> sin(t) - t*cos(t);
y:= t -> cos(t) + t*sin(t);
z:= t -> t^2;
s1:= spacecurve([x(t),y(t),z(t)], t=0..6*Pi, numpoints = 120, axes=NORMAL):
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
dz:= t -> D(z)(t);
t0:= 3*Pi/2:
s2:=spacecurve([x(t0)+t*dx(t0),y(t0)+t*dy(t0),z(t0)+t*dz(t0),t=-2..2]):
display({s1,s2},title = `Space Curve and Tangent Line at t0=3 Pi/2`);
```

Mathematica:
```
Clear[x,y,z,t]
r[t_] = {x[t],y[t],z[t]}
x[t_] = Sin[t] - t Cos[t]
y[t_] = Cos[t] + t Sin[t]
z[t_] = t^2
{a,b} = {0, 6 Pi};
t0 = 3/2 Pi;
p1 = ParametricPlot3D[ {x[t],y[t],z[t]}, {t,a,b} ]
v[t_] = r'[t]
v0 = v[t0]
```

```
line[t_] = r[t0] + t v0
p2 = ParametricPlot3D[ Evaluate[ line[t] ], {t,-2,2} ]
Show[ p1, p2 ]
```

51-52. Example CAS commands:

Maple:

```
with(plots):
x:= t -> cos(a*t):
y:= t -> sin(a*t):
z:= t -> b*t:  a:=2: b:= 1:
s1:=spacecurve([x(t),y(t),z(t)], t=0..4*Pi, numpoints = 400, axes=NORMAL):
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
dz:= t -> D(z)(t);
t0:= 3*Pi/2:
s2:=spacecurve([x(t0)+t*dx(t0),y(t0)+t*dy(t0),z(t0)+t*dz(t0),t=-2..2]):
display({s1,s2},title = ` Helix With a = 2 and b = 1`);
```

Mathematica:

```
Clear[a,b]
x[t_] = Cos[a t]
y[t_] = Sin[a t]
z[t_] = b t
t0 = 3/2 Pi;
v[t_] = r'[t]
v0 = v[t0]
line[t_] = r[t0] + t v0
b = 1
a = 2
p1 = ParametricPlot3D[ {x[t],y[t],z[t]}, {t,0,4Pi} ]
p2 = ParametricPlot3D[ Evaluate[ line[t] ], {t,-2,2} ]
Show[ p1, p2 ]
```

10.6 ARC LENGTH AND THE UNIT TANGENT VECTOR T

1. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2} = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = 3;\ \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(-\frac{2}{3}\sin t\right)\mathbf{i} + \left(\frac{2}{3}\cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$ and Length $= \int_0^{\pi} |\mathbf{v}|\,dt = \int_0^{\pi} 3\,dt = [3t]_0^{\pi} = 3\pi$

2. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} + 5\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = 13;\ \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and Length $= \int_0^{\pi} |\mathbf{v}|\,dt = \int_0^{\pi} 13\,dt = [13t]_0^{\pi} = 13\pi$

3. $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + \left(t^{1/2}\right)^2} = \sqrt{1+t}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$

and Length $= \int_0^8 \sqrt{1+t}\,dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$

4. $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

and Length $= \int_0^3 \sqrt{3}\,dt = \left[\sqrt{3}t\right]_0^3 = 3\sqrt{3}$

5. $\mathbf{r} = \left(\cos^3 t\right)\mathbf{j} + \left(\sin^3 t\right)\mathbf{k} \Rightarrow \mathbf{v} = \left(-3\cos^2 t \sin t\right)\mathbf{j} + \left(3\sin^2 t \cos t\right)\mathbf{k} \Rightarrow |\mathbf{v}|$

$= \sqrt{\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t\cos t\right)^2} = \sqrt{\left(9\cos^2 t \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)} = 3\,|\cos t \sin t|$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3\,|\cos t \sin t|}\mathbf{j} + \frac{3\sin^2 t\cos t}{3\,|\cos t \sin t|}\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, if $0 \le t \le \frac{\pi}{2}$, and

Length $= \int_0^{\pi/2} 3\,|\cos t \sin t|\,dt = \int_0^{\pi/2} 3\cos t \sin t\,dt = \int_0^{\pi/2} \frac{3}{2}\sin 2t\,dt = \left[-\frac{3}{4}\cos 2t\right]_0^{\pi/2} = \frac{3}{2}$

6. $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(18t^2\right)^2 + \left(-6t^2\right)^2 + \left(-9t^2\right)^2} = \sqrt{441t^4} = 21t^2$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\mathbf{i} - \frac{6t^2}{21t^2}\mathbf{j} - \frac{9t^2}{21t^2}\mathbf{k} = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$ and Length $= \int_1^2 21t^2\,dt = \left[7t^3\right]_1^2 = 49$

7. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}\,t^{1/2}\right)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2t}\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1$, if $t \ge 0$;

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}\,t^{1/2}}{t+1}\right)\mathbf{k}$ and Length $= \int_0^{\pi} (t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$

8. $\mathbf{r} = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t\cos t - \sin t)\mathbf{i} + (\cos t - t\sin t - \cos t)\mathbf{j}$

$= (t\cos t)\mathbf{i} - (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (-t\sin t)^2} = \sqrt{t^2} = |t| = t$ if $\sqrt{2} \le t \le 2$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{t\cos t}{t}\right)\mathbf{i} - \left(\frac{t\sin t}{t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and Length $= \int_{\sqrt{2}}^{2} t\,dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^{2} = 1$

9. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (5\cos t)\mathbf{i} - (5\sin t)\mathbf{j} + 12\mathbf{k}$ and $26\pi = \int_0^{t_0} \sqrt{25\cos^2 t + 25\sin^2 t + 144}\,dt$

$= \int_0^{t_0} 13\,dt = 13t_0 \Rightarrow t_0 = 2\pi$, and the point is $P(2\pi) = (5\sin 2\pi, 5\cos 2\pi, 24\pi) = (5, 0, 24\pi)$

10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and

$$-13\pi = \int_0^{t_0} \sqrt{144\cos^2 t + 144\sin^2 t + 25}\, dt = \int_0^{t_0} 13\, dt = 13t_0 \Rightarrow t_0 = -\pi, \text{ and the point is}$$

$$P(-\pi) = (12\sin(-\pi), -12\cos(-\pi), -5\pi) = (0, 12, -5\pi)$$

11. $\mathbf{r} = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$

$$= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5\, d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

12. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j}$

$$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\cos t)^2} = \sqrt{t^2} = t, \text{ since } \frac{\pi}{2} \le t \le \pi \Rightarrow s(t) = \int_0^t \tau\, d\tau = \frac{t^2}{2}$$

$$\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$$

13. $\mathbf{r} = \left(e^t\cos t\right)\mathbf{i} + \left(e^t\sin t\right)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v} = \left(e^t\cos t - e^t\sin t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t\right)\mathbf{j} + e^t\mathbf{k}$

$$\Rightarrow |\mathbf{v}| = \sqrt{\left(e^t\cos t - e^t\sin t\right)^2 + \left(e^t\sin t + e^t\cos t\right)^2 + \left(e^t\right)^2} = \sqrt{3e^{2t}} = \sqrt{3}\,e^t \Rightarrow s(t) = \int_0^t \sqrt{3}\,e^\tau\, d\tau$$

$$= \sqrt{3}\,e^t - \sqrt{3} \Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - \left(\sqrt{3}\,e^{-\ln 4} - \sqrt{3}\right) = \frac{3\sqrt{3}}{4}$$

14. $\mathbf{r} = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7\, d\tau = 7t$

$$\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$$

15. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since

$$-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j};\ \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{\sec t}\right) \cdot 1 = \cos t$$

16. $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t,$

since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j};\ \frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{\sec t}\right)\cdot 1 = \cos t$$

17. $\mathbf{r} = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1+t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$

$$= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j};\ \frac{d\mathbf{T}}{dt} = \frac{-t}{\left(\sqrt{1+t^2}\right)^3} - \frac{1}{\left(\sqrt{1+t^2}\right)^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{\left(\sqrt{1+t^2}\right)^3}\right)^2 + \left(-\frac{1}{\left(\sqrt{1+t^2}\right)^3}\right)^2}$$

$$= \sqrt{\frac{1}{\left(1+t^2\right)^2}} = \frac{1}{1+t^2} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{2\sqrt{1+t^2}}\right)\left(\frac{1}{1+t^2}\right) = \frac{1}{2\left(1+t^2\right)^{3/2}}$$

18. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t|$

$= t$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j};\ \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j};$$

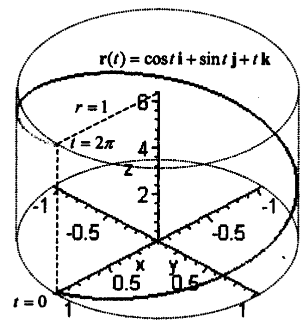
$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{t}\right)\cdot 1 = \frac{1}{t}$$

$$= [(t\cos t)(\sin t + t\cos t) - (t\sin t)(\cos t - t\sin t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{\left(t^2\right)^2} = t^2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{t^3} = \frac{1}{t}$$

19. $\mathbf{r} = \left(\sqrt{2}t\right)\mathbf{i} + \left(\sqrt{2}t\right)\mathbf{j} + \left(1-t^2\right)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + (-2t)^2} = \sqrt{4+4t^2}$

$$= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1+t^2}\,dt = \left[2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$$

20. Let the helix make one complete turn from $t = 0$ to $t = 2\pi$. Note that the radius of the cylinder is $1 \Rightarrow$ the circumference of the base is 2π. When $t = 2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $= 2\pi$ and a height equal to 2π, the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t = 0$ to $t = 2\pi$ is its diagonal.



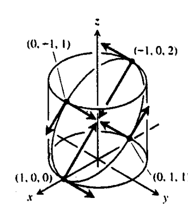
21. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow x = \cos t$, $y = \sin t$, $z = 1 - \cos t \Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, a right circular cylinder with the z-axis as the axis and radius $= 1$. Therefore $P(\cos t, \sin t, 1 - \cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1, 0, 0)$ is on the curve; $t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k}$

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} = 2\mathbf{i} + 2\mathbf{k}$$ is a vector normal to the plane of P, Q, and R. Then the plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

(b) $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$$= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1 + \sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j},\ \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}},\ \mathbf{T}(\pi) = -\mathbf{j},\ \mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$$

(c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1 \Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t = 0 \Rightarrow \mathbf{a}$ is orthogonal to $\mathbf{n} \Rightarrow \mathbf{a}$ is parallel to the plane; $\mathbf{a}(0) = -\mathbf{i} + \mathbf{k}$, $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j}$, $\mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



(d) $|\mathbf{v}| = \sqrt{1 + \sin^2 t}$ (See part (b) $\Rightarrow L = \int_0^{2\pi} \sqrt{1 + \sin^2 t}\, dt$

(e) $L \approx 7.64$ (by *Mathematica*)