32. Across the cap: $g(x,y,z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$

$\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 2z$ since $z \geq 0 \Rightarrow d\sigma = \frac{10}{2z}\, dA$

$\Rightarrow \text{Flux}_{\text{cap}} = \iint\limits_{\text{cap}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint\limits_{R} \left(\frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}\right)\left(\frac{5}{z}\right) dA = \iint\limits_{R} (x^2 + y^2 + 1)\, dx\, dy = \int_0^{2\pi} \int_0^4 (r^2 + 1)\, r\, dr\, d\theta$

$= \int_0^{2\pi} 72\, d\theta = 144\pi.$

Across the bottom: $g(x,y,z) = z = 3 \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1$

$\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint\limits_{\text{bottom}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint\limits_{R} -1\, d\sigma = \iint\limits_{R} -1\, dA = -1(\text{Area of the circular region})$

$= -16\pi$. Therefore, $\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} = 128\pi$

33. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \geq 0 \Rightarrow d\sigma = \frac{2a}{2z}\, dA$

$= \frac{a}{z}\, dA$; $M = \iint\limits_{S} \delta\, d\sigma = \frac{\delta}{8}(\text{surface area of sphere}) = \frac{\delta\pi a^2}{2}$; $M_{xy} = \iint\limits_{S} z\delta\, d\sigma = \delta \iint\limits_{R} z\left(\frac{a}{z}\right) dA$

$= a\delta \iint\limits_{R} dA = a\delta \int_0^{\pi/2} \int_0^a r\, dr\, d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}$. Because of symmetry, $\bar{x} = \bar{y}$

$= \frac{a}{2} \Rightarrow$ the centroid is $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

34. $\nabla f = 2y\mathbf{i} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4(y^2 + z^2)} = 6$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z$ since $z \geq 0 \Rightarrow d\sigma = \frac{6}{2z}\, dA$

$= \frac{3}{z}\, dA$; $M = \iint\limits_{S} 1\, d\sigma = \int_{-3}^{3} \int_0^3 \frac{3}{z}\, dx\, dy = \int_{-3}^{3} \int_0^3 \frac{3}{\sqrt{9 - y^2}}\, dx\, dy = 9\left[\sin^{-1} \frac{y}{3}\right]_{-3}^{3} = 9\pi$; $M_{xy} = \iint\limits_{S} z\, d\sigma$

$= \int_{-3}^{3} \int_0^3 z\left(\frac{3}{z}\right) dx\, dy = 54$; $M_{xz} = \iint\limits_{S} y\, d\sigma = \int_{-3}^{3} \int_0^3 y\left(\frac{3}{z}\right) dx\, dy = \int_{-3}^{3} \int_0^3 \frac{3y}{\sqrt{9 - y^2}}\, dx\, dy = 0$;

$M_{yz} = \iint\limits_{S} x\, d\sigma = \int_{-3}^{3} \int_0^3 \frac{3x}{\sqrt{9 - y^2}}\, dx\, dy = \frac{27}{2}\pi$. Therefore, $\bar{x} = \frac{\left(\frac{27}{2}\pi\right)}{9\pi} = \frac{3}{2}$, $\bar{y} = 0$, and $\bar{z} = \frac{54}{9\pi} = \frac{6}{\pi}$

35. Because of symmetry, $\bar{x} = \bar{y} = 0$; $M = \iint\limits_{S} \delta\, d\sigma = \delta \iint\limits_{S} d\sigma = (\text{Area of S})\delta = 3\pi\sqrt{2}\,\delta$; $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2 + y^2 + z^2}}{2z}\, dA$

$$= \frac{\sqrt{x^2+y^2+(x^2+y^2)}}{z}\,dA = \frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\,dA \Rightarrow M_{xy} = \delta \iint_S z\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right) dA$$

$$= \delta \iint_S \sqrt{2}\sqrt{x^2+y^2}\,dA = \delta \int_0^{2\pi}\int_1^2 \sqrt{2}\,r^2\,dr\,d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \bar{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\,\delta} = \frac{14}{9}$$

$$\Rightarrow (\bar{x},\bar{y},\bar{z}) = \left(0,0,\tfrac{14}{9}\right). \text{ Next, } I_z = \iint_S (x^2+y^2)\delta\,d\sigma = \iint_S (x^2+y^2)\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right)\delta\,dA$$

$$= \delta\sqrt{2} \iint_S (x^2+y^2)\,dA = \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$$

36. $f(x,y,z) = 4x^2+4y^2-z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i}+8y\mathbf{j}-2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2+64y^2+4z^2}$

$$= 2\sqrt{16x^2+16y^2+z^2} = 2\sqrt{4z^2+z^2} = 2\sqrt{5}\,z \text{ since } z \ge 0;\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f\cdot\mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}\,z}{2z}\,dA = \sqrt{5}\,dA$$

$$\Rightarrow I_z = \iint_S (x^2+y^2)\,\delta\,d\sigma = \delta\sqrt{5}\iint_R (x^2+y^2)\,dx\,dy = \delta\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \frac{3\sqrt{5}\pi\delta}{2}$$

37. (a) Let the diameter lie on the z-axis and let $f(x,y,z) = x^2+y^2+z^2 = a^2$, $z \ge 0$ be the upper hemisphere

$$\Rightarrow \nabla f = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2a,\ a>0;\ \mathbf{p}=\mathbf{k} \Rightarrow |\nabla f\cdot\mathbf{p}| = 2z \text{ since } z \ge 0$$

$$\Rightarrow d\sigma = \tfrac{a}{z}\,dA \Rightarrow I_z = \iint_S \delta(x^2+y^2)\left(\tfrac{a}{z}\right)d\sigma = a\delta\iint_R \frac{x^2+y^2}{\sqrt{a^2-(x^2+y^2)}}\,dA = a\delta\int_0^{2\pi}\int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$= a\delta\int_0^{2\pi}\left[-r^2\sqrt{a^2-r^2}-\tfrac{2}{3}\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta = a\delta\int_0^{2\pi}\tfrac{2}{3}a^3\,d\theta = \tfrac{4\pi}{3}a^4\delta \Rightarrow \text{the moment of inertia is } \tfrac{8\pi}{3}a^4\delta \text{ for}$$

the whole sphere

(b) $I_L = I_{c.m.} + mh^2$, where m is the mass of the body and h is the distance between the parallel lines; now,

$$I_{c.m.} = \tfrac{8\pi}{3}a^4\delta \text{ (from part a) and } \tfrac{m}{2} = \iint_S \delta\,d\sigma = \delta\iint_R \left(\tfrac{a}{z}\right)dA = a\delta\iint_R \frac{1}{\sqrt{a^2-(x^2+y^2)}}\,dy\,dx$$

$$= a\delta\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = a\delta\int_0^{2\pi}\left[-\sqrt{a^2-r^2}\right]_0^a d\theta = a\delta\int_0^{2\pi} a\,d\theta = 2\pi a^2\delta \text{ and } h = a$$

$$\Rightarrow I_L = \tfrac{8\pi}{3}a^4\delta + 4\pi a^2\delta a^2 = \tfrac{20\pi}{3}a^4\delta$$

38. (a) Let $z = \frac{h}{a}\sqrt{x^2+y^2}$ be the cone from $z=0$ to $z=h$, $h>0$. Because of symmetry, $\bar{x}=0$ and $\bar{y}=0$;

$$z = \tfrac{h}{a}\sqrt{x^2+y^2} \Rightarrow f(x,y,z) = \frac{h^2}{a^2}(x^2+y^2)-z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2}\mathbf{i}+\frac{2yh^2}{a^2}\mathbf{j}-2z\mathbf{k}$$

$\Rightarrow |\nabla f| = \sqrt{\frac{4x^2h^4}{a^4} + \frac{4y^2h^4}{a^4} + 4z^2} = 2\sqrt{\frac{h^4}{a^4}(x^2+y^2) + \frac{h^2}{a^2}(x^2+y^2)} = 2\sqrt{\left(\frac{h^2}{a^2}\right)(x^2+y^2)\left(\frac{h^2}{a^2}+1\right)}$

$= 2\sqrt{z^2\left(\frac{h^2+a^2}{a^2}\right)} = \left(\frac{2z}{a}\right)\sqrt{h^2+a^2}$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right)\sqrt{h^2+a^2}}{2z}\,dA$

$= \frac{\sqrt{h^2+a^2}}{a}\,dA$; $M = \iint_S d\sigma = \iint_R \frac{\sqrt{h^2+a^2}}{a}\,dA = \frac{\sqrt{h^2+a^2}}{a}(\pi a^2) = \pi a\sqrt{h^2+a^2}$;

$M_{xy} = \iint_S z\delta\,d\sigma = \iint_R z\left(\frac{\sqrt{h^2+a^2}}{a}\right)dA = \frac{\sqrt{h^2+a^2}}{a}\iint_R \frac{h}{a}\sqrt{x^2+y^2}\,dx\,dy = \frac{h\sqrt{h^2+a^2}}{a^2}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta$

$= \frac{2\pi ah\sqrt{h^2+a^2}}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow$ the centroid is $\left(0,0,\frac{2h}{3}\right)$

(b) The base is a circle of radius a and center at $(0,0,h) \Rightarrow (0,0,h)$ is the centroid of the base and the mass is $M = \iint_S d\sigma = \pi a^2$. In Pappus' formula, let $\mathbf{c}_1 = \frac{2h}{3}\mathbf{k}$, $\mathbf{c}_2 = h\mathbf{k}$, $m_1 = \pi a\sqrt{h^2+a^2}$, and $m_2 = \pi a^2$

$\Rightarrow \mathbf{c} = \frac{\pi a\sqrt{h^2+a^2}\left(\frac{2h}{3}\right)\mathbf{k} + \pi a^2h\mathbf{k}}{\pi a\sqrt{h^2+a^2} + \pi a^2} = \frac{2h\sqrt{h^2+a^2}+3ah}{3\left(\sqrt{h^2+a^2}+a\right)}\mathbf{k} \Rightarrow$ the centroid is $\left(0,0,\frac{2h\sqrt{h^2+a^2}+3ah}{3\left(\sqrt{h^2+a^2}+a\right)}\right)$

(c) If the hemisphere is sitting so its base is in the plane $z = h$, then its centroid is $\left(0,0,h+\frac{a}{2}\right)$ and its mass is $2\pi a^2$. In Pappus' formula, let $\mathbf{c}_1 = \frac{2h}{3}\mathbf{k}$, $\mathbf{c}_2 = \left(h+\frac{a}{2}\right)\mathbf{k}$, $m_1 = \pi a\sqrt{h^2+a^2}$, and $m_2 = 2\pi a^2$

$\Rightarrow \mathbf{c} = \frac{\pi a\sqrt{h^2+a^2}\left(\frac{2h}{3}\right)\mathbf{k} + 2\pi a^2\left(h+\frac{a}{2}\right)\mathbf{k}}{\pi a\sqrt{h^2+a^2}+2\pi a^2} = \frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)}\mathbf{k} \Rightarrow$ the centroid is

$\left(0,0,\frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)}\right)$. Thus, for the centroid to be in the plane of the bases we must have $z = h$

$\Rightarrow \frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)} = h \Rightarrow 2h\sqrt{h^2+a^2}+6ah+3a^2 = 3h\sqrt{h^2+a^2}+6ah \Rightarrow 3a^2 = h\sqrt{h^2+a^2}$

$\Rightarrow 9a^4 = h^2\left(h^2+a^2\right) \Rightarrow h^4 + a^2h^2 - 9a^4 = 0 \Rightarrow h^2 = \frac{\left(\sqrt{37}-1\right)a^2}{2}$ (the positive root) $\Rightarrow h = \frac{\sqrt{2\sqrt{37}-2}}{2}\,a$

39. $f_x(x,y) = 2x$, $f_y(x,y) = 2y \Rightarrow \sqrt{f_x^2+f_y^2+1} = \sqrt{4x^2+4y^2+1} \Rightarrow \text{Area} = \iint_R \sqrt{4x^2+4y^2+1}\,dx\,dy$

$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4r^2+1}\,r\,dr\,d\theta = \frac{\pi}{6}\left(13\sqrt{13}-1\right)$

40. $f_y(y,z) = -2y$, $f_z(y,z) = -2z \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4y^2 + 4z^2 + 1}\, dy\, dz$

$= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$

41. $f_x(x,y) = \frac{x}{\sqrt{x^2 + y^2}}$, $f_y(x,y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2}$

$\Rightarrow \text{Area} = \iint_{R_{xy}} \sqrt{2}\, dx\, dy = \sqrt{2}(\text{Area between the ellipse and the circle}) = \sqrt{2}(6\pi - \pi) = 5\pi\sqrt{2}$

42. Over R_{xy}: $z = 2 - \frac{2}{3}x - 2y \Rightarrow f_x(x,y) = -\frac{2}{3}$, $f_y(x,y) = -2 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{4}{9} + 4 + 1} = \frac{7}{3}$

$\Rightarrow \text{Area} = \iint_{R_{xy}} \frac{7}{3}\, dA = \frac{7}{3}(\text{Area of the shadow triangle in the xy-plane}) = \left(\frac{7}{3}\right)\left(\frac{1}{2}\right) = \frac{7}{2}.$

Over R_{xz}: $y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x,z) = -\frac{1}{3}$, $f_z(x,z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6}$

$\Rightarrow \text{Area} = \iint_{R_{xz}} \frac{7}{6}\, dA = \frac{7}{6}(\text{Area of the shadow triangle in the xz-plane}) = \left(\frac{7}{6}\right)(3) = \frac{7}{2}.$

Over R_{yz}: $x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y,z) = -3$, $f_z(y,z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2}$

$\Rightarrow \text{Area} = \iint_{R_{yz}} \frac{7}{2}\, dA = \frac{7}{2}(\text{Area of the shadow triangle in the yz-plane}) = \left(\frac{7}{2}\right)(1) = \frac{7}{2}.$

43. $y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x,z) = 0$, $f_z(x,z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z + 1}$; $y = \frac{16}{3} \Rightarrow \frac{16}{3} = \frac{2}{3}z^{3/2} \Rightarrow z = 4$

$\Rightarrow \text{Area} = \int_0^4 \int_0^1 \sqrt{z + 1}\, dx\, dz = \int_0^4 \sqrt{z + 1}\, dz = \frac{2}{3}\left(5\sqrt{5} - 1\right)$

44. $y = 4 - z \Rightarrow f_x(x,z) = 0$, $f_z(x,z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint_{R_{xz}} \sqrt{2}\, dA = \int_0^2 \int_0^{4-z^2} \sqrt{2}\, dx\, dz$

$= \sqrt{2} \int_0^2 \left(4 - z^2\right) dz = \frac{16\sqrt{2}}{3}$

13.6 PARAMETRIZED SURFACES

1. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \left(\sqrt{x^2 + y^2}\right)^2 = r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

2. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 9 - x^2 - y^2 = 9 - r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$; $z \ge 0 \Rightarrow 9 - r^2 \ge 0 \Rightarrow r^2 \le 9 \Rightarrow -3 \le r \le 3$, $0 \le \theta \le 2\pi$. But $-3 \le r \le 0$ gives the same points as $0 \le r \le 3$, so let $0 \le r \le 3$.

3. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{\sqrt{x^2+y^2}}{2} \Rightarrow z = \frac{r}{2}$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}$. For $0 \le z \le 3$, $0 \le \frac{r}{2} \le 3 \Rightarrow 0 \le r \le 6$; to get only the first octant, let $0 \le \theta \le \frac{\pi}{2}$.

4. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 2\sqrt{x^2+y^2} \Rightarrow z = 2r$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$. For $2 \le z \le 4$, $2 \le 2r \le 4 \Rightarrow 1 \le r \le 2$, and let $0 \le \theta \le 2\pi$.

5. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$ since $x^2 + y^2 + z^2 = 9 \Rightarrow z^2 = 9 - (x^2 + y^2) = 9 - r^2$ $\Rightarrow z = \sqrt{9 - r^2}$, $z \ge 0$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{9 - r^2}\,\mathbf{k}$. Let $0 \le \theta \le 2\pi$. For the domain of r: $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 9 \Rightarrow 2(x^2+y^2) = 9 \Rightarrow 2r^2 = 9$ $\Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow 0 \le r \le \frac{3}{\sqrt{2}}$.

6. In cylindrical coordinates, $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{4 - r^2}\,\mathbf{k}$ (see Exercise 5 above with $x^2 + y^2 + z^2 = 4$, instead of $x^2 + y^2 + z^2 = 9$). For the first octant, let $0 \le \theta \le \frac{\pi}{2}$. For the domain of r: $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 4 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 4 \Rightarrow 2(x^2+y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$. Thus, let $\sqrt{2} \le r \le 2$ (to get the portion of the sphere between the cone and the xy-plane).

7. In spherical coordinates, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$ $\Rightarrow z = \sqrt{3}\cos\phi$ for the sphere; $z = \frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$; $z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$. Then $\mathbf{r}(r,\theta) = (\sqrt{3}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{3}\sin\phi\sin\theta)\mathbf{j} + (\sqrt{3}\cos\phi)\mathbf{k}$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ and $0 \le \theta \le 2\pi$.

8. In spherical coordinates, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 8 \Rightarrow \rho = \sqrt{8} = 2\sqrt{2}$ $\Rightarrow x = 2\sqrt{2}\sin\phi\cos\theta$, $y = 2\sqrt{2}\sin\phi\sin\theta$, and $z = 2\sqrt{2}\cos\phi$. Thus let $\mathbf{r}(r,\theta) = (2\sqrt{2}\sin\phi\cos\theta)\mathbf{i} + (2\sqrt{2}\sin\phi\sin\theta)\mathbf{j} + (2\sqrt{2}\cos\phi)\mathbf{k}$; $z = -2 \Rightarrow -2 = 2\sqrt{2}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}$; $z = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2}\cos\phi \Rightarrow \cos\phi = 1 \Rightarrow \phi = 0$. Thus $0 \le \phi \le \frac{3\pi}{4}$ and $0 \le \theta \le 2\pi$.

9. Since $z = 4 - y^2$, we can let $\mathbf{r}$ be a function of x and y $\Rightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$. Then $z = 0$ $\Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2$. Thus, let $-2 \le y \le 2$ and $0 \le x \le 2$.

10. Since $y = x^2$, we can let $\mathbf{r}$ be a function of x and z $\Rightarrow \mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. Then $y = 2$ $\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. Thus, let $-\sqrt{2} \le x \le \sqrt{2}$ and $0 \le z \le 3$.

11. When $x = 0$, let $y^2 + z^2 = 9$ be the circular section in the yz-plane. Use polar coordinates in the yz-plane $\Rightarrow y = 3\cos\theta$ and $z = 3\sin\theta$. Thus let $x = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = x\mathbf{i} + (3\cos v)\mathbf{j} + (3\sin v)\mathbf{k}$ where $0 \le u \le 3$, and $0 \le v \le 2\pi$.

12. When $y = 0$, let $x^2 + z^2 = 4$ be the circular section in the xz-plane. Use polar coordinates in the xz-plane $\Rightarrow x = 2\cos\theta$ and $z = 2\sin\theta$. Thus let $y = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = (2\cos v)\mathbf{i} + u\mathbf{j} + (3\sin v)\mathbf{k}$ where $-2 \le u \le 2$, and $0 \le v \le \pi$ (since we want the portion <u>above</u> the xy-plane).

13. (a) $x + y + z = 1 \Rightarrow z = 1 - x - y$. In cylindrical coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow z = 1 - r\cos\theta - r\sin\theta \Rightarrow \mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r\cos\theta - r\sin\theta)\mathbf{k}$, $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

    (b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2 + z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with $(x,0,0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u\cos v - u\sin v$, then $\mathbf{r}$ is a function of u and v $\Rightarrow \mathbf{r}(u,v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le 3$ and $0 \le v \le 2\pi$.

14. (a) In a fashion similar to cylindrical coordinates, but working in the xz-plane instead of the xy-plane, let $x = u\cos v$, $z = u\sin v$ where $u = \sqrt{x^2 + z^2}$ and v is the angle formed by (x,y,z), $(y,0,0)$, and $(x,y,0)$ with vertex $(y,0,0)$. Since $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$, then $\mathbf{r}(u,v)$ $= (u\cos v)\mathbf{i} + (u\cos v + 2u\sin v - 2)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{3}$ and $0 \le v \le 2\pi$.

    (b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2 + z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with vertex $(x,0,0)$. Since $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$, then $\mathbf{r}(u,v)$ $= (u\cos v - 2u\sin v + 2)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{2}$ and $0 \le v \le 2\pi$.

15. Let $x = w\cos v$ and $z = w\sin v$. Then $(x-2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0 \Rightarrow w^2\cos^2 v - 4w\cos v + w^2\sin^2 v$ $= 0 \Rightarrow w^2 - 4w\cos v = 0 \Rightarrow w = 0$ or $w - 4\cos v = 0 \Rightarrow w = 0$ or $w = 4\cos v$. Now $w = 0 \Rightarrow x = 0$ and $y = 0$, which is a line not a cylinder. Therefore, let $w = 4\cos v \Rightarrow x = (4\cos v)(\cos v) = 4\cos^2 v$ and $z = 4\cos v\sin v$. Finally, let $y = u$. Then $\mathbf{r}(u,v) = (4\cos^2 v)\mathbf{i} + u\mathbf{j} + (4\cos v\sin v)\mathbf{k}$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$ and $0 \le u \le 3$.

16. Let $y = w\cos v$ and $z = w\sin v$. Then $y^2 + (z-5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$ $\Rightarrow w^2\cos^2 v + w^2\sin^2 v - 10w\sin v = 0 \Rightarrow w^2 - 10w\sin v = 0 \Rightarrow w(w - 10\sin v) = 0 \Rightarrow w = 0$ or $w = 10\sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10\sin v$ $\Rightarrow y = 10\sin v\cos v$ and $z = 10\sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u,v) = u\mathbf{i} + (10\sin v\cos v)\mathbf{j} + (10\sin^2 v)\mathbf{k}$,

$0 \le u \le 10$ and $0 \le v \le \pi$.

17. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2 - r\sin\theta}{2}\right)\mathbf{k}$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - \left(\frac{\sin\theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} - \left(\frac{r\cos\theta}{2}\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{\sin\theta}{2} \\ -r\sin\theta & r\cos\theta & -\frac{r\cos\theta}{2} \end{vmatrix}$$

$$= \left(\frac{-r\sin\theta\cos\theta}{2} + \frac{(\sin\theta)(r\cos\theta)}{2}\right)\mathbf{i} + \left(\frac{r\sin^2\theta}{2} + \frac{r\cos^2\theta}{2}\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}\,r}{2} \Rightarrow A = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}\,r}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{5}\,r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{\sqrt{5}}{4}\,d\theta = \frac{\pi\sqrt{5}}{2}$$

18. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= \left(r\sin^2\theta + r\cos^2\theta\right)\mathbf{i} + (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} = r\mathbf{i} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\pi\sqrt{2}$$

19. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r$, $1 \le r \le 3$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k}$$

$$= (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5}\,d\theta = 8\pi\sqrt{5}$$

20. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = \frac{\sqrt{x^2+y^2}}{3} = \frac{r}{3}$, $3 \le r \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{3}\right)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \left(\frac{1}{3}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-\tfrac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\tfrac{1}{3}r\sin\theta\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k}$$

$$= \left(-\tfrac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\tfrac{1}{3}r\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\tfrac{1}{9}r^2\cos^2\theta + \tfrac{1}{9}r^2\sin^2\theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3}$$

$$\Rightarrow A = \int_0^{2\pi}\int_3^4 \frac{r\sqrt{10}}{3}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{10}}{6}\right]_3^4 d\theta = \int_0^{2\pi}\frac{7\sqrt{10}}{6}\,d\theta = \frac{7\pi\sqrt{10}}{3}$$

21. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 = 1$, $1 \le z \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(z,\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$ and $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\,\mathbf{j} = |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^4 1\,dr\,d\theta = \int_0^{2\pi} 3\,d\theta = 6\pi$$

22. Let $x = u\cos v$ and $z = u\sin v \Rightarrow u^2 = x^2 + y^2 = 10$, $-1 \le y \le 1$, $0 \le v \le 2\pi$. Then $\mathbf{r}(y,v) = (u\cos v)\mathbf{i} + y\mathbf{j} + (u\sin v)\mathbf{k} = \left(\sqrt{10}\cos v\right)\mathbf{i} + y\mathbf{j} + \left(\sqrt{10}\sin v\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_v = \left(-\sqrt{10}\sin v\right)\mathbf{i} + \left(\sqrt{10}\cos v\right)\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \left(-\sqrt{10}\cos v\right)\mathbf{i} - \left(\sqrt{10}\sin v\right)\mathbf{k} = |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10} \Rightarrow A = \int_0^{2\pi}\int_{-1}^{1}\sqrt{10}\,du\,dv = \int_0^{2\pi}\left[\sqrt{10}u\right]_{-1}^{1} dv$$

$$= \int_0^{2\pi} 2\sqrt{10}\,dv = 4\pi\sqrt{10}$$

23. $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2+y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$ or $z = 1$. Since $z = \sqrt{x^2+y^2} \ge 0$, we get $z = 1$ where the cone intersects the paraboloid. When $x = 0$ and $y = 0$, $z = 2 \Rightarrow$ the vertex of the paraboloid is $(0,0,2)$. Therefore, z ranges from 1 to 2 on the "cap" $\Rightarrow$ r ranges from 1 (when $x^2 + y^2 = 1$) to 0 (when $x = 0$ and $y = 0$ at the vertex). Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2 - r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(2 - r^2\right)\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and

$$\mathbf{r}_\theta = (-r \sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2+1}$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{5\sqrt{5}-1}{12}\right)d\theta = \tfrac{\pi}{6}(5\sqrt{5}-1)$$

24. Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = x^2+y^2 = r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $1 \le r \le 2$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} - (2r^2\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2+1} \Rightarrow A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_1^2 d\theta$$

$$= \int_0^{2\pi}\left(\frac{17\sqrt{17}-5\sqrt{5}}{12}\right)d\theta = \tfrac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$$

25. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = \sqrt{2}$ on the sphere. Next, $x^2+y^2+z^2 = 2$ and $z = \sqrt{x^2+y^2} \Rightarrow z^2+z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \ge 0 \Rightarrow \phi = \frac{\pi}{4}$. For the lower portion of the sphere cut by the cone, we get $\phi = \pi$. Then

$\mathbf{r}(\phi,\theta) = (\sqrt{2}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{2}\sin\phi\sin\theta)\mathbf{j} + (\sqrt{2}\cos\phi)\mathbf{k}$, $\frac{\pi}{4} \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (\sqrt{2}\cos\phi\cos\theta)\mathbf{i} + (\sqrt{2}\cos\phi\sin\theta)\mathbf{j} - (\sqrt{2}\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-\sqrt{2}\sin\phi\sin\theta)\mathbf{i} + (\sqrt{2}\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (2\sin^2\phi\cos\theta)\mathbf{i} + (2\sin^2\phi\sin\theta)\mathbf{j} + (2\sin\phi\cos\phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi\cos^2\theta + 4\sin^4\phi\sin^2\theta + 4\sin^2\phi\cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$$

$$\Rightarrow A = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}(2+\sqrt{2})\,d\theta = (4+2\sqrt{2})\pi$$

26. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = 2$ on the sphere. Next, $z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$; $z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$. Then

$\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k},\ \frac{\pi}{6} \le \phi \le \frac{2\pi}{3},\ 0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = \sqrt{16\sin^2\phi} = 4|\sin\phi| = 4\sin\phi$$

$$\Rightarrow A = \int_0^{2\pi}\int_{\pi/6}^{2\pi/3} 4\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left(2+2\sqrt{3}\right) d\theta = \left(4+4\sqrt{3}\right)\pi$$

27. Let the parametrization be $\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^3\int_0^2 x\sqrt{4x^2+1}\, dx\, dz = \int_0^3 \left[\tfrac{1}{12}\left(4x^2+1\right)^{3/2}\right]_0^2 dz$$

$$= \int_0^3 \tfrac{1}{12}\left(17\sqrt{17}-1\right) dz = \frac{17\sqrt{17}-1}{4}$$

28. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{4-y^2}\,\mathbf{k},\ -2 \le y \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4-y^2}}\mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4-y^2}} \end{vmatrix} = \frac{y}{\sqrt{4-y^2}}\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4-y^2}+1} = \frac{2}{\sqrt{4-y^2}}$$

$$\Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_1^4\int_{-2}^2 \sqrt{4-y^2}\left(\frac{2}{\sqrt{4-y^2}}\right) dy\, dx = 24$$

29. Let the parametrization be $\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 1$ on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_\phi = (\cos\phi\cos\theta)\mathbf{i} + (\cos\phi\sin\theta)\mathbf{j} - (\sin\phi)\mathbf{k}$ and

$$\mathbf{r}_\theta = (-\sin\phi\sin\theta)\mathbf{i} + (\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (\sin^2\phi\cos\theta)\mathbf{i} + (\sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$

$= \sin\phi$; $x = \sin\phi\cos\theta \Rightarrow G(x,y,z) = \cos^2\theta\sin^2\phi \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^{\pi} (\cos^2\theta\sin^2\phi)(\sin\phi)\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} (\cos^2\theta)(1-\cos^2\phi)(\sin\phi)\, d\phi\, d\theta; \begin{bmatrix} u = \cos\phi \\ du = -\sin\phi\, d\phi \end{bmatrix} \to \int_0^{2\pi}\int_{-1}^{1} (\cos^2\theta)(u^2-1)\, du\, d\theta$

$= \int_0^{2\pi} (\cos^2\theta)\left[\frac{u^3}{3} - u\right]_1^{-1} d\theta = \frac{4}{3}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{4}{3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{4\pi}{3}$

30. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (since $z \ge 0$), $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (a^2\sin^2\phi\cos\theta)\mathbf{i} + (a^2\sin^2\phi\sin\theta)\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} = a^2\sin\phi$; $z = a\cos\phi$

$\Rightarrow G(x,y,z) = a^2\cos^2\phi \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^{\pi/2} (a^2\cos^2\phi)(a^2\sin\phi)\, d\phi\, d\theta = \frac{2}{3}\pi a^4$

31. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-x-y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint_S F(x,y,z)\, d\sigma = \int_0^1\int_0^1 (4-x-y)\sqrt{3}\, dy\, dx$

$= \int_0^1 \sqrt{3}\left[4y - xy - \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \sqrt{3}\left(\frac{7}{2} - x\right) dx = \sqrt{3}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = 3\sqrt{3}$

32. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2}$; $z = r$ and $x = r\cos\theta$

$\Rightarrow F(x,y,z) = r - r\cos\theta \Rightarrow \iint\limits_S F(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right) dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 (1-\cos\theta)\,r^2\,dr\,d\theta$
$= \frac{2\pi\sqrt{2}}{3}$

33. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(1-r^2\right)\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\left(2r^2\cos\theta\right)^2 + \left(2r^2\sin\theta\right)^2 + r^2} = r\sqrt{1+4r^2}$; $z = 1 - r^2$ and

$x = r\cos\theta \Rightarrow H(x,y,z) = \left(r^2\cos^2\theta\right)\sqrt{1+4r^2} \Rightarrow \iint\limits_S H(x,y,z)\,d\sigma$

$= \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(\sqrt{1+4r^2}\right)\left(r\sqrt{1+4r^2}\right) dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\left(1+4r^2\right)\cos^2\theta\,dr\,d\theta = \frac{11\pi}{12}$

34. Let the parametrization be $\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \le \phi \le \frac{\pi}{4}$; $x^2 + y^2 + z^2 = 4$ and $z = \sqrt{x^2+y^2} \Rightarrow z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \ge 0$) $\Rightarrow 2\cos\phi = \sqrt{2} \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$, $0 \le \theta \le 2\pi$; $\mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$

and $\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$

$= \left(4\sin^2\phi\cos\theta\right)\mathbf{i} + \left(4\sin^2\phi\sin\theta\right)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$; $y = 2\sin\phi\sin\theta$ and

$z = 2\cos\phi \Rightarrow H(x,y,z) = 4\cos\phi\sin\phi\sin\theta \Rightarrow \iint\limits_S H(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi/4} (4\cos\phi\sin\phi\sin\theta)(4\sin\phi)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/4} 16\sin^2\phi\cos\phi\sin\theta\,d\phi\,d\theta = 0$

35. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \left(4-y^2\right)\mathbf{k}$, $0 \le x \le 1$, $-2 \le y \le 2$; $z = 0 \Rightarrow 0 = 4 - y^2$

$\Rightarrow y = \pm 2$; $\mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma$

$= \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| \, dy \, dx = (2xy - 3z) \, dy \, dx = [2xy - 3(4 - y^2)] \, dy \, dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$

$= \int_0^1 \int_{-2}^2 (2xy + 3y^2 - 12) \, dy \, dx = \int_0^1 [xy^2 + y^3 - 12y]_{-2}^2 \, dx = \int_0^1 -32 \, dx = -32$

36. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $-1 \le x \le 1$, $0 \le z \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$

$\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} |\mathbf{r}_x \times \mathbf{r}_z| \, dz \, dx = -x^2 \, dz \, dx$

$\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_{-1}^1 \int_0^2 -x^2 \, dz \, dx = -\frac{4}{3}$

37. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (for the first octant)
$\Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix}$

$= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\theta \, d\phi$

$= a^3 \cos^2 \phi \sin \phi \, d\theta \, d\phi$ since $\mathbf{F} = z\mathbf{k} = (a \cos \phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \cos^2 \phi \sin \phi \, d\phi \, d\theta = \frac{\pi a^3}{6}$

38. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_\phi = (a \cos \phi \cos \theta)\mathbf{i} + (a \cos \phi \sin \theta)\mathbf{j} - (a \sin \phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a \sin \phi \sin \theta)\mathbf{i} + (a \sin \phi \cos \theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix}$

$= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} \, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\theta \, d\phi$

$= (a^3 \sin^3 \phi \cos^2 \phi + a^3 \sin^3 \phi \sin^2 \theta + a^3 \sin \phi \cos^2 \phi) \, d\theta \, d\phi = a^3 \sin \phi \, d\theta \, d\phi$ since $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$= (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^{\pi} a^3 \sin \phi \, d\phi \, d\theta = 4\pi a^3$

39. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$, $0 \le x \le a$, $0 \le y \le a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx$$

$= [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\, dy\, dx$ since $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$

$= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$$= \int_0^a \int_0^a [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\, dy\, dx = \int_0^a \int_0^a \left(4ay - 2y^2 + 4ax - 2x^2 - 2xy\right) dy\, dx$$

$$= \int_0^a \left(\tfrac{4}{3}a^3 + 3a^2x - 2ax^2\right) dx = \left(\tfrac{4}{3} + \tfrac{3}{2} - \tfrac{2}{3}\right)a^4 = \frac{13a^4}{6}$$

40. Let the parametrization be $\mathbf{r}(\theta, z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$, $0 \le z \le a$, $0 \le \theta \le 2\pi$ (where $r = \sqrt{x^2 + y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|} |\mathbf{r}_\theta \times \mathbf{r}_z|\, dz\, d\theta = \left(\cos^2\theta + \sin^2\theta\right) dz\, d\theta = dz\, d\theta$, since $\mathbf{F} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$

$$\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^a 1\, dz\, d\theta = 2\pi a$$

41. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|} |\mathbf{r}_\theta \times \mathbf{r}_r|\, d\theta\, dr = \left(r^3 \sin\theta\cos^2\theta + r^2\right) d\theta\, dr$ since

$$\mathbf{F} = \left(r^2 \sin\theta\cos\theta\right)\mathbf{i} - r\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^1 \left(r^3 \sin\theta\cos^2\theta + r^2\right) dr\, d\theta = \int_0^{2\pi} \left(\tfrac{1}{4}\sin\theta\cos^2\theta + \tfrac{1}{3}\right) d\theta$$

$$= \left[-\tfrac{1}{12}\cos^3\theta + \tfrac{\theta}{3}\right]_0^{2\pi} = \frac{2\pi}{3}$$

42. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$, $0 \le r \le 2$ (since $0 \le z \le 2$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2 \end{vmatrix}$$

$$= (2r\cos\theta)\mathbf{i} + (2r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|}|\mathbf{r}_\theta \times \mathbf{r}_r|\,d\theta\,dr$$

$= \left(2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r\right)d\theta\,dr$ since

$$\mathbf{F} = \left(r^2\sin^2\theta\right)\mathbf{i} + \left(2r^2\cos\theta\right)\mathbf{j} - \mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 \left(2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{1}{2}\sin^2\theta\cos\theta + \cos\theta\sin\theta + \tfrac{1}{2}\right)d\theta = \left[\tfrac{1}{6}\sin^3\theta + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi$$

43. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $1 \le r \le 2$ (since $1 \le z \le 2$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|}|\mathbf{r}_\theta \times \mathbf{r}_r|\,d\theta\,dr = \left(-r^2\cos^2\theta - r^2\sin^2\theta - r^3\right)d\theta\,dr$$

$$= \left(-r^2 - r^3\right)d\theta\,dr \text{ since } \mathbf{F} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r^2\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_1^2 \left(-r^2 - r^3\right)dr\,d\theta = -\frac{73\pi}{6}$$

44. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}$$

$$= \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|}|\mathbf{r}_\theta \times \mathbf{r}_r|\,d\theta\,dr = \left(8r^3\cos^2\theta + 8r^3\sin^2\theta - 2r\right)d\theta\,dr$$

$$= \left(8r^3 - 2r\right)d\theta\,dr \text{ since } \mathbf{F} = (4r\cos\theta)\mathbf{i} + (4r\sin\theta)\mathbf{j} + 2\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 \left(8r^3 - 2r\right)dr\,d\theta = 2\pi$$

45. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $1 \le r \le 2$ (since $1 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_r| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$. The mass is

$M = \iint_S \delta\, d\sigma = \int_0^{2\pi}\int_1^2 \delta r\sqrt{2}\, dr\, d\theta = (3\sqrt{2})\pi\delta$; the first moment is $M_{yz} = \iint_S \delta x\, d\sigma = \int_0^{2\pi}\int_1^2 \delta r(r\sqrt{2})\, dr\, d\theta$

$= \frac{(14\sqrt{2})\pi\delta}{3} \Rightarrow \bar{x} = \frac{\left(\frac{(14\sqrt{2})\pi\delta}{3}\right)}{(3\sqrt{2})\pi\delta} = \frac{14}{9} \Rightarrow$ the center of mass is located at $\left(0,0,\frac{14}{9}\right)$ by symmetry. The

moment of inertia is $I_z = \iint_S \delta(x^2+y^2)\, d\sigma = \int_0^{2\pi}\int_1^2 \delta r^2(r\sqrt{2})\, dr\, d\theta = \frac{(15\sqrt{2})\pi\delta}{2} \Rightarrow$ the radius of gyration is

$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{2}}$

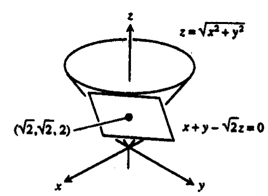
46. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_r| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$. The moment of inertia is

$I_z = \iint_S \delta(x^2+y^2)\, d\sigma = \int_0^{2\pi}\int_0^1 \delta r^2(r\sqrt{2})\, dr\, d\theta = \frac{\pi\delta\sqrt{2}}{2}$

47. The parametrization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$
at $P_0 = (\sqrt{2},\sqrt{2},2) \Rightarrow \theta = \frac{\pi}{4}$, $r = 2$,
$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$



$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$

$= -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k} \Rightarrow$ the tangent plane is

$(-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k})\cdot\left[(x-\sqrt{2})\mathbf{i} + (y-\sqrt{2})\mathbf{j} + (z-2)\mathbf{k}\right] = \sqrt{2}x + \sqrt{2}y - 2z = 0$, or $x + y - \sqrt{2}z = 0$. The parametrization $\mathbf{r}(r,\theta) \Rightarrow x = r\cos\theta$, $y = r\sin\theta$ and $z = r \Rightarrow x^2 + y^2 = r^2 = z^2 \Rightarrow$ the surface is $z = \sqrt{x^2+y^2}$.

48. The parametrization $\mathbf{r}(\phi, \theta)$

$= (4 \sin \phi \cos \theta)\mathbf{i} + (4 \sin \phi \sin \theta)\mathbf{j} + (4 \cos \phi)\mathbf{k}$

at $P_0 = \left(\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right) \Rightarrow \rho = 4$ and $z = 2\sqrt{3}$

$= 4 \cos \phi \Rightarrow \ = \frac{\pi}{6}$; also $x = \sqrt{2}$ and $y = \sqrt{2}$

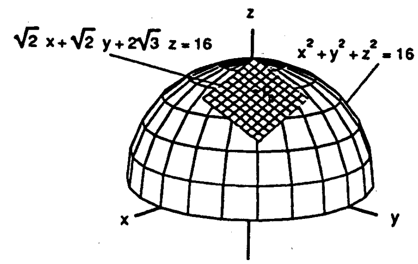
$\Rightarrow \theta = \frac{\pi}{4}$. Then $\mathbf{r}_\phi$

$= (4 \cos \phi \cos \theta)\mathbf{i} + (r \cos \phi \sin \theta)\mathbf{j} - (4 \sin \phi)\mathbf{k}$

$= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k}$ and

$\mathbf{r}_\theta = (-4 \sin \phi \sin \theta)\mathbf{i} + (4 \sin \phi \cos \theta)\mathbf{j}$

$= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$ at P_0



$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k} \Rightarrow \text{the tangent plane is}$$

$\left(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}\right) \cdot \left[\left(x - \sqrt{2}\right)\mathbf{i} + \left(y - \sqrt{2}\right)\mathbf{j} + \left(z - 2\sqrt{3}\right)\mathbf{k}\right] = 0 \Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16$,

or $x + y + \sqrt{6}z = 8\sqrt{2}$. The parametrization $\Rightarrow x = 4 \sin \phi \cos \theta$, $y = 4 \sin \phi \sin \theta$, $z = 4 \cos \phi$

$\Rightarrow$ the surface is $x^2 + y^2 + z^2 = 16$, $z \geq 0$.

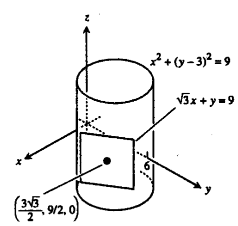
49. The parametrization $\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + \left(6 \sin^2 \theta\right)\mathbf{j} + z\mathbf{k}$

at $P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta = \frac{\pi}{3}$ and $z = 0$. Then

$\mathbf{r}_\theta = (6 \cos 2\theta)\mathbf{i} + (12 \sin \theta \cos \theta)\mathbf{j}$

$= -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$ at P_0

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j} \Rightarrow \text{the tangent}$$



plane is $\left(3\sqrt{3}\mathbf{i} + 3\mathbf{j}\right) \cdot \left[\left(x - \frac{3\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{9}{2}\right)\mathbf{j} + (z - 0)\mathbf{k}\right] = 0$

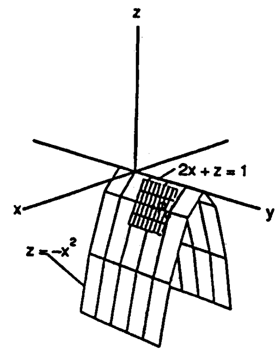
$\Rightarrow \sqrt{3}x + y = 9$. The parametrization $\Rightarrow x = 3 \sin 2\theta$ and $y = 6 \sin^2 \theta \Rightarrow x^2 + y^2 = 9 \sin^2 2\theta + \left(6 \sin^2 \theta\right)^2$

$= 9\left(4 \sin^2 \theta \cos^2 \theta\right) + 36 \sin^4 \theta = 6\left(6 \sin^2 \theta\right) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y - 3)^2 = 9$

50. The parametrization $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$ at $P_0 = (1,2,-1)$ $\Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j}$ at $P_0 \Rightarrow \mathbf{r}_x \times \mathbf{r}_y$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow \text{the tangent plane is}$$

$(2\mathbf{i} + \mathbf{k}) \cdot [(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}] = 0 \Rightarrow 2x + z = 1.$ The parametrization $\Rightarrow x = x$, $y = y$ and $z = -x^2 \Rightarrow$ the surface is $z = -x^2$



51. (a) An arbitrary point on the circle C is $(x,z) = (R + r\cos u, r\sin u) \Rightarrow (x,y,z)$ is on the torus with $x = (R + r\cos u)\cos v$, $y = (R + r\cos u)\sin v$, and $z = r\sin u$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$

(b) $\mathbf{r}_u = (-r\sin u\cos v)\mathbf{i} - (r\sin u\sin v)\mathbf{j} + (r\cos u)\mathbf{k}$ and $\mathbf{r}_v = (-(R + r\cos u)\sin v)\mathbf{i} + ((R + r\cos u)\cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$$

$$= -(R + r\cos u)(r\cos v\cos u)\mathbf{i} - (R + r\cos u)(r\sin v\cos u)\mathbf{j} + (-r\sin u)(R + r\cos u)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r\cos u)^2\left(r^2\cos^2 v\cos^2 u + r^2\sin^2 v\cos^2 u + r^2\sin^2 u\right) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u)$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi} \left(rR + r^2\cos u\right) du\, dv = \int_0^{2\pi} 2\pi rR\, dv = 4\pi^2 rR$$

52. (a) The point (x,y,z) is on the surface for fixed $x = f(u)$ when $y = g(u)\sin\left(\frac{\pi}{2} - v\right)$ and $z = g(u)\cos\left(\frac{\pi}{2} - v\right)$ $\Rightarrow x = f(u)$, $y = g(u)\cos v$, and $z = g(u)\sin v \Rightarrow \mathbf{r}(u,v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $a \le u \le b$

(b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u,v) = u^2\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $0 \le u$

53. (a) Let $w^2 + \frac{z^2}{c^2} = 1$ where $w = \cos\phi$ and $\frac{z}{c} = \sin\phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi \Rightarrow \frac{x}{a} = \cos\phi\cos\theta$ and $\frac{y}{b} = \cos\phi\sin\theta$ $\Rightarrow x = a\cos\theta\cos\phi$, $y = b\sin\theta\cos\phi$, and $z = c\sin\phi$ $\Rightarrow \mathbf{r}(\theta,\phi) = (a\cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$

(b) $\mathbf{r}_\theta = (-a\sin\theta\cos\phi)\mathbf{i} + (b\cos\theta\cos\phi)\mathbf{j}$ and $\mathbf{r}_\phi = (-a\cos\theta\sin\phi)\mathbf{i} - (b\sin\theta\sin\phi)\mathbf{j} + (c\cos\phi)\mathbf{k}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$$= \left(bc\cos\theta\cos^2\phi\right)\mathbf{i} + \left(ac\sin\theta\cos^2\phi\right)\mathbf{j} + (ab\sin\phi\cos\phi)\mathbf{k}$$

$$\Rightarrow \left|\mathbf{r}_\theta \times \mathbf{r}_\phi\right|^2 = b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^{\pi} \left|\mathbf{r}_\theta \times \mathbf{r}_\phi\right| d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \left(a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^4\phi\cos^2\theta + a^2c^2\cos^4\phi\sin^2\theta\right)^{1/2} d\phi\, d\theta$$

13.7 STOKES' THEOREM

1. $\text{curl }\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl }\mathbf{F}\cdot\mathbf{n} = 2 \Rightarrow d\sigma = dx\,dy$

$$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R 2\, dA = 2(\text{Area of the ellipse}) = 4\pi$$

2. $\text{curl }\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3-2)\mathbf{k} = \mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl }\mathbf{F}\cdot\mathbf{n} = 1 \Rightarrow d\sigma = dx\,dy$

$$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R dx\,dy = \text{Area of circle} = 9\pi$$

3. $\text{curl }\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \Rightarrow \text{curl }\mathbf{F}\cdot\mathbf{n}$

$$= \frac{1}{\sqrt{3}}(-3x+z-1) \Rightarrow d\sigma = \frac{\sqrt{3}}{1}\, dA \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x+z-1)\sqrt{3}\, dA$$

$$= \int_0^1\int_0^{1-x} [-3x + (1-x-y) - 1]\, dy\, dx = \int_0^1\int_0^{1-x} (-4x-y)\, dy\, dx = \int_0^1 -\left[4x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx$$

$$= -\int_0^1 \left(\tfrac{1}{2} + 3x - \tfrac{7}{2}x^2\right) dx = -\tfrac{5}{6}$$

4. $\text{curl }\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$\Rightarrow$ curl $\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{3}}(2y - 2z + 2z - 2x + 2x - 2y) = 0 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S d\sigma = 0$

5. curl $\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \mathbf{k}$

$\Rightarrow$ curl $\mathbf{F}\cdot\mathbf{n} = 2x - 2y \Rightarrow d\sigma = dx\,dy \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\int_{-1}^{1} (2x-2y)\,dx\,dy = \int_{-1}^{1} \left[x^2 - 2xy\right]_{-1}^{1} dy$

$= \int_{-1}^{1} -4y\,dy = 0$

6. curl $\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k}$ and $\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4}$

$\Rightarrow$ curl $\mathbf{F}\cdot\mathbf{n} = -\frac{3}{4}x^2y^2z$; $d\sigma = \frac{4}{z}\,dA$ (Section 14.5, Example 5, with $a = 4$) $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r}$

$= \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA = -3\int_0^{2\pi}\int_0^{2} \left(r^2\cos^2\theta\right)\left(r^2\sin^2\theta\right) r\,dr\,d\theta = -3\int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^2 (\cos\theta\sin\theta)^2\,d\theta$

$= -32\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\,d\theta = -4\int_0^{4\pi}\sin^2 u\,du = -4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$

7. $x = 3\cos t$ and $y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + \left(9\cos^2 t\right)\mathbf{j} + \left(9\cos^2 t + 16\sin^4 t\right)\sin e^{\sqrt{(6\sin t\cos t)(0)}}\mathbf{k}$ at the base of the shell; $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -6\sin^2 t + 18\cos^3 t$

$\Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\left(-6\sin^2 t + 18\cos^3 t\right) dt = \left[-3t + \frac{3}{2}\sin 2t + 6(\sin t)\left(\cos^2 t + 2\right)\right]_0^{2\pi} = -6\pi$

8. curl $\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}$; $f(x,y,z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$

$\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = |\nabla f|\,dA$; $\nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\nabla f|}(-2\mathbf{j}\cdot\nabla f) = \frac{-2}{|\nabla f|}$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -2\,dA \Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R -2\,dA = -2(\text{Area of R}) = -2(\pi\cdot 1\cdot 2) = -4\pi$, where R is the elliptic region in the xz-plane enclosed by $4x^2 + z^2 = 4$.

9. Flux of $\nabla \times \mathbf{F} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$, so let C be parametrized by $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ay \sin t + ax \cos t = a^2 \sin^2 t + a^2 \cos^2 t = a^2$

$\Rightarrow$ Flux of $\nabla \times \mathbf{F} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2\, dt = 2\pi a^2$

10. $\nabla \times (y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}$; $\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\Rightarrow \nabla \times (y\mathbf{i}) \cdot \mathbf{n} = -z$; $d\sigma = \frac{1}{z}\, dA$ (Section 14.5, Example 5, with $a = 1$) $\Rightarrow \iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n}\, d\sigma$

$= \iint_R (-z)\left(\frac{1}{z}\, dA\right) = -\iint_R dA = -\pi$, where R is the circle $x^2 + y^2 = 1$ in the xy-plane.

11. Let S_1 and S_2 be oriented surfaces that span C and that induce the same positive direction on C. Then $\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1\, d\sigma_1 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2\, d\sigma_2$

12. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, and since S_1 and S_2 are joined by the simple closed curve C, each of the above integrals will be equal to a circulation integral on C. But for one surface the circulation will be counterclockwise, and for the other surface the circulation will be clockwise. Since the integrands are the same, the sum will be $0 \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$.

13. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin\theta)\mathbf{i} + (r \cos\theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2 \cos\theta)\mathbf{i} + (2r^2 \sin\theta)\mathbf{j} + r\mathbf{k}$; $\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|}$ and $d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta = (10r^2 \cos\theta + 4r^2 \sin\theta + 3r)\, dr\, d\theta \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$= \int_0^{2\pi} \int_0^2 (10r^2 \cos\theta + 4r^2 \sin\theta + 3r)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{10}{3} r^3 \cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2\right]_0^2 d\theta$

$$= \int_0^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta = 6(2\pi) = 12\pi$$

14. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x+z \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} + r\mathbf{k}$ and

$\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$$= \int_0^{2\pi}\int_0^3 \left(2r^2\cos\theta - 2r^2\sin\theta - 2r\right) dr\, d\theta = \int_0^{2\pi} \left[\frac{2}{3}r^3\cos\theta - \frac{2}{3}r^3\sin\theta - r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{54}{3}\cos\theta - \frac{54}{3}\sin\theta - 9\right) d\theta = -9(2\pi) = -18\pi$$

15. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above)

$$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \left(2ry^3\cos\theta - rx^2\right) dr\, d\theta = \int_0^{2\pi}\int_0^1 \left(2r^4\sin\theta\cos\theta - r^3\cos^2\theta\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{5}\sin\theta\cos\theta - \frac{1}{4}\cos^2\theta\right) d\theta = \left[\frac{1}{5}\sin^2\theta - \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi} = -\frac{\pi}{4}$$

16. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above)

$$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta + r)\, dr\, d\theta = \int_0^{2\pi} \left[(\cos\theta + \sin\theta + 1)\frac{r^2}{2}\right]_0^5 d\theta = \left(\frac{25}{2}\right)(2\pi) = 25\pi$$

17. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}$;

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (3\sin^2\phi\cos\theta)\mathbf{i} + (3\sin^2\phi\sin\theta)\mathbf{j} + (3\sin\phi\cos\phi)\mathbf{k}$; $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\phi\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^{\pi/2} -15\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left[\frac{15}{2}\cos^2\phi\right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{15}{2}\, d\theta = -15\pi$

18. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}$;

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$; $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\phi\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \left(-8z\sin^2\phi\cos\theta - 4\sin^2\phi\sin\theta - 8y\sin\phi\cos\theta\right) d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(-16\sin^2\phi\cos\phi\cos\theta - 4\sin^2\phi\sin\theta - 16\sin^2\phi\sin\theta\cos\theta\right) d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{16}{3}\sin^3\phi\cos\theta - 4\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta) - 16\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta\cos\theta)\right]_0^{\pi/2} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{16}{3}\cos\theta - \pi\sin\theta - 4\pi\sin\theta\cos\theta\right) d\theta = \left[-\frac{16}{3}\sin\theta + \pi\cos\theta - 2\pi\sin^2\theta\right]_0^{2\pi} = 0$$

19. (a) $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S d\sigma = 0$

(b) Let $f(x,y,z) = xy^2z^3 \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ $= \iint_S 0\, d\sigma = 0$

(c) $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

(d) $\mathbf{F} = \nabla f \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

20. $\mathbf{F} = \nabla f = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\mathbf{i} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\mathbf{j} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\mathbf{k}$

$= -x(x^2+y^2+z^2)^{-3/2}\mathbf{i} - y(x^2+y^2+z^2)^{-3/2}\mathbf{j} - z(x^2+y^2+z^2)^{-3/2}\mathbf{k}$

(a) $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -x(x^2+y^2+z^2)^{-3/2}(-a \sin t) - y(x^2+y^2+z^2)^{-3/2}(a \cos t)$

$= \left(-\frac{a \cos t}{a^3}\right)(-a \sin t) - \left(\frac{a \sin t}{a^3}\right)(a \cos t) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \nabla \times \nabla f \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

21. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k};\ \mathbf{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = -2 \Rightarrow \oint_C 2y\, dx + 3z\, dy - x\, dz = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S -2\, d\sigma$

$= -2 \iint_S d\sigma$, where $\iint_S d\sigma$ is the area of the region enclosed by C on the plane S: $2x + 2y + z = 2$

22. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

23. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ exists such that $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$

$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) = \frac{\partial}{\partial x}(x) \Rightarrow \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} = 1$. Likewise, $\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}(y)$

$\Rightarrow \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} = 1$ and $\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\partial}{\partial z}(z) \Rightarrow \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 1$. Summing the calculated equations

$\Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial y \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y}\right) = 3$ or $0 = 3$ (assuming the second mixed partials are equal). This result is a contradiction, so there is no field $\mathbf{F}$ such that curl $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

24. Yes: If $\nabla \times \mathbf{F} = \mathbf{0}$, then the circulation of $\mathbf{F}$ around the boundary C of any oriented surface S in the domain of $\mathbf{F}$ is zero. The reason is this: By Stokes' theorem, circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma$ $= 0$.

25. $r = \sqrt{x^2+y^2} \Rightarrow r^4 = (x^2+y^2)^2 \Rightarrow \mathbf{F} = \nabla(r^4) = 4x(x^2+y^2)\mathbf{i} + 4y(x^2+y^2)\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

$\Rightarrow \oint_C \nabla(r^4)\cdot\mathbf{n}\,ds = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \iint_R [4(x^2+y^2) + 8x^2 + 4(x^2+y^2) + 8y^2]\,dA = \iint_R 16(x^2+y^2)\,dA = 16\iint_R x^2\,dA + 16\iint_R y^2\,dA$

$= 16I_y + 16I_x$.

26. $\frac{\partial P}{\partial y} = 0$, $\frac{\partial N}{\partial z} = 0$, $\frac{\partial M}{\partial z} = 0$, $\frac{\partial P}{\partial x} = 0$, $\frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial M}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \text{curl }\mathbf{F} = \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right]\mathbf{k} = \mathbf{0}$.

However, $x^2+y^2 = 1 \Rightarrow \mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} = \left(\frac{-a\sin t}{a^2}\right)\mathbf{i} + \left(\frac{a\cos t}{a^2}\right)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \frac{a^2\sin^2 t}{a^2} + \frac{a^2\cos^2 t}{a^2} = 1 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_0^{2\pi} 1\,dt = 2\pi$ which is not zero.

13.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

1. $\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}} \Rightarrow \text{div }\mathbf{F} = \frac{xy - xy}{(x^2+y^2)^{3/2}} = 0$

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \text{div }\mathbf{F} = 1 + 1 = 2$

3. $\mathbf{F} = -\frac{GM(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \text{div }\mathbf{F} = -GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$

$-GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right] - GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$

$= -GM\left[\frac{3(x^2+y^2+z^2)^2 - 3(x^2+y^2+z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}\right] = 0$

4. $z = a^2 - r^2$ in cylindrical coordinates $\Rightarrow z = a^2 - (x^2+y^2) \Rightarrow \mathbf{v} = (a^2 - x^2 - y^2)\mathbf{k} \Rightarrow \text{div }\mathbf{v} = 0$

5. $\frac{\partial}{\partial x}(y-x) = -1$, $\frac{\partial}{\partial y}(z-y) = -1$, $\frac{\partial}{\partial z}(y-x) = 0 \Rightarrow \nabla\cdot\mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\,dx\,dy\,dz = -2(2^3)$

$= -16$

6. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(y^2) = 2y$, $\frac{\partial}{\partial x}(z^2) = 2z \Rightarrow \nabla\cdot\mathbf{F} = 2x + 2y + 2z$

(a) Flux $= \int_0^1 \int_0^1 \int_0^1 (2x+2y+2z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \left[x^2+2x(y+z)\right]_0^1 dy\,dz = \int_0^1 \int_0^1 (1+2y+2z)\,dy\,dz$

$= \int_0^1 \left[y(1+2z)+y^2\right]_0^1 dz = \int_0^1 (2+2z)\,dz = \left[2z+z^2\right]_0^1 = 3$

(b) Flux $= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x+2y+2z)\,dx\,dy\,dz = \int_{-1}^1 \int_{-1}^1 \left[x^2+2x(y+z)\right]_{-1}^1 dy\,dz = \int_{-1}^1 \int_{-1}^1 (4y+4z)\,dy\,dz$

$= \int_{-1}^1 \left[2y^2+4yz\right]_{-1}^1 dz = \int_{-1}^1 8z\,dz = \left[4z^2\right]_{-1}^1 = 0$

(c) In cylindrical coordinates, Flux $= \iiint_D (2x+2y+2z)\,dx\,dy\,dz$

$= \int_0^1 \int_0^{2\pi} \int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \left[\tfrac{2}{3}r^3\cos\theta + \tfrac{2}{3}r^3\sin\theta + zr^2\right]_0^2 d\theta\,dz$

$= \int_0^1 \int_0^{2\pi} \left(\tfrac{16}{3}\cos\theta + \tfrac{16}{3}\sin\theta + 4z\right) d\theta\,dz = \int_0^1 \left[\tfrac{16}{3}\sin\theta - \tfrac{16}{3}\cos\theta + 4z\theta\right]_0^{2\pi} dz = \int_0^1 8\pi z\,dz = \left[4\pi z^2\right]_0^1 = 4\pi$

7. $\frac{\partial}{\partial x}(y) = 0$, $\frac{\partial}{\partial y}(xy) = x$, $\frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla\cdot\mathbf{F} = x-1$; $z = x^2+y^2 \Rightarrow z = r^2$ in cylindrical coordinates

$\Rightarrow$ Flux $= \iiint_D (x-1)\,dz\,dy\,dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r\cos\theta - 1)\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(r^3\cos\theta - r^2\right) r\,dr\,d\theta$

$= \int_0^{2\pi} \left[\tfrac{r^5}{5}\cos\theta - \tfrac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} \left(\tfrac{32}{5}\cos\theta - 4\right) d\theta = \left[\tfrac{32}{5}\sin\theta - 4\theta\right]_0^{2\pi} = -8\pi$

8. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(xz) = 0$, $\frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla\cdot\mathbf{F} = 2x+3 \Rightarrow$ Flux $= \iiint_D (2x+3)\,dV$

$= \int_0^{2\pi} \int_0^{\pi} \int_0^2 (2\rho\sin\phi\cos\theta + 3)\left(\rho^2\sin\phi\right) d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^{\pi} \left[\tfrac{\rho^4}{2}\sin\phi\cos\theta + \rho^3\right]_0^2 \sin\phi\,d\phi\,d\theta$

$= \int_0^{2\pi} \int_0^{\pi} (8\sin\phi\cos\theta + 8)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} \left[8\left(\tfrac{\phi}{2} - \tfrac{\sin 2\phi}{4}\right)\cos\theta - 8\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} (4\pi\cos\theta + 16)\,d\theta$

$= 32\pi$

9. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(-2xy) = -2x$, $\frac{\partial}{\partial z}(3xz) = 3x \Rightarrow$ Flux $= \iiint_D 3x\,dx\,dy\,dz$

$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (3\rho\sin\phi\cos\theta)\left(\rho^2\sin\phi\right) d\rho\,d\phi\,d\theta = \int_0^{\pi/2} \int_0^{\pi/2} 12\sin^2\phi\cos\theta\,d\phi\,d\theta = \int_0^{\pi/2} 3\pi\cos\theta\,d\theta = 3\pi$

10. $\frac{\partial}{\partial x}(6x^2+2xy)=12x+2y$, $\frac{\partial}{\partial y}(2y+x^2z)=2$, $\frac{\partial}{\partial z}(4x^2y^3)=0 \Rightarrow \nabla\cdot\mathbf{F}=12x+2y+2$

$$\Rightarrow \text{Flux}=\iiint_D (12x+2y+2)\,dV=\int_0^3\int_0^{\pi/2}\int_0^2 (12r\cos\theta+2r\sin\theta+2)\,r\,dr\,d\theta\,dz$$

$$=\int_0^3\int_0^{\pi/2}\left(32\cos\theta+\tfrac{16}{3}\sin\theta+4\right)d\theta\,dz=\int_0^3\left(32+2\pi+\tfrac{16}{3}\right)dz=112+6\pi$$

11. $\frac{\partial}{\partial x}(2xz)=2z$, $\frac{\partial}{\partial y}(-xy)=-x$, $\frac{\partial}{\partial z}(-z^2)=-2z \Rightarrow \nabla\cdot\mathbf{F}=-x \Rightarrow \text{Flux}=\iiint_D -x\,dV$

$$=\int_0^2\int_0^{\sqrt{16-4x^2}}\int_0^{4-y} -x\,dz\,dy\,dx=\int_0^2\int_0^{\sqrt{16-4x^2}}(xy-4x)\,dy\,dx=\int_0^2\left[\tfrac{1}{2}x\left(16-4x^2\right)-4x\sqrt{16-4x^2}\right]dx$$

$$=\left[4x^2-\tfrac{1}{2}x^4+\tfrac{1}{3}\left(16-4x^2\right)^{3/2}\right]_0^2=-\tfrac{40}{3}$$

12. $\frac{\partial}{\partial x}(x^3)=3x^2$, $\frac{\partial}{\partial y}(y^3)=3y^2$, $\frac{\partial}{\partial z}(z^3)=3z^2 \Rightarrow \nabla\cdot\mathbf{F}=3x^2+3y^2+3z^2 \Rightarrow \text{Flux}=\iiint_D 3\left(x^2+y^2+z^2\right)dV$

$$=3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=3\int_0^{2\pi}\int_0^{\pi}\tfrac{a^5}{5}\sin\phi\,d\phi\,d\theta=3\int_0^{2\pi}\tfrac{2a^5}{5}\,d\theta=\tfrac{12\pi a^5}{5}$$

13. Let $\rho=\sqrt{x^2+y^2+z^2}$. Then $\frac{\partial\rho}{\partial x}=\frac{x}{\rho}$, $\frac{\partial\rho}{\partial y}=\frac{y}{\rho}$, $\frac{\partial\rho}{\partial z}=\frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x)=\left(\frac{\partial\rho}{\partial x}\right)x+\rho=\frac{x^2}{\rho}+\rho$, $\frac{\partial}{\partial y}(\rho y)=\left(\frac{\partial\rho}{\partial y}\right)y+\rho$

$=\frac{y^2}{\rho}+\rho$, $\frac{\partial}{\partial z}(\rho z)=\left(\frac{\partial\rho}{\partial z}\right)z+\rho=\frac{z^2}{\rho}+\rho \Rightarrow \nabla\cdot\mathbf{F}=\frac{x^2+y^2+z^2}{\rho}+3\rho=4\rho$, since $\rho=\sqrt{x^2+y^2+z^2}$

$$\Rightarrow \text{Flux}=\iiint_D 4\rho\,dV=\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}(4\rho)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}3\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}6\,d\theta=12\pi$$

14. Let $\rho=\sqrt{x^2+y^2+z^2}$. Then $\frac{\partial\rho}{\partial x}=\frac{x}{\rho}$, $\frac{\partial\rho}{\partial y}=\frac{y}{\rho}$, $\frac{\partial\rho}{\partial z}=\frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right)=\frac{1}{\rho}-\left(\frac{x}{\rho^2}\right)\frac{\partial\rho}{\partial x}=\frac{1}{\rho}-\frac{x^2}{\rho^3}$. Similarly,

$\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right)=\frac{1}{\rho}-\frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right)=\frac{1}{\rho}-\frac{z^2}{\rho^3} \Rightarrow \nabla\cdot\mathbf{F}=\frac{3}{\rho}-\frac{x^2+y^2+z^2}{\rho^3}=\frac{2}{\rho}$

$$\Rightarrow \text{Flux}=\iiint_D \tfrac{2}{\rho}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_1^{2}\left(\tfrac{2}{\rho}\right)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}15\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}30\,d\theta=60\pi$$

15. $\frac{\partial}{\partial x}(5x^3 + 12xy^2) = 15x^2 + 12y^2$, $\frac{\partial}{\partial y}(y^3 + e^y \sin z) = 3y^2 + e^y \sin z$, $\frac{\partial}{\partial z}(5z^3 + e^y \cos z) = 15z^2 - e^y \sin z$

$\Rightarrow \nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2 \Rightarrow \text{Flux} = \iiint_D 15\rho^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (15\rho^2)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} (12\sqrt{2} - 3) \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} (24\sqrt{2} - 6)\, d\theta = (48\sqrt{2} - 12)\pi$

16. $\frac{\partial}{\partial x}[\ln(x^2 + y^2)] = \frac{2x}{x^2 + y^2}$, $\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2 + y^2}$, $\frac{\partial}{\partial z}\left(z\sqrt{x^2 + y^2}\right) = \sqrt{x^2 + y^2}$

$\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2} \Rightarrow \text{Flux} = \iiint_D \left(\frac{2x}{x^2 + y^2} - \frac{2z}{x^2 + y^2} + \sqrt{x^2 + y^2}\right) dz\, dy\, dx$

$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2} \left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}} \left(6\cos\theta - \frac{3}{r} + 3r^2\right) dr\, d\theta$

$= \int_0^{2\pi} \left[6(\sqrt{2} - 1)\cos\theta - 3\ln\sqrt{2} + 2\sqrt{2} - 1\right] d\theta = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2} - 1\right)$

17. (a) $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \Rightarrow \nabla \times \mathbf{G} = \text{curl } \mathbf{G} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{k} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \Rightarrow \nabla \cdot \nabla \times \mathbf{G}$

$= \text{div}(\text{curl } \mathbf{G}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0$ if all first and second partial derivatives are continuous

(b) By the Divergence Theorem, the outward flux of $\nabla \times \mathbf{G}$ across a closed surface is zero because

outward flux of $\nabla \times \mathbf{G} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, d\sigma$

$= \iiint_D \nabla \cdot \nabla \times \mathbf{G}\, dV$ [Divergence Theorem with $\mathbf{F} = \nabla \times \mathbf{G}$]

$= \iiint_D (0)\, dV = 0$ [by part (a)]

18. (a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k} \Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2$

$= (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k} \Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2)$

$= \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$

$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a $\Rightarrow$ $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2)$

$$= \left[\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i} + \left[\left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right) - \left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$$

$$+ \left[\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x}\right) - \left(a\frac{\partial M_1}{\partial y} + b\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k} = a\left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\mathbf{k}\right]$$

$$+ b\left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\mathbf{k}\right] = a\,\nabla \times \mathbf{F}_1 + b\,\nabla \times \mathbf{F}_2$$

(c) $$\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2)$$

$$= \nabla \cdot [(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial y}(M_1P_2 - P_1M_2) + \frac{\partial}{\partial z}(M_1N_2 - N_1M_2) = \left(P_2\frac{\partial N_1}{\partial x} + N_1\frac{\partial P_2}{\partial x} - N_2\frac{\partial P_1}{\partial x} - P_1\frac{\partial N_2}{\partial x}\right)$$

$$- \left(M_1\frac{\partial P_2}{\partial y} + P_2\frac{\partial M_1}{\partial y} - P_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial y}\right) + \left(M_1\frac{\partial N_2}{\partial z} + N_2\frac{\partial M_1}{\partial z} - N_1\frac{\partial M_2}{\partial z} - M_2\frac{\partial N_1}{\partial z}\right)$$

$$= M_2\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right) + P_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) + M_1\left(\frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial y}\right) + N_1\left(\frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial z}\right)$$

$$+ P_1\left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\right) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$$

19. (a) $$\operatorname{div}(g\mathbf{F}) = \nabla \cdot g\mathbf{F} = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) = \left(g\frac{\partial M}{\partial x} + M\frac{\partial g}{\partial x}\right) + \left(g\frac{\partial N}{\partial y} + N\frac{\partial g}{\partial y}\right) + \left(g\frac{\partial P}{\partial z} + P\frac{\partial g}{\partial x}\right)$$

$$= \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} + P\frac{\partial g}{\partial z}\right) + g\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) = g\,\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$$

(b) $$\nabla \times (g\mathbf{F}) = \left[\frac{\partial}{\partial y}(gP) - \frac{\partial}{\partial z}(gN)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(gM) - \frac{\partial}{\partial x}(gP)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(gN) - \frac{\partial}{\partial y}(gM)\right]\mathbf{k}$$

$$= \left(P\frac{\partial g}{\partial y} + g\frac{\partial P}{\partial y} - N\frac{\partial g}{\partial z} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} + g\frac{\partial M}{\partial z} - P\frac{\partial g}{\partial x} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} + g\frac{\partial N}{\partial x} - M\frac{\partial g}{\partial y} - g\frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$= \left(P\frac{\partial g}{\partial y} - N\frac{\partial g}{\partial z}\right)\mathbf{i} + \left(g\frac{\partial P}{\partial y} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} - P\frac{\partial g}{\partial x}\right)\mathbf{j} + \left(g\frac{\partial M}{\partial z} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} - M\frac{\partial g}{\partial y}\right)\mathbf{k}$$

$$+ \left(g\frac{\partial N}{\partial x} - g\frac{\partial M}{\partial y}\right)\mathbf{k} = g\,\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$$

20. Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$.

(a) $\mathbf{F}_1 \times \mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$

$$= \left[\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial x}(M_1N_2 - N_1M_2)\right]\mathbf{j}$$

$+\left[\frac{\partial}{\partial x}(P_1M_2 - M_1P_2) - \frac{\partial}{\partial y}(N_1P_2 - P_1N_2)\right]\mathbf{k}$

and consider the **i**-component only: $\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)$

$= N_2\frac{\partial M_1}{\partial y} + M_1\frac{\partial N_2}{\partial y} - M_2\frac{\partial N_1}{\partial y} - N_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial z} - P_1\frac{\partial M_2}{\partial z} + P_2\frac{\partial M_1}{\partial z} + M_1\frac{\partial P_2}{\partial z}$

$= \left(N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1 - \left(\frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$

$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1$

$-\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$. Now, **i**-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial}{\partial x} + N_2\frac{\partial}{\partial y} + P_2\frac{\partial}{\partial z}\right)M_1$

$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$; likewise, **i**-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$;

i-comp of $(\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 = \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1$ and **i**-comp of $(\nabla \cdot \mathbf{F}_1)\mathbf{F}_2 = \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$.

Similar results hold for the **j** and **k** components of $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result

$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla \times (\mathbf{F}_1 \cdot \mathbf{F}_2)$

$= \frac{\partial}{\partial x}(M_1M_2 + N_1N_2 + P_1P_2) = \left(M_1\frac{\partial M_2}{\partial x} + M_2\frac{\partial M_1}{\partial x} + N_1\frac{\partial N_2}{\partial x} + N_2\frac{\partial N_1}{\partial x} + P_1\frac{\partial P_2}{\partial x} + P_2\frac{\partial P_1}{\partial x}\right)$

$=$ **i**-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$,

i-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$,

i-comp of $\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) = N_1\left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right) - P_1\left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)$, and

i-comp of $\mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) - P_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)$.

Since corresponding components are equal, we see that

$\nabla \times (\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$, as claimed.

21. The integral's value never exceeds the surface area of S. Since $|\mathbf{F}| \le 1$, we have $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}||\mathbf{n}| \le (1)(1) = 1$ and

$$\iiint_D \nabla \cdot \mathbf{F}\, d\sigma = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{[Divergence Theorem]}$$

$$\le \iint_S |\mathbf{F} \cdot \mathbf{n}|\, d\sigma \qquad \text{[A property of integrals]}$$

$$\le \iint_S (1)\, d\sigma \qquad [|\mathbf{F} \cdot \mathbf{n}| \le 1]$$

$= \text{Area of S}.$

22. Yes, the outward flux through the top is 5. The reason is this: Since $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}$ $= 1 - 2 + 1 = 0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5. Therefore the flux across the top is 5.

23. (a) $\frac{\partial}{\partial x}(x) = 1,\ \frac{\partial}{\partial y}(y) = 1,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux} = \iiint\limits_D 3\,dV = 3\iiint\limits_D dV$
$= 3(\text{Volume of the solid})$

(b) If $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of S, then $\mathbf{F}\cdot\mathbf{n} = 0$ everywhere $\Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 0$.
But the flux is 3(Volume of the solid) $\neq 0$, so $\mathbf{F}$ is not orthogonal to $\mathbf{n}$ at every point.

24. $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12)\,dz\,dy\,dx$
$= \int_0^a \int_0^b (-2x - 4y + 9)\,dy\,dx = \int_0^a \left(-2xb - 2b^2 + 9b\right) dx = -a^2b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a,b);$
$\frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$ and $a(-a - 4b + 9) = 0 \Rightarrow b = 0$ or $-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0$ $\Rightarrow \text{Flux} = 0$; $-2a - 2b + 9 = 0$ and $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f\left(3, \frac{3}{2}\right) = \frac{27}{2}$ is the maximum flux.

25. $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F}\,dV = \iiint\limits_D 3\,dV \Rightarrow \frac{1}{3}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint\limits_D dV = \text{Volume of D}$

26. $\mathbf{F} = \mathbf{C} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F}\,dV = \iiint\limits_D 0\,dV = 0$

27. (a) From the Divergence Theorem, $\iint\limits_S \nabla f\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \nabla \cdot \nabla f\,dV = \iiint\limits_D \nabla^2 f\,dV = \iiint\limits_D 0\,dV = 0$

(b) From the Divergence Theorem, $\iint\limits_S f\nabla f\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \nabla \cdot f\nabla f\,dV$. Now,

$$f\nabla f = \left(f\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial f}{\partial z}\right)\mathbf{k} \Rightarrow \nabla \cdot f\nabla f = \left[f\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right]$$

$= f\nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2$ since f is harmonic $\Rightarrow \iint\limits_S f\nabla f\cdot\mathbf{n}\,d\sigma = \iiint\limits_D |\nabla f|^2\,dV$, as claimed.

28. From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right) dV$. Now,

$f(x,y,z) = \ln \sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2+y^2+z^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}, \frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2}$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}, \Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$

$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} \Rightarrow \iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \frac{\rho^2 \sin\phi}{\rho^2}\, d\rho\, d\phi\, d\theta$

$= \int_0^{\pi/2}\int_0^{\pi/2} a \sin\phi\, d\phi\, d\theta = \int_0^{\pi/2} \left[-a\cos\phi\right]_0^{\pi/2} d\theta = \int_0^{\pi/2} a\, d\theta = \frac{\pi a}{2}$

29. $\iint_S f\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f\nabla g\, dV = \iiint_D \nabla \cdot \left(f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}\right) dV$

$= \iiint_D \left(f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) dV$

$= \iiint_D \left[f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right] dV = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

30. $\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, d\sigma$

$= \iiint_D \nabla \cdot (f\nabla g - g\nabla f)\, dV$

$= \iiint_D (\nabla \cdot f\nabla g - \nabla \cdot g\nabla f)\, dV$ (Exercise 18a)

$= \iiint_D (f\nabla \cdot \nabla g + \nabla f \cdot \nabla g - g\nabla \cdot \nabla f - \nabla g \cdot \nabla f)\, dV$ (Exercise 19a)

$= \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV$, since $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$

31. (a) The integral $\iiint_D p(t,x,y,z)\, dV$ represents the mass of the fluid at any time t. The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D: the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting $\mathbf{n}$ as the outward pointing unit normal to the surface).

(b) $\iiint_D \frac{\partial p}{\partial t}\, dV = \frac{d}{dt}\iiint_D p\, dV = -\iint_S p\mathbf{v} \cdot \mathbf{n}\, d\sigma = -\iiint_D \nabla \cdot p\mathbf{v}\, dV \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot p\mathbf{v}$

$\Rightarrow \nabla \cdot p\mathbf{v} + \frac{\partial p}{\partial t} = 0$, as claimed

32. (a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow$ ∇T points away from the point $\Rightarrow -\nabla T$ points toward the point $\Rightarrow -\nabla T$ points in the direction the heat flows.

(b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k\nabla T = \mathbf{v}$ and $c\rho T = p$, we have $\frac{d}{dt}\iiint\limits_{D} c\rho T \, dV = -\iint\limits_{S} -k\nabla T \cdot \mathbf{n}\, d\sigma \Rightarrow$ the continuity equation, $\nabla \cdot (-k\nabla T) + \frac{\partial}{\partial t}(c\rho T) = 0$ $\Rightarrow c\rho\frac{\partial T}{\partial t} = -\nabla \cdot (-k\nabla T) = k\nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c\rho}\nabla^2 T = K\nabla^2 T$, as claimed

CHAPTER 13 PRACTICE EXERCISES

1. Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t,\ y = t,\ z = t,\ 0 \le t \le 1 \Rightarrow f(g(t),h(t),k(t)) = 3 - 3t^2$ and $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 1,$ $\frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{3}\, dt \Rightarrow \int\limits_{C} f(x,y,z)\, ds = \int_0^1 \sqrt{3}\left(3 - 3t^2\right) dt = 2\sqrt{3}$

Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = 0 \Rightarrow f(g(t),h(t),k(t)) = 2t - 3t^2 + 3$ and $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 1,$ $\frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{2}\, dt \Rightarrow \int\limits_{C_1} f(x,y,z)\, ds = \int_0^1 \sqrt{2}\left(2t - 3t^2 + 3\right) dt = 3\sqrt{2};$

$\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1,\ y = 1,\ z = t \Rightarrow f(g(t),h(t),k(t)) = 2 - 2t$ and $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 1$ $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int\limits_{C_2} f(x,y,z)\, ds = \int_0^1 (2 - 2t)\, dt = 1$ $\Rightarrow \int\limits_{C} f(x,y,z)\, ds = \int\limits_{C_1} f(x,y,z)\, ds + \int\limits_{C_2} f(x,y,z) = 3\sqrt{2} + 1$

2. Path 1: $\mathbf{r}_1 = t\mathbf{i} \Rightarrow x = t,\ y = 0,\ z = 0 \Rightarrow f(g(t),h(t),k(t)) = t^2$ and $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 0$ $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int\limits_{C_1} f(x,y,z)\, ds = \int_0^1 t^2\, dt = \frac{1}{3};$

$\mathbf{r}_2 = \mathbf{i} + t\mathbf{j} \Rightarrow x = 1,\ y = t,\ z = 0 \Rightarrow f(g(t),h(t),k(t)) = 1 + t$ and $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = 0$ $\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int\limits_{C_2} f(x,y,z)\, ds = \int_0^1 (1 + t)\, dt = \frac{3}{2};$

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1,\ y = 1,\ z = t \Rightarrow f(g(t),h(t),k(t)) = 2 - t$ and $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_3} f(x,y,z)\,ds = \int_0^1 (2-t)\,dt = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 1}} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \int_{C_3} f(x,y,z)\,ds = \frac{10}{3}$

Path 2: $\mathbf{r}_4 = t\mathbf{i} + t\mathbf{j} \Rightarrow x = t,\ y = t,\ z = 0 \Rightarrow f(g(t),h(t),k(t)) = t^2 + t$ and $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{2}\,dt \Rightarrow \int_{C_4} f(x,y,z)\,ds = \int_0^1 \sqrt{2}\left(t^2+t\right)dt = \frac{5}{6}\sqrt{2};$

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$ (see above) $\Rightarrow \int_{C_3} f(x,y,z)\,ds = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 2}} f(x,y,z)\,ds = \int_{C_3} f(x,y,z)\,ds + \int_{C_4} f(x,y,z)\,ds = \frac{5}{6}\sqrt{2} + \frac{3}{2} = \frac{5\sqrt{2}+9}{6}$

Path 3: $\mathbf{r}_5 = t\mathbf{k} \Rightarrow x = 0,\ y = 0,\ z = t,\ 0 \le t \le 1 \Rightarrow f(g(t),h(t),k(t)) = -t$ and $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_5} f(x,y,z)\,ds = \int_0^1 -t\,dt = -\frac{1}{2};$

$\mathbf{r}_6 = t\mathbf{j} + \mathbf{k} \Rightarrow x = 0,\ y = t,\ z = 1,\ 0 \le t \le 1 \Rightarrow f(g(t),h(t),k(t)) = t - 1$ and $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_6} f(x,y,z)\,ds = \int_0^1 (t-1)\,dt = -\frac{1}{2};$

$\mathbf{r}_7 = t\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow x = t,\ y = 1,\ z = 1,\ 0 \le t \le 1 \Rightarrow f(g(t),h(t),k(t)) = t^2$ and $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_7} f(x,y,z)\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$

$\Rightarrow \int_{\text{Path 3}} f(x,y,z)\,ds = \int_{C_5} f(x,y,z)\,ds + \int_{C_6} f(x,y,z)\,ds + \int_{C_7} f(x,y,z)\,ds = -\frac{1}{2} - \frac{1}{2} + \frac{1}{3} = -\frac{2}{3}$

3. $\mathbf{r} = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k} \Rightarrow x = 0,\ y = a\cos t,\ z = a\sin t \Rightarrow f(g(t),h(t),k(t)) = \sqrt{a^2\sin^2 t} = a\,|\sin t|$ and

$\frac{dx}{dt} = 0,\ \frac{dy}{dt} = -a\sin t,\ \frac{dz}{dt} = a\cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = a\,dt$

$\Rightarrow \int_C f(x,y,z)\,ds = \int_0^{2\pi} a^2\,|\sin t|\,dt = \int_0^{\pi} a^2\sin t\,dt + \int_{\pi}^{2\pi} -a^2\sin t\,dt = 4a^2$

4. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow x = \cos t + t\sin t,\ y = \sin t - t\cos t,\ z = 0$

$\Rightarrow f(g(t),h(t),k(t)) = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2} = \sqrt{1+t^2}$ and $\frac{dx}{dt} = -\sin t + \sin t + t\cos t$

$= t\cos t,\ \frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t,\ \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = |t|\,dt = t\,dt$ since $0 \le t \le \sqrt{3} \Rightarrow \int_C f(x,y,z)\,ds = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,dt = \frac{7}{3}$

5. $\frac{\partial P}{\partial y} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x+y+z}} \Rightarrow f(x,y,z) = 2\sqrt{x+y+z} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x+y+z}} + \frac{\partial g}{\partial y}$

$= \frac{1}{\sqrt{x+y+z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = 2\sqrt{x+y+z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x+y+z}} + h'(z)$

$= \frac{1}{\sqrt{x+y+z}} \Rightarrow h'(x) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = 2\sqrt{x+y+z} + C \Rightarrow \int_{(-1,1,1)}^{(4,-3,0)} \frac{dx+dy+dz}{\sqrt{x+y+z}}$

$= f(4,-3,0) - f(-1,1,1) = 2\sqrt{1} - 2\sqrt{1} = 0$

6. $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 1 \Rightarrow f(x,y,z)$

$= x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y,z) = -2\sqrt{yz} + h(z) \Rightarrow f(x,y,z) = x - 2\sqrt{yz} + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x - 2\sqrt{yz} + C$

$\Rightarrow \int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\,dy - \sqrt{\frac{y}{z}}\,dz = f(10,3,3) - f(1,1,1) = (10 - 2\cdot 3) - (1 - 2\cdot 1) = 4 + 1 = 5$

7. $\frac{\partial M}{\partial z} = -y\cos z \ne y\cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$ is not conservative; $\mathbf{r} = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} - \mathbf{k},\ 0 \le t \le 2\pi$

$\Rightarrow d\mathbf{r} = -2\sin t\mathbf{i} - 2\cos t\mathbf{j} \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} [-(-2\sin t)(\sin(-1))(-2\sin t) + (2\cos t)(\sin(-1))(-2\cos t)]\,dt$

$= 4\sin(1)\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 8\pi\sin(1)$

8. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = 0$

9. Let $M = 8x\sin y$ and $N = -8y\cos x \Rightarrow \frac{\partial M}{\partial y} = 8x\cos y$ and $\frac{\partial N}{\partial x} = 8y\sin x \Rightarrow \int_C 8x\sin y\,dx - 8y\cos x\,dy$

$= \iint_R (8y\sin x - 8x\cos y)\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2} (8y\sin x - 8x\cos y)\,dy\,dx = \int_0^{\pi/2} \left(\pi^2\sin x - 8x\right)dx$

$= -\pi^2 + \pi^2 = 0$

10. Let $M = y^2$ and $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dx\,dy$

$= \int_0^{2\pi}\int_0^2 (2r\cos\theta - 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \frac{16}{3}(\cos\theta - \sin\theta)\,d\theta = 0$

11. Let $z = 1 - x - y \Rightarrow f_x(x,y) = -1$ and $f_y(x,y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow$ Surface Area $= \iint_R \sqrt{3}\,dx\,dy$

$= \sqrt{3}$(Area of the circular region in the xy-plane) $= \pi\sqrt{3}$

12. $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 3$

$\Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{9 + 4y^2 + 4z^2}}{3}\,dy\,dz = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\sqrt{9 + 4r^2}}{3}\,r\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi} \left(\frac{7}{4}\sqrt{21} - \frac{9}{4}\right)d\theta = \frac{\pi}{6}(7\sqrt{21} - 9)$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = |2z| = 2z$ since

$z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{2}{2z}\,dA = \iint_R \frac{1}{z}\,dA = \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - r^2}}\,r\,dr\,d\theta$

$\int_0^{2\pi} \left[-\sqrt{1 - r^2}\right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \left(1 - \frac{1}{\sqrt{2}}\right)d\theta = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right)$

14. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since

$z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{4}{2z}\,dA = \iint_R \frac{2}{z}\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{2}{\sqrt{4 - r^2}}\,r\,dr\,d\theta = 4\pi - 8$

(b) $r = 2\cos\theta \Rightarrow dr = -2\sin\theta\,d\theta$; $ds^2 = r^2\,d\theta^2 + dr^2$ (Arc length in polar coordinates)

$\Rightarrow ds^2 = (2\cos\theta)^2\,d\theta^2 + dr^2 = 4\cos^2\theta\,d\theta^2 + 4\sin^2\theta\,d\theta^2 = 4\,d\theta^2 \Rightarrow ds = 2\,d\theta$; the height of the cylinder is $z = \sqrt{4 - r^2} = \sqrt{4 - 4\cos^2\theta} = 2|\sin\theta| = 2\sin\theta$ if $0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow$ Surface Area $= \int_{-\pi/2}^{\pi/2} h\,ds = 2\int_0^{\pi/2} (2\sin\theta)(2\,d\theta) = 8$

15. $f(x,y,z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c}$

since $c > 0 \Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}{\left(\frac{1}{c}\right)}\,dA = c\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}\iint_R dA = \frac{1}{2}abc\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$,

since the area of the triangular region R is $\frac{1}{2}ab$

16. (a) $\nabla f = 2y\mathbf{j} - \mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\, dx\,dy$

$$\Rightarrow \iint_S g(x,y,z)\, d\sigma = \iint_R \frac{yz}{\sqrt{4y^2+1}} \sqrt{4y^2+1}\, dx\,dy = \iint_R y\left(y^2-1\right) dx\,dy = \int_{-1}^{1}\int_0^3 \left(y^3-y\right) dx\,dy$$

$$= \int_{-1}^{1} 3\left(y^3-y\right) dy = 3\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{1} = 0$$

(b) $$\iint_S g(x,y,z)\, d\sigma = \iint_R \frac{z}{\sqrt{4y^2+1}} \sqrt{4y^2+1}\, dx\,dy = \int_{-1}^{1}\int_0^3 \left(y^2-1\right) dx\,dy = \int_{-1}^{1} 3\left(y^2-1\right) dy$$

$$= 3\left[\frac{y^3}{3} - y\right]_{-1}^{1} = -4$$

17. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+4z^2} = 2\sqrt{y^2+z^2} = 10$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$

$$\Rightarrow d\sigma = \frac{10}{2z}\, dx\,dy = \frac{5}{z}\, dx\,dy = \iint_S g(x,y,z)\, d\sigma = \iint_R \left(x^4y\right)\left(y^2+z^2\right)\left(\frac{5}{z}\right) dx\,dy$$

$$= \iint_R \left(x^4y\right)(25)\left(\frac{5}{\sqrt{25-y^2}}\right) dx\,dy = \int_0^4\int_0^1 \frac{125y}{\sqrt{25-y^2}}\, x^4\, dx\,dy = \int_0^4 \frac{25y}{\sqrt{25-y^2}}\, dy = 50$$

18. Define the coordinate system so that the origin is at the center of the earth, the z-axis is the earth's axis (north is the positive z direction), and the xz-plane contains the earth's prime meridian. Let S denote the surface which is Wyoming so then S is part of the surface $z = \left(R^2-x^2-y^2\right)^{1/2}$. Let R_{xy} be the projection of S onto the xy-plane. The surface area of Wyoming is $\iint_S 1\, d\sigma = \iint_{R_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$

$$\iint_{R_{xy}} \sqrt{\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}+1}\, dA = \iint_{R_{xy}} \frac{R}{\left(R^2-x^2-y^2\right)^{1/2}}\, dA = \int_{\theta_1}^{\theta_2}\int_{R\sin 45^\circ}^{R\sin 49^\circ} R\left(R^2-r^2\right)^{-1/2} r\, dr\, d\theta$$

(where θ_1 and θ_2 are the radian equivalent to $104^\circ 3'$ and $111^\circ 3'$, respectively)

$$= \int_{\theta_1}^{\theta_2} -R\left(R^2-r^2\right)^{1/2}\Big|_{R\sin 45^\circ}^{R\sin 49^\circ} = \int_{\theta_1}^{\theta_2} R\left(R^2-R^2\sin^2 45^\circ\right)^{1/2} - R\left(R^2-R^2\sin^2 49^\circ\right)^{1/2} d\theta$$

$$= \left(\theta_2-\theta_1\right)R^2(\cos 45^\circ - \cos 49^\circ) = \frac{7\pi}{180}R^2(\cos 45^\circ - \cos 49^\circ) = \frac{7\pi}{180}(3959)^2(\cos 45^\circ - \cos 49^\circ)$$

$\approx 97{,}751$ sq. mi.

19. A possible parametrization is $\mathbf{r}(\phi,\theta) = (6\sin\phi\cos\theta)\mathbf{i} + (6\sin\phi\sin\theta)\mathbf{j} + (6\cos\phi)\mathbf{k}$ (spherical coordinates); now $\rho = 6$ and $z = -3 \Rightarrow -3 = 6\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ and $z = 3\sqrt{3} \Rightarrow 3\sqrt{3} = 6\cos\phi$ $\Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$; also $0 \le \theta \le 2\pi$

20. A possible parametrization is $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - \left(\frac{r^2}{2}\right)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2+y^2} \Rightarrow z = -\frac{r^2}{2}$ and $-2 \le z \le 0 \Rightarrow -2 \le -\frac{r^2}{2} \le 0 \Rightarrow 4 \ge r^2 \ge 0 \Rightarrow 0 \le r \le 2$ since $r \ge 0$; also $0 \le \theta \le 2\pi$

21. A possible parametrization is $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1+r)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2+y^2} \Rightarrow z = 1+r$ and $1 \le z \le 3 \Rightarrow 1 \le 1+r \le 3 \Rightarrow 0 \le r \le 2$; also $0 \le \theta \le 2\pi$

22. A possible parametrization is $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \left(3 - x - \frac{y}{2}\right)\mathbf{k}$ for $0 \le x \le 2$ and $0 \le y \le 2$

23. Let $x = u\cos v$ and $z = u\sin v$, where $u = \sqrt{x^2+z^2}$ and v is the angle in the xz-plane with the x-axis $\Rightarrow \mathbf{r}(u,v) = (u\cos v)\mathbf{i} + 2u^2\mathbf{j} + (u\sin v)\mathbf{k}$ is a possible parametrization; $0 \le y \le 2 \Rightarrow 2u^2 \le 2 \Rightarrow u^2 \le 1$ $\Rightarrow 0 \le u \le 1$ since $u \ge 0$; also, for just the upper half of the paraboloid, $0 \le v \le \pi$

24. A possible parametrization is $\left(\sqrt{10}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{10}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{10}\cos\phi\right)\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le \frac{\pi}{2}$

25. $\mathbf{r}_u = \mathbf{i}+\mathbf{j}$, $\mathbf{r}_v = \mathbf{i}-\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6}$

$$\Rightarrow \text{Surface Area} = \iint_{R_{uv}} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^1\int_0^1 \sqrt{6}\,du\,dv = \sqrt{6}$$

26. $$\iint_S (xy - z^2)\,d\sigma = \int_0^1\int_0^1 \left[(u+v)(u-v) - v^2\right]\sqrt{6}\,du\,dv = \sqrt{6}\int_0^1\int_0^1 (u^2 - 2v^2)\,du\,dv$$
$$= \sqrt{6}\int_0^1 \left[\frac{u^3}{3} - 2uv^2\right]_0^1 dv = \sqrt{6}\int_0^1 \left(\frac{1}{3} - 2v^2\right) dv = \sqrt{6}\left[\frac{1}{3}v - \frac{2}{3}v^3\right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$$

27. $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$

$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2} \Rightarrow \text{Surface Area} = \iint_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln\left(r + \sqrt{1+r^2}\right)\right]_0^1 d\theta = \int_0^{2\pi} \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right] d\theta$$
$$= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

28. $\iint_S \sqrt{x^2+y^2+1}\, d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1}\,\sqrt{1+r^2}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (1+r^2)\, dr\, d\theta$

$= \int_0^{2\pi} \left[r + \frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8}{3}\pi$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

30. $\frac{\partial P}{\partial y} = \frac{-3zy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-3xz}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-3xy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

31. $\frac{\partial P}{\partial y} = 0 \neq ye^z = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

32. $\frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

33. $\frac{\partial f}{\partial x} = 2 \Rightarrow f(x,y,z) = 2x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z \Rightarrow g(y,z) = y^2 + zy + h(z)$

$\Rightarrow f(x,y,z) = 2x + y^2 + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = y + h'(z) = y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + C$

$\Rightarrow f(x,y,z) = 2x + y^2 + zy + z$

34. $\frac{\partial f}{\partial x} = z\cos xz \Rightarrow f(x,y,z) = \sin xz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \Rightarrow g(y,z) = e^y + h(z)$

$\Rightarrow f(x,y,z) = \sin xz + e^y + h(z) \Rightarrow \frac{\partial f}{\partial z} = x\cos xz + h'(z) = x\cos xz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x,y,z) = \sin xz + e^y$

35. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = t$ and $d\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$\Rightarrow \mathbf{F}\cdot d\mathbf{r} = (3t^2+1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2+1)\,dt = 2;$

Over Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = 0$ and $d\mathbf{r}_1 = (\mathbf{i}+\mathbf{j})\,dt \Rightarrow \mathbf{F}_1 = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$\Rightarrow \mathbf{F}_1\cdot d\mathbf{r}_1 = (2t^2+1)\,dt \Rightarrow \text{Work}_1 = \int_0^1 (2t^2+1)\,dt = \frac{5}{3};\ \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = 1,\ y = 1,\ z = t$ and

$d\mathbf{r}_2 = \mathbf{k}\,dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}_2\cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3} + 1 = \frac{8}{3}$

36. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = t$ and $d\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F}\cdot d\mathbf{r} = (3t^2+1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2+1)\,dt = 2;$

Over Path 2: Since f is conservative, $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ around any simple closed curve C. Thus consider

$\int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the path from $(0,0,0)$ to $(1,1,0)$ to $(1,1,1)$ and C_2 is the path from $(1,1,1)$ to $(0,0,0)$. Now, from Path 1 above, $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + (-2)$

$\Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2$

37. (a) $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} \Rightarrow x = e^t \cos t,\ y = e^t \sin t$ from $(1,0)$ to $(e^{2\pi},0) \Rightarrow 0 \le t \le 2\pi$

$\Rightarrow \frac{d\mathbf{r}}{dt} = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j}$ and $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{(x^2+y^2)^{3/2}} = \frac{(e^t\cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}}{(e^{2t}\cos^2 t + e^{2t}\sin^2 t)^{3/2}}$

$= \left(\frac{\cos t}{e^{2t}}\right)\mathbf{i} + \left(\frac{\sin t}{e^{2t}}\right)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(\frac{\cos^2 t}{e^t} - \frac{\sin t\cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t \cos t}{e^t}\right) = e^{-t}$

$\Rightarrow \text{Work} = \int_0^{2\pi} e^{-t}\,dt = 1 - e^{-2\pi}$

(b) $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{(x^2+y^2)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} + \frac{\partial g}{\partial y}$

$= \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow g(y,z) = C \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2}$ is a potential function for $\mathbf{F} \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r}$

$= f(e^{2\pi},0) - f(1,0) = 1 - e^{-2\pi}$

38. (a) $\mathbf{F} = \nabla(x^2 z e^y) \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for <u>any</u> closed path C

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(1,0,0)}^{(1,0,2\pi)} \nabla(x^2ze^y)\cdot d\mathbf{r} = (x^2ze^y)\big|_{(1,0,2\pi)} - (x^2ze^y)\big|_{(1,0,0)} = 2\pi - 0 = 2\pi$

39. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k}$; unit normal to the plane is $\mathbf{n} = \frac{2\mathbf{i}+6\mathbf{j}-3\mathbf{k}}{\sqrt{4+36+9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{6}{7}y$; $\mathbf{p} = \mathbf{k}$ and $f(x,y,z) = 2x + 6y - 3z \Rightarrow |\nabla f\cdot\mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \frac{7}{3}\,dA$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{6}{7}y\,d\sigma = \iint_R \left(\frac{6}{7}y\right)\left(\frac{7}{3}\,dA\right) = \iint_R 2y\,dA = \int_0^{2\pi}\int_0^1 2r\sin\theta\, r\,dr\,d\theta = \int_0^{2\pi} \frac{2}{3}\sin\theta\,d\theta = 0$

40. $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y & x+y & 4y^2-z\end{vmatrix}=8y\mathbf{i}$; the circle lies in the plane $f(x,y,z)=y+z=0$ with unit normal

$\mathbf{n}=\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\Rightarrow\nabla\times\mathbf{F}\cdot\mathbf{n}=0\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_R\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_R 0\,d\sigma=0$

41. (a) $\mathbf{r}=\sqrt{2}t\mathbf{i}+\sqrt{2}t\mathbf{j}+(4-t^2)\mathbf{k},\ 0\le t\le 1\Rightarrow x=\sqrt{2}t,\ y=\sqrt{2}t,\ z=4-t^2\Rightarrow\frac{dx}{dt}=\sqrt{2},\ \frac{dy}{dt}=\sqrt{2},\ \frac{dz}{dt}=-2t$

$\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{4+4t^2}\,dt\Rightarrow M=\int_C\delta(x,y,z)\,ds=\int_0^1 3t\sqrt{4+4t^2}\,dt=\left[\frac{1}{4}(4+4t)^{3/2}\right]_0^1$

$=4\sqrt{2}-2$

(b) $M=\int_C\delta(x,y,z)\,ds=\int_0^1\sqrt{4+4t^2}\,dt=\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

42. $\mathbf{r}=t\mathbf{i}+2t\mathbf{j}+\frac{2}{3}t^{3/2}\mathbf{k},\ 0\le t\le 2\Rightarrow x=t,\ y=2t,\ z=\frac{2}{3}t^{3/2}\Rightarrow\frac{dx}{dt}=1,\ \frac{dy}{dt}=2,\ \frac{dz}{dt}=t^{1/2}$

$\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{t+5}\,dt\Rightarrow M=\int_C\delta(x,y,z)\,ds=\int_0^2 3\sqrt{5+t}\,\sqrt{t+5}\,dt$

$=\int_0^2 3(t+5)\,dt=36;\ M_{yz}=\int_C x\delta\,ds=\int_0^2 3t(t+5)\,dt=38;\ M_{xz}=\int_C y\delta\,ds=\int_0^2 6t(t+5)\,dt=76;$

$M_{xy}=\int_C z\delta\,ds=\int_0^2 2t^{3/2}(t+5)\,dt=\frac{144}{7}\sqrt{2}\Rightarrow\bar{x}=\frac{M_{yz}}{M}=\frac{38}{36}=\frac{19}{18},\ \bar{y}=\frac{M_{xz}}{M}=\frac{76}{36}=\frac{19}{9},\ \bar{z}=\frac{M_{xy}}{M}=\frac{\left(\frac{144}{7}\sqrt{2}\right)}{36}$

$=\frac{4}{7}\sqrt{2}$

43. $\mathbf{r}=t\mathbf{i}+\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j}+\left(\frac{t^2}{2}\right)\mathbf{k},\ 0\le t\le 2\Rightarrow x=t,\ y=\frac{2\sqrt{2}}{3}t^{3/2},\ z=\frac{t^2}{2}\Rightarrow\frac{dx}{dt}=1,\ \frac{dy}{dt}=\sqrt{2}t^{1/2},\ \frac{dz}{dt}=t$

$\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{1+2t+t^2}\,dt=\sqrt{(t+1)^2}\,dt=|t+1|\,dt=(t+1)\,dt$ on the domain given.

Then $M=\int_C\delta\,ds=\int_0^2\left(\frac{1}{t+1}\right)(t+1)\,dt=\int_0^2 dt=2;\ M_{yz}=\int_C x\delta\,ds=\int_0^2 t\left(\frac{1}{t+1}\right)(t+1)\,dt=\int_0^2 t\,dt=2;$

$M_{xz}=\int_C y\delta\,ds=\int_0^2\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt=\int_0^2\frac{2\sqrt{2}}{3}t^{3/2}\,dt=\frac{32}{15};\ M_{xy}=\int_C z\delta\,ds$

$=\int_0^2\left(\frac{t^2}{2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt=\int_0^2\frac{t^2}{2}\,dt=\frac{4}{3}\Rightarrow\bar{x}=\frac{M_{yz}}{M}=\frac{2}{2}=1;\ \bar{y}=\frac{M_{xz}}{M}=\frac{\left(\frac{32}{15}\right)}{2}=\frac{16}{15};\ \bar{z}=\frac{M_{xy}}{M}$

$= \frac{\left(\frac{4}{3}\right)}{2} = \frac{2}{3}$; $I_x = \int_C (y^2+z^2)\delta\, ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{t^4}{4}\right) dt = \frac{232}{45}$; $I_y = \int_C (x^2+z^2)\delta\, ds = \int_0^2 \left(t^2 + \frac{t^4}{4}\right) dt = \frac{64}{15}$;

$I_z = \int_C (y^2+x^2)\delta\, ds = \int_0^2 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{56}{9}$; $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{\left(\frac{232}{45}\right)}{2}} = \frac{2\sqrt{29}}{3\sqrt{5}}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{\left(\frac{64}{15}\right)}{2}} = \frac{4\sqrt{2}}{\sqrt{15}}$;

$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\left(\frac{56}{9}\right)}{2}} = \frac{2\sqrt{7}}{3}$

44. $\overline{z} = 0$ because the arch is in the xy-plane, and $\overline{x} = 0$ because the mass is distributed symmetrically with respect to the y-axis; $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le \pi \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = a\, dt$, since $a \ge 0$; $M = \int_C \delta\, ds = \int_C (2a - y)\, ds = \int_0^\pi (2 - a\sin t)\, a\, dt$

$= 2a\pi - 2a^2$; $M_{xz} = \int_C y\delta\, dt = \int_C y(2a-y)\, ds = \int_0^\pi (a\sin t)(2a - a\sin t)\, dt = \int_0^\pi \left(2a^2 \sin t - a^2\sin^2 t\right) dt$

$= \left[-2a^2\cos t - a^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^\pi = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \overline{y} = \frac{\left(4a^2 - \frac{a^2\pi}{2}\right)}{2a\pi - 2a^2} = \frac{8a - a\pi}{4\pi - 4a} \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, \frac{8a - a\pi}{4\pi - 4a}, 0\right)$

45. $\mathbf{r}(t) = \left(e^t\cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} + e^t\mathbf{k}$, $0 \le t \le \ln 2 \Rightarrow x = e^t\cos t$, $y = e^t \sin t$, $z = e^t \Rightarrow \frac{dx}{dt} = \left(e^t\cos t - e^t\sin t\right)$,

$\frac{dy}{dt} = \left(e^t\sin t + e^t\cos t\right)$, $\frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \sqrt{\left(e^t\cos t - e^t\sin t\right)^2 + \left(e^t\sin t + e^t\cos t\right)^2 + \left(e^t\right)^2}\, dt = \sqrt{3e^{2t}}\, dt = \sqrt{3}\, e^t\, dt$; $M = \int_C \delta\, ds = \int_0^{\ln 2} \sqrt{3}\, e^t\, dt$

$= \sqrt{3}$; $M_{xy} = \int_C z\delta\, ds = \int_0^{\ln 2} \left(\sqrt{3}\, e^t\right)\left(e^t\right) dt = \int_0^{\ln 2} \sqrt{3}\, e^{2t}\, dt = \frac{3\sqrt{3}}{2} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{3}} = \frac{3}{2}$;

$I_z = \int_C (x^2+y^2)\delta\, ds = \int_0^{\ln 2} \left(e^{2t}\cos^2 t + e^{2t}\sin^2 t\right)\left(\sqrt{3}\, e^t\right) dt = \int_0^{\ln 2} \sqrt{3}\, e^{3t}\, dt = \frac{7\sqrt{3}}{3} \Rightarrow R_z = \sqrt{\frac{I_z}{M}}$

$= \sqrt{\frac{7\sqrt{3}}{3\sqrt{3}}} = \sqrt{\frac{7}{3}}$

46. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 3t\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow x = 2\sin t$, $y = 2\cos t$, $z = 3t \Rightarrow \frac{dx}{dt} = 2\cos t$, $\frac{dy}{dt} = -2\sin t$,

$\frac{dz}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{4+9}\, dt = \sqrt{13}\, dt$; $M = \int_C \delta\, ds = \int_0^{2\pi} \rho\sqrt{13}\, dt = 2\pi\rho\sqrt{13}$;

$M_{xy} = \int_C z\delta\, ds = \int_0^{2\pi} (3t)\left(\rho\sqrt{13}\right) dt = 6\rho\pi^2\sqrt{13}$; $M_{yz} = \int_C x\delta\, ds = \int_0^{2\pi} (2\sin t)\left(\rho\sqrt{13}\right) dt = 0$;

$M_{xz} = \int_C y\delta\, ds = \int_0^{2\pi} (2\cos t)\left(\rho\sqrt{13}\right) dt = 0 \Rightarrow \overline{x} = \overline{y} = 0$ and $\overline{z} = \frac{M_{xy}}{M} = \frac{6\rho\pi^2\sqrt{13}}{2\rho\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi)$ is the center of mass

47. Because of symmetry $\overline{x} = \overline{y} = 0$. Let $f(x,y,z) = x^2 + y^2 + z^2 - 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$, since $z \geq 0 \Rightarrow M = \iint_R \delta(x,y,z)\, d\sigma$

$= \iint_R z\left(\frac{10}{2z}\right) dA = \iint_R 5\, dA = 5(\text{Area of the circular region}) = 80\pi$; $M_{xy} = \iint_R z\delta\, d\sigma = \iint_R 5z\, dA$

$= \iint_R 5\sqrt{25 - x^2 - y^2}\, dx\, dy = \int_0^{2\pi}\int_0^4 \left(5\sqrt{25 - r^2}\right) r\, dr\, d\theta = \int_0^{2\pi} \frac{490}{3}\, d\theta = \frac{980}{3}\pi \Rightarrow \overline{z} = \frac{\left(\frac{980}{3}\pi\right)}{80\pi} = \frac{49}{12}$

$\Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{49}{12}\right)$; $I_z = \iint_R (x^2 + y^2)\delta\, d\sigma = \iint_R 5(x^2 + y^2)\, dx\, dy = \int_0^{2\pi}\int_0^4 5r^3\, dr\, d\theta = \int_0^{2\pi} 320\, d\theta = 640\pi$;

$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{640\pi}{80\pi}} = 2\sqrt{2}$

48. On the face $z = 1$: $g(x,y,z) = z = 1$ and $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (x^2 + y^2)\, dA = 2\int_0^{\pi/4}\int_0^{\sec\theta} r^3\, dr\, d\theta = \frac{2}{3}$; On the face $z = 0$: $g(x,y,z) = z = 0 \Rightarrow \nabla g = \mathbf{k}$ and $\mathbf{p} = \mathbf{k}$

$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + y^2)\, dA = \frac{2}{3}$; On the face $y = 0$: $g(x,y,z) = y = 0$

$\Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + 0)\, dA = \int_0^1\int_0^1 x^2\, dx\, dz = \frac{1}{3}$;

On the face $y = 1$: $g(x,y,z) = y = 1 \Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (x^2 + 1^2)\, dA = \int_0^1\int_0^1 (x^2 + 1)\, dx\, dz = \frac{4}{3}$; On the face $x = 1$: $g(x,y,z) = x = 1 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i}$

$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (1^2 + y^2)\, dA = \int_0^1\int_0^1 (1 + y^2)\, dy\, dz = \frac{4}{3}$; On the face $x = 0$: $g(x,y,z) = x = 0 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (0^2 + y^2)\, dA = \int_0^1\int_0^1 y^2\, dy\, dz = \frac{1}{3} \Rightarrow I_z = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = \frac{14}{3}$

49. $M = 2xy + x$ and $N = xy - y \Rightarrow \frac{\partial M}{\partial x} = 2y + 1,\ \frac{\partial M}{\partial y} = 2x,\ \frac{\partial N}{\partial x} = y,\ \frac{\partial N}{\partial y} = x - 1 \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \iint_R (2y + 1 + x - 1)\,dy\,dx = \int_0^1 \int_0^1 (2y + x)\,dy\,dx = \frac{3}{2}$; $\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R (y - 2x)\,dy\,dx = \int_0^1 \int_0^1 (y - 2x)\,dy\,dx = -\frac{1}{2}$

50. $M = y - 6x^2$ and $N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = -12x,\ \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \iint_R (-12x + 2y)\,dx\,dy = \int_0^1 \int_y^1 (-12x + 2y)\,dx\,dy = \int_0^1 (4y^2 + 2y - 6)\,dy = -\frac{11}{3}$;

$\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 - 1)\,dx\,dy = 0$

51. $M = -\frac{\cos y}{x}$ and $N = \ln x \sin y \Rightarrow \frac{\partial M}{\partial y} = \frac{\sin y}{x}$ and $\frac{\partial N}{\partial x} = \frac{\sin y}{x} \Rightarrow \oint_C \ln x \sin y\,dy - \frac{\cos y}{x}\,dx$

$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x}\right) dx\,dy = 0$

52. (a) Let $M = x$ and $N = y \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \iint_R (1 + 1)\,dx\,dy = 2 \iint_R dx\,dy = 2(\text{Area of the region})$

(b) Let C be a closed curve to which Green's Theorem applies and let $\mathbf{n}$ be the unit normal vector to C. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and assume $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of C. Then the flux density of $\mathbf{F}$ at every point of C is 0 since $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of C $\Rightarrow \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$ at every point of C

$\Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$. But part (a) above states that the flux is 2(Area of the region) $\Rightarrow$ the area of the region would be 0 $\Rightarrow$ contradiction. Therefore, $\mathbf{F}$ cannot be orthogonal to $\mathbf{n}$ at every point of C.

53. $\frac{\partial}{\partial x}(2xy) = 2y,\ \frac{\partial}{\partial y}(2yz) = 2z,\ \frac{\partial}{\partial z}(2xz) = 2x \Rightarrow \nabla \cdot \mathbf{F} = 2y + 2z + 2x \Rightarrow \text{Flux} = \iiint_D (2x + 2y + 2z)\,dV$

$= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz = \int_0^1 \int_0^1 (1 + 2y + 2z)\,dy\,dz = \int_0^1 (2 + 2z)\,dz = 3$