CHAPTER 9 VECTORS IN THE PLANE AND POLAR FUNCTIONS

9.1 VECTORS IN THE PLANE

1. (a) $\langle 3(3), 3(-2)\rangle = \langle 9, -6\rangle$

   (b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$

   (b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

3. (a) $\langle 3 + (-2), -2 + 5\rangle = \langle 1, 3\rangle$

   (b) $\sqrt{1^2 + 3^2} = \sqrt{10}$

4. (a) $\langle 3 - (-2), -2 - 5\rangle = \langle 5, -7\rangle$

   (b) $\sqrt{5^2 + (-7)^2} = \sqrt{74}$

5. (a) $2\mathbf{u} = \langle 2(3), 2(-2)\rangle = \langle 6, -4\rangle$
   $3\mathbf{v} = \langle 3(-2), 3(5)\rangle = \langle -6, 15\rangle$
   $2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-6), -4 - 15\rangle = \langle 12, -19\rangle$

   (b) $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

6. (a) $-2\mathbf{u} = \langle -2(3), -2(-2)\rangle = \langle -6, 4\rangle$
   $5\mathbf{v} = \langle 5(-2), 5(5)\rangle = \langle -10, 25\rangle$
   $-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25\rangle = \langle -16, 29\rangle$

   (b) $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2)\right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
   $\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5)\right\rangle = \left\langle -\frac{8}{5}, 4\right\rangle$
   $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4\right\rangle = \left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$

   (b) $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{14}(-2)\right\rangle = -\left\langle \frac{15}{13}, \frac{10}{13}\right\rangle$
   $\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5)\right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13}\right\rangle$
   $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13}\right\rangle = \left\langle -3, \frac{70}{13}\right\rangle$

   (b) $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

9. $\langle 2 - 1, -1 - 3\rangle = \langle 1, -4\rangle$

10. $\left\langle \frac{2 + (-4)}{2} - 0, \frac{-1 + 3}{2} - 0\right\rangle = \langle -1, 1\rangle$

11. $\langle 0 - 2, 0 - 3\rangle = \langle -2, -3\rangle$

12. $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1)\rangle = \langle 1, 1\rangle$
    $\overrightarrow{CD} = \langle -2 - (-1), 2 - 3\rangle = \langle -1, -1\rangle$
    $\overrightarrow{AB} + \overrightarrow{CD} = \langle 1 + (-1), 1 + (-1)\rangle = \langle 0, 0\rangle$

13. $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$

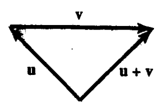
14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

15. This is the unit vector which makes an angle of $120^\circ + 90^\circ = 210^\circ$ with the positive x-axis;
    $\langle \cos 210^\circ, \sin 210^\circ\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

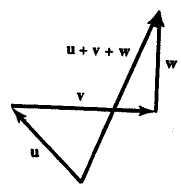
16. $\langle \cos 135^\circ, \sin 135^\circ\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

17. The vector **v** is horizontal and 1 in. long. The vectors **u** and **w** are $\frac{11}{16}$ in. long. **w** is vertical and **u** makes a 45° angle with the horizontal. All vectors must be drawn to scale.

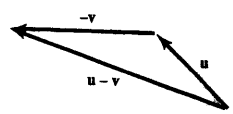
(a)



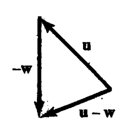
(b)



(c)

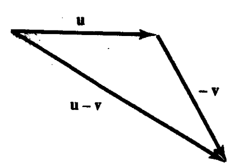


(d)

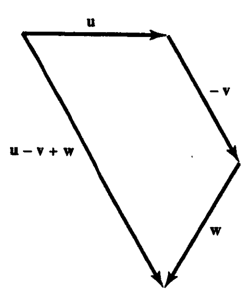


18. The angle between the vectors is 120° and vector **u** is horizontal. They are all 1 in. long. Draw to scale.

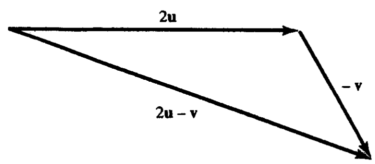
(a)



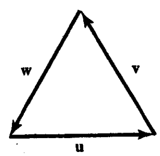
(b)



(c)

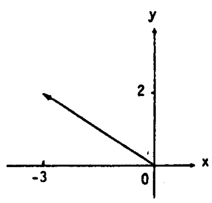


(d)

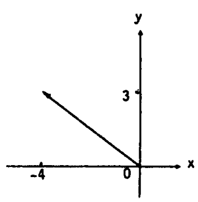


$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$

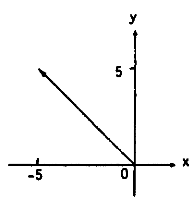
19. $\overrightarrow{P_1P_2} = (2-5)\mathbf{i} + (9-7)\mathbf{j} = -3\mathbf{i} + 2\mathbf{j}$



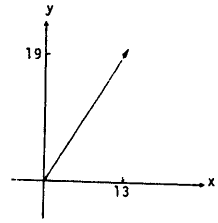
20. $\overrightarrow{P_1P_2} = (-3-1)\mathbf{i} + (5-2)\mathbf{j} = -4\mathbf{i} + 3\mathbf{j}$



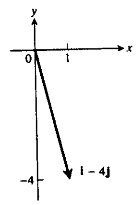
21. $\overrightarrow{AB} = (-10-(-5))\mathbf{i} + (8-3)\mathbf{j} = -5\mathbf{i} + 5\mathbf{j}$



22. $\overrightarrow{AB} = (6-(-7))\mathbf{i} + (11-(-8))\mathbf{j} = 13\mathbf{i} + 19\mathbf{j}$

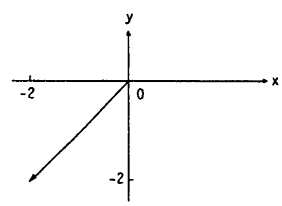


23. $\overrightarrow{P_1P_2} = (2-1)\mathbf{i} + (-1-3)\mathbf{j} = \mathbf{i} - 4\mathbf{j}$



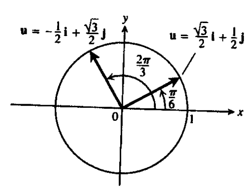
24. P_4 is $\left(\frac{2-4}{2}, \frac{-1+3}{2}\right) = (-1, 1)$

$\Rightarrow \overrightarrow{P_3P_4} = (-1-1)\mathbf{i} + (1-3)\mathbf{j} = -2\mathbf{i} - 2\mathbf{j}$



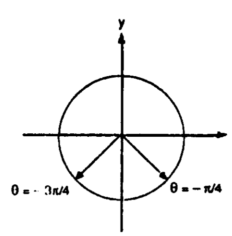
25. $\mathbf{u} = \left(\cos\frac{\pi}{6}\right)\mathbf{i} + \left(\sin\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;

$\mathbf{u} = \left(\cos\frac{2\pi}{3}\right)\mathbf{i} + \left(\sin\frac{2\pi}{3}\right)\mathbf{j} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

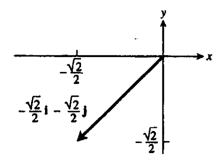


26. $\mathbf{u} = \left(\cos\left(-\frac{\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(-\frac{\pi}{4}\right)\right)\mathbf{j} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$;

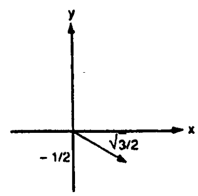
$\mathbf{u} = \left(\cos\left(-\frac{3\pi}{4}\right)\right)\mathbf{i} + \left(\sin\left(-\frac{3\pi}{4}\right)\right)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$



27. $\mathbf{u}=\left(\cos\left(\frac{\pi}{2}+\frac{3\pi}{4}\right)\right)\mathbf{i}+\left(\sin\left(\frac{\pi}{2}+\frac{3\pi}{4}\right)\right)\mathbf{j}$
$=\left(\cos\left(\frac{5\pi}{4}\right)\right)\mathbf{i}+\left(\sin\left(\frac{5\pi}{4}\right)\right)\mathbf{j}$
$=-\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j}$



28. $\mathbf{u}=\left(\cos\left(\frac{\pi}{2}-\frac{2\pi}{3}\right)\right)\mathbf{i}+\left(\sin\left(\frac{\pi}{2}-\frac{2\pi}{3}\right)\right)\mathbf{j}$
$=\left(\cos\left(-\frac{\pi}{6}\right)\right)\mathbf{i}+\left(\sin\left(-\frac{\pi}{6}\right)\right)\mathbf{j}$
$=\frac{\sqrt{3}}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$



29. $\sqrt{3^2+4^2}=5;\ \frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5},\frac{4}{5}\right\rangle$

30. $\sqrt{4^2+(-3)^2}=5;\ \frac{1}{5}\langle 4,-3\rangle=\left\langle\frac{4}{5},-\frac{3}{5}\right\rangle$

31. $\sqrt{(-15)^2+8^2}=17;\ \frac{1}{17}\langle -15,8\rangle=\left\langle-\frac{15}{17},\frac{8}{17}\right\rangle$

32. $\sqrt{(-5)^2+(-2)^2}=\sqrt{29};$
$\frac{1}{\sqrt{29}}\langle -5,-2\rangle=\left\langle-\frac{5}{\sqrt{29}},-\frac{2}{\sqrt{29}}\right\rangle$

33. $|6\mathbf{i}-8\mathbf{j}|=\sqrt{36+64}=10\Rightarrow\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{6}{10}\mathbf{i}-\frac{8}{10}\mathbf{j}=\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$

34. $|-\mathbf{i}+3\mathbf{j}|=\sqrt{1+9}=\sqrt{10}\Rightarrow\frac{\mathbf{v}}{|\mathbf{v}|}=-\frac{1}{\sqrt{10}}\mathbf{i}+\frac{3}{\sqrt{10}}\mathbf{j}$

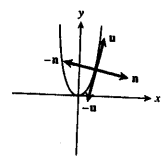
35. $\mathbf{v}=5\mathbf{i}+12\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{25+144}=13\Rightarrow\mathbf{v}=|\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)=13\left(\frac{5}{13}\mathbf{i}+\frac{12}{13}\mathbf{j}\right)$

36. $\mathbf{v}=2\mathbf{i}-3\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{4+9}=\sqrt{13}\Rightarrow\mathbf{v}=|\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)=\sqrt{13}\left(\frac{2}{\sqrt{13}}\mathbf{i}-\frac{3}{\sqrt{13}}\mathbf{j}\right)$

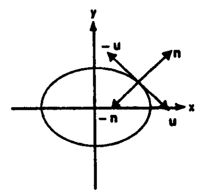
37. $\mathbf{v}=3\mathbf{i}-4\mathbf{j}\Rightarrow|\mathbf{v}|=\sqrt{9+16}=5\Rightarrow\mathbf{u}=\pm\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)=\pm\frac{1}{5}(3\mathbf{i}-4\mathbf{j})$

38. $\mathbf{A}=-\mathbf{i}+2\mathbf{j}\Rightarrow|\mathbf{A}|=\sqrt{1+4}=\sqrt{5}\Rightarrow\mathbf{v}=-2\frac{\mathbf{A}}{|\mathbf{A}|}=-2\left(-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{j}\right)=\frac{2}{\sqrt{5}}\mathbf{i}-\frac{4}{\sqrt{5}}\mathbf{j}$ is a vector of length 2 whose direction is opposite to $\mathbf{A}$; there is only one such vector

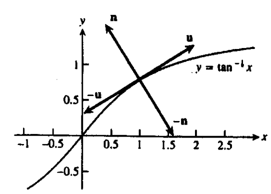
39. $\frac{dy}{dx} = 2x\,|_{x=2} = 4 \Rightarrow \mathbf{i} + 4\mathbf{j}$ is tangent to the curve at $(2,4)$
$\Rightarrow \mathbf{u} = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{4}{\sqrt{17}}\mathbf{i} - \frac{1}{\sqrt{17}}\mathbf{j}$ and $-\mathbf{n} = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$ are unit normal vectors



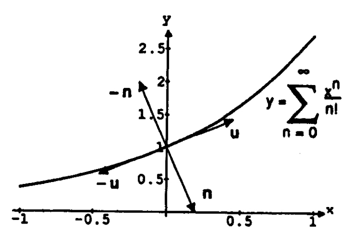
40. $2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{4y}\Big|_{(2,1)} = -1 \Rightarrow \mathbf{i} - \mathbf{j}$ is tangent to the curve at $(2,1) \Rightarrow \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are unit normal vectors



41. $\frac{dy}{dx} = \frac{1}{1+x^2}\Big|_{x=1} = \frac{1}{2} \Rightarrow \mathbf{i} + \frac{1}{2}\mathbf{j}$ is tangent to the curve at $(1,1) \Rightarrow 2\mathbf{i} + \mathbf{j}$ is tangent $\Rightarrow \mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$ and $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ and $-\mathbf{n} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$ are unit normal vectors



42. $\frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x\,|_{(0,1)} = 1 \Rightarrow \mathbf{i} + \mathbf{j}$ is tangent to the curve at $(0,1) \Rightarrow \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are unit tangent vectors; $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ are unit normal vectors



43. $6x + 8y + 8x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x+4y}{4x+2y}\Big|_{(1,0)} = -\frac{3}{4} \Rightarrow 4\mathbf{i} - 3\mathbf{j}$ is tangent to the curve at $(1,0)$
$\Rightarrow \mathbf{u} = \pm\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ are unit normal vectors

44. $2x - 6y - 6x\frac{dy}{dx} + 16y\frac{dy}{dx} - 2 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-3y-1}{8y-3x}\Big|_{(1,1)} = \frac{3}{5} \Rightarrow 5\mathbf{i} + 3\mathbf{j}$ is tangent to the curve at $(1,1)$
$\Rightarrow \mathbf{u} = \pm\frac{1}{\sqrt{34}}(5\mathbf{i} + 3\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{\sqrt{34}}(-3\mathbf{i} + 5\mathbf{j})$ are unit normal vectors

45. $\frac{dy}{dx} = \sqrt{3+x^4}\big|_{(0,0)} = \sqrt{3} \Rightarrow \mathbf{i} + \sqrt{3}\mathbf{j}$ is tangent to the curve at $(0,0) \Rightarrow \mathbf{u} = \pm\frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$ are unit tangent vectors and $\mathbf{v} = \pm\frac{1}{2}(-\sqrt{3}\mathbf{i} + \mathbf{j})$ are unit normal vectors

46. $\frac{dy}{dx} = \ln(\ln x)\big|_{(e,0)} = \ln 1 = 0 \Rightarrow \mathbf{u} = \pm\mathbf{i}$ are unit tangent vectors and $\mathbf{v} = \pm\mathbf{j}$ are unit normal vectors

47. $2\mathbf{i} + \mathbf{j} = a(\mathbf{i}+\mathbf{j}) + b(\mathbf{i}-\mathbf{j}) = (a+b)\mathbf{i} + (a-b)\mathbf{j} \Rightarrow a+b = 2$ and $a-b = 1 \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$ and $b = a - 1 = \frac{1}{2}$

48. $\mathbf{i} - 2\mathbf{j} = a(2\mathbf{i}+3\mathbf{j}) + b(\mathbf{i}+\mathbf{j}) = (2a+b)\mathbf{i} + (3a+b)\mathbf{j} \Rightarrow 2a+b = 1$ and $3a+b = -2 \Rightarrow a = -3$ and $b = 1 - 2a = 7 \Rightarrow \mathbf{u}_1 = a(2\mathbf{i}+3\mathbf{j}) = -6\mathbf{i} - 9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i}+\mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$

49. If $|x|$ is the magnitude of the x-component, then $\cos 30° = \frac{|x|}{|\mathbf{F}|} \Rightarrow |x| = |\mathbf{F}| \cos 30° = (10)\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$ lb $\Rightarrow \mathbf{F}_x = 5\sqrt{3}\,\mathbf{i}$;
if $|y|$ is the magnitude of the y-component, then $\sin 30° = \frac{|y|}{|\mathbf{F}|} \Rightarrow |y| = |\mathbf{F}| \sin 30° = (10)\left(\frac{1}{2}\right) = 5$ lb $\Rightarrow \mathbf{F}_y = 5\,\mathbf{j}$.

50. If $|x|$ is the magnitude of the x-component, then $\cos 45° = \frac{|x|}{|\mathbf{F}|} \Rightarrow |x| = |\mathbf{F}| \cos 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb $\Rightarrow \mathbf{F}_x = -6\sqrt{2}\,\mathbf{i}$ (the negative sign is indicated by the diagram);
if $|y|$ is the magnitude of the y-component, then $\sin 45° = \frac{|y|}{|\mathbf{F}|} \Rightarrow |y| = |\mathbf{F}| \sin 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb $\Rightarrow \mathbf{F}_y = -6\sqrt{2}\,\mathbf{j}$ (the negative sign is indicated by the diagram).

51. 25° west of north is $90° + 25° = 115°$ north of east.
$800\langle \cos 115°, \sin 115° \rangle \approx \langle -338.095, 725.045 \rangle$

52. 10° east of south is $270° + 10° = 280°$ "north" of east.
$600\langle \cos 280°, \sin 280° \rangle \approx \langle 104.189, -590.885 \rangle$

53. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5 \cos 60°)\mathbf{i} + (5 \sin 60°)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10 \cos 315°)\mathbf{i} + (10 \sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$
$\Rightarrow Q = \left(\frac{5+\sqrt{2}}{2}, \frac{5\sqrt{3} - 10\sqrt{2}}{2}\right)$

54. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (7 \cos 45°)\mathbf{i} + (7 \sin 45°)\mathbf{j} = \frac{7\sqrt{2}}{2}\mathbf{i} + \frac{7\sqrt{2}}{2}\mathbf{j}$
$\Rightarrow P = \left(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$

(b) The telephone pole is located at the point Q which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$= \left(\frac{7\sqrt{2}}{2}\mathbf{i} + \frac{7\sqrt{2}}{2}\mathbf{j}\right) + (8 \cos 210°)\mathbf{i} + (8 \sin 210°)\mathbf{j} = \left(\frac{7\sqrt{2}}{2} - \frac{8\sqrt{3}}{2}\right)\mathbf{i} + \left(\frac{7\sqrt{2}}{2} - \frac{8}{2}\right)\mathbf{j}$

$\Rightarrow Q = \left(\frac{7\sqrt{2}}{2} - 4\sqrt{3}, \frac{7\sqrt{2}}{2} - 4\right)$

9.2 DOT PRODUCTS

NOTE: In Exercises 1-6 below we calculate $\text{proj}_{\mathbf{v}}\, \mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}| \cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
1.	-12	$2\sqrt{5}$	$2\sqrt{5}$	$-\frac{3}{5}$	$-\frac{6\sqrt{5}}{5}$	$-\frac{6}{5}\mathbf{i} + \frac{12}{5}\mathbf{j}$
2.	24	$2\sqrt{26}$	$2\sqrt{2}$	$\frac{3\sqrt{13}}{13}$	$\frac{6\sqrt{26}}{13}$	$\frac{3}{13}(2\mathbf{i} + 10\mathbf{j})$
3.	$\sqrt{3} - \sqrt{2}$	$\sqrt{2}$	$\sqrt{5}$	$\frac{\sqrt{30} - \sqrt{20}}{10}$	$\frac{\sqrt{6} - 2}{2}$	$\frac{\sqrt{3} - \sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$
4.	$10 + \sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10 + \sqrt{17}}{\sqrt{546}}$	$\frac{10 + \sqrt{17}}{\sqrt{26}}$	$\frac{10 + \sqrt{17}}{26}(5\mathbf{i} + \mathbf{j})$
5.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$
6.	-1	1	1	-1	-1	$-\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

7. $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{(2)(1) + (1)(2)}{\sqrt{2^2+1^2}\,\sqrt{1^2+2^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{4}{5}\right) \approx 0.64$ rad

8. $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{u} = 3\mathbf{i} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$, $|\mathbf{u}| = 3$, and $\mathbf{v}\cdot\mathbf{u} = 2(3) + (-2)0 = 6$
$\Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $6 = (2\sqrt{2})(3)\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \approx 0.79$

9. $\mathbf{v} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{u} = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{3}\right)^2 + (-7)^2} = 2\sqrt{13}$, $|\mathbf{u}| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$, and
$\mathbf{v}\cdot\mathbf{u} = \left(\sqrt{3}\right)\left(\sqrt{3}\right) + (-7)(1) = -4 \Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $-4 = \left(2\sqrt{13}\right)(2)\cos\theta$
$\Rightarrow \theta = \cos^{-1}\left(-\frac{\sqrt{13}}{13}\right) \approx 1.85$

10. $\mathbf{v} = \mathbf{i} + \sqrt{2}\mathbf{j}$, $\mathbf{u} = -\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{3}$, $|\mathbf{u}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, and
$\mathbf{v}\cdot\mathbf{u} = (1)(-1) + \left(\sqrt{2}\right)(1) = -1 + \sqrt{2} \Rightarrow \mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta$ gives $-1 + \sqrt{2} = \left(\sqrt{3}\right)\left(\sqrt{2}\right)\cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\frac{-1+\sqrt{2}}{\sqrt{6}}\right) \approx 1.40$

11. $\overrightarrow{AB} = \langle 3,1\rangle$, $\overrightarrow{BC} = \langle -1,-3\rangle$, and $\overrightarrow{AC} = \langle 2,-2\rangle$. $\overrightarrow{BA} = \langle -3,-1\rangle$, $\overrightarrow{CB} = \langle 1,3\rangle$, and $\overrightarrow{CA} = \langle -2,2\rangle$.

$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{10}$, $|\overrightarrow{BC}| = |\overrightarrow{CB}| = \sqrt{10}$, and $|\overrightarrow{AC}| = |\overrightarrow{CA}| = 2\sqrt{2}$.

Angle at A $= \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$,

Angle at B $= \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{(\sqrt{10})(\sqrt{10})}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$, and

Angle at C $= \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{|\overrightarrow{CB}||\overrightarrow{CA}|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$,

12. $\overrightarrow{AC} = \langle 2,4\rangle$ and $\overrightarrow{BD} = \langle 4,-2\rangle$

$\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4)+4(-2) = 0$, so the angle measures 90°.

13. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1+\mathbf{v}_2\cdot\mathbf{v}_1-\mathbf{v}_1\cdot\mathbf{v}_2-\mathbf{v}_2\cdot\mathbf{v}_2 = |\mathbf{v}_1|^2-|\mathbf{v}_2|^2 = 0$

14. $\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

15. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$ $\Rightarrow \mathbf{d}_1\cdot\mathbf{d}_2 = (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

16. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1\cdot\mathbf{d}_2 = 0 \Leftrightarrow (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{v}|^2-|\mathbf{u}|^2 = 0$ $\Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|)(|\mathbf{v}|-|\mathbf{u}|) = 0 \Leftrightarrow (|\mathbf{v}|+|\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}|-|\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}| \Rightarrow$ the rectangle is a square.

17. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the opposite sides of a parallelogram be the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$. The equal diagonals of the parallelogram are

$d_1 = (v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})$ and $d_2 = (v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})$. Hence $|d_1| = |d_2| = |(v_1\mathbf{i}+v_2\mathbf{j})+(u_1\mathbf{i}+u_2\mathbf{j})|$ $= |(v_1\mathbf{i}+v_2\mathbf{j})-(u_1\mathbf{i}+u_2\mathbf{j})| \Rightarrow |(v_1+u_1)\mathbf{i}+(v_2+u_2)\mathbf{j}| = |(v_1-u_1)\mathbf{i}+(v_2-u_2)\mathbf{j}|$

$\Rightarrow \sqrt{(v_1+u_1)^2+(v_2+u_2)^2} = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2} \Rightarrow v_1^2+2v_1u_1+u_1^2+v_2^2+2v_2u_2+u_2^2$ $= v_1^2-2v_1u_1+u_1^2+v_2^2-2v_2u_2+u_2^2 \Rightarrow 2(v_1u_1+v_2u_2) = -2(v_1u_1+v_2u_2) \Rightarrow v_1u_1+v_2u_2 = 0$

$\Rightarrow (v_1\mathbf{i}+v_2\mathbf{j})\cdot(u_1\mathbf{i}+u_2\mathbf{j}) = 0 \Rightarrow$ the vectors $(v_1\mathbf{i}+v_2\mathbf{j})$ and $(u_1\mathbf{i}+u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

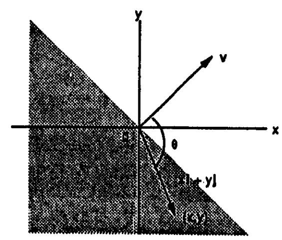
18. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u}+\mathbf{v}$ is the indicated diagonal, then $(\mathbf{u}+\mathbf{v})\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u} = |\mathbf{u}|^2+\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2$ $= \mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}|}\right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{v}|}\right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

19. horizontal component: $1200\cos(8^\circ) \approx 1188$ ft/s; vertical component: $1200\sin(8^\circ) \approx 167$ ft/s

20. $|\mathbf{w}|\cos(33^\circ - 15^\circ) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5 \text{ lb}}{\cos 18^\circ}$. Then $\mathbf{w} = \frac{2.5 \text{ lb}}{\cos 18^\circ}\langle\cos 33^\circ, \sin 33^\circ\rangle \approx \langle 2.205, 1.432\rangle$.

21. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta| \le |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|$.
(b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

22. $(x\mathbf{i}+y\mathbf{j})\cdot\mathbf{v} = |x\mathbf{i}+y\mathbf{j}||\mathbf{v}|\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



23. $\mathbf{v}\cdot\mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2)\cdot\mathbf{u}_1 = a\mathbf{u}_1\cdot\mathbf{u}_1 + b\mathbf{u}_2\cdot\mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2\cdot\mathbf{u}_1) = a(1)^2 + b(0) = a$

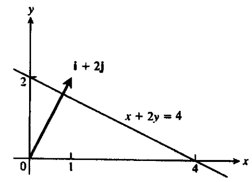
24. No, $\mathbf{v}_1$need not equal $\mathbf{v}_2$. For example, $\mathbf{i}+\mathbf{j} \ne \mathbf{i}+2\mathbf{j}$ but $\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}) = \mathbf{i}\cdot\mathbf{i}+\mathbf{i}\cdot\mathbf{j} = 1+0 = 1$ and $\mathbf{i}\cdot(\mathbf{i}+2\mathbf{j}) = \mathbf{i}\cdot\mathbf{i}+2\mathbf{i}\cdot\mathbf{j} = 1+2\cdot 0 = 1$.

25. $P(x_1, y_1) = P\left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$ are any two points P and Q on the line with $b \ne 0$ $\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_2 - x_1)\mathbf{j} \Rightarrow \overrightarrow{PQ}\cdot\mathbf{v} = \left[(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_2 - x_1)\mathbf{j}\right]\cdot(a\mathbf{i}+b\mathbf{j}) = a(x_2 - x_1) + b\left(\frac{a}{b}\right)(x_2 - x_1)$ $= 0 \Rightarrow \mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \ne 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$.

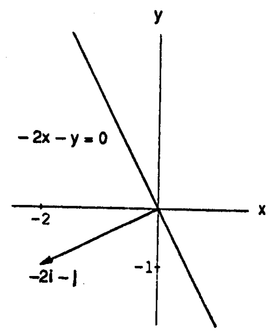
Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

26. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \ne 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $ax - by = c$.

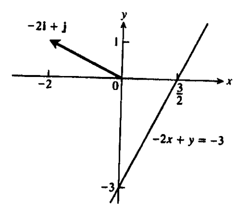
27. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;
    $P(2,1)$ on the line $\Rightarrow 2 + 2 = c \Rightarrow x + 2y = 4$



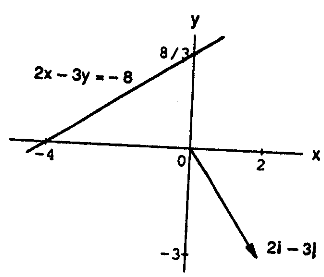
28. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;
    $P(-1,2)$ on the line $\Rightarrow (-2)(-1) - 2 = c \Rightarrow -2x - y = 0$



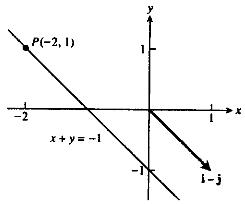
29. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
    $P(-2,-7)$ on the line $\Rightarrow (-2)(-2) - 7 = c \Rightarrow -2x + y = -3$



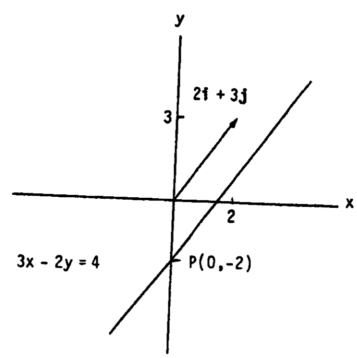
30. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
    $P(11,10)$ on the line $\Rightarrow (2)(11) - (3)(10) = c$
    $\Rightarrow 2x - 3y = -8$



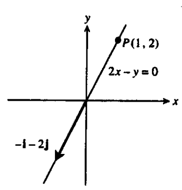
31. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $x + y = c$;
$P(-2, 1)$ on the line $\Rightarrow -2 + 1 = c \Rightarrow x + y = -1$



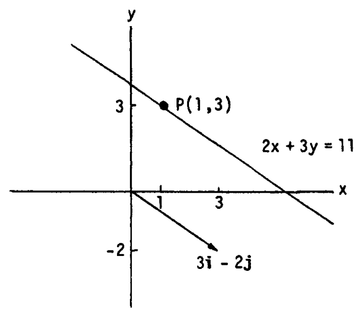
32. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;
$P(0, -2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$



33. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $2x - y = c$;
$P(1, 2)$ on the line $\Rightarrow (2)(1) - 2 = c \Rightarrow 2x - y = 0$



34. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $2x + 3y = c$; $P(1,3)$ on the line $\Rightarrow (2)(1) + (3)(3) = c$ $\Rightarrow 2x + 3y = 11$



35. $P(0,0)$, $Q(1,1)$ and $\mathbf{F} = 5\mathbf{j} \Rightarrow \overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5\ \mathrm{N \cdot m} = 5\ \mathrm{J}$

36. $\mathbf{W} = |\mathbf{F}|\,(\text{distance})\cos\theta = (602{,}148\ \mathrm{N})(605\ \mathrm{km})(\cos 0) = 364{,}299{,}540\ \mathrm{N \cdot km} = (364{,}299{,}540)(1000)\ \mathrm{N \cdot m}$ $= 3.6429954 \times 10^{11}\ \mathrm{J}$

37. $\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right|\cos\theta = (200)(20)(\cos 30^\circ) = 2000\sqrt{3} = 3464.10\ \mathrm{N \cdot m} = 3464.10\ \mathrm{J}$

38. $\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right|\cos\theta = (1000)(5280)(\cos 60^\circ) = 2{,}640{,}000\ \mathrm{ft \cdot lb}$

In Exercises 39-44 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

39. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

40. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

41. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

42. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = (1-\sqrt{3})\mathbf{i} + (1+\sqrt{3})\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$

$$= \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\,\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right) = \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

43. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\,\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14\ \mathrm{rad}$

44. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18\ \mathrm{rad}$

45. The angle between the corresponding normals is equal to the angle between the corresponding tangents. The points of intersection are $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$. At $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} \Rightarrow y = -\sqrt{3}x - \frac{3}{4}$, and the tangent line for $f(x) = \left(\frac{3}{2}\right) - x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$. The corresponding normals are $\mathbf{n}_1 = \sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; the angles are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. At $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$ and the tangent line for $f(x) = \frac{3}{2} - x^2$ is $y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4}$ $= -\sqrt{3}x - \frac{3}{4}$. The corresponding normals are $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; the angles are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

46. The points of intersection are $\left(0, \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2}\right)$. The curve $x = \frac{3}{4} - y^2$ has derivative $\frac{dy}{dx} = -\frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - 0) \Rightarrow \mathbf{n}_1 = \frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve at that point. The curve $x = y^2 - \frac{3}{4}$ has derivative $\frac{dy}{dx} = \frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - 0)$ $\Rightarrow \mathbf{n}_2 = -\frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve. The angle between the curves is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-\frac{1}{3}+1}{\sqrt{\frac{1}{3}+1}\sqrt{\frac{1}{3}+1}}\right) = \cos^{-1}\left(\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\frac{2\pi}{3}$. Because of symmetry the angles between the curves at the two points of intersection are the same.

47. The curves intersect when $y = x^3 = \left(y^2\right)^3 = y^6 \Rightarrow y = 0$ or $y = 1$. The points of intersection are $(0,0)$ and $(1,1)$. Note that $y \geq 0$ since $y = y^6$. At $(0,0)$ the tangent line for $y = x^3$ is $y = 0$ and the tangent line for $y = \sqrt{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$ is $\frac{\pi}{2}$. At $(1,1)$ the tangent line for $y = x^3$ is $y = 3x - 2$ and the tangent line for $y = \sqrt{x}$ is $y = \frac{1}{2}x + \frac{1}{2}$. The corresponding normal vectors are $\mathbf{n}_1 = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\frac{1}{2}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$; the angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

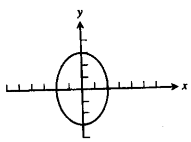
48. The points of intersection for the curves $y = -x^2$ and $y = \sqrt[3]{x}$ are $(0,0)$ and $(-1,-1)$. At $(0,0)$ the tangent line for $y = -x^2$ is $y = 0$ and the tangent line for $y = \sqrt[3]{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$

is $\frac{\pi}{2}$. At $(-1,-1)$ the tangent line for $y=-x^2$ is $y=2x+1$ and the tangent line for $y=\sqrt[3]{x}$ is $y=\frac{1}{3}x-\frac{2}{3}$. The corresponding normal vectors are $\mathbf{n}_1=2\mathbf{i}-\mathbf{j}$ and $\mathbf{n}_2=\frac{1}{3}\mathbf{i}-\mathbf{j} \Rightarrow \theta=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$

$$=\cos^{-1}\left(\frac{\frac{2}{3}+1}{\sqrt{5}\sqrt{\frac{1}{9}+1}}\right)=\cos^{-1}\left(\frac{\left(\frac{5}{3}\right)}{\frac{\sqrt{5}\sqrt{10}}{3}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}\text{; the angles are }\frac{\pi}{4}\text{ and }\frac{3\pi}{4}.$$

9.3 VECTOR-VALUED FUNCTIONS

1. (a)

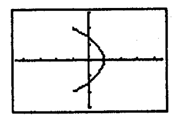


(b) $\mathbf{v}(t)=\frac{d}{dt}(2\cos t)\mathbf{i}+\frac{d}{dt}(3\sin t)\mathbf{j}$
$=(-2\sin t)\mathbf{i}+(3\cos t)\mathbf{j}$
$\mathbf{a}(t)=\frac{d}{dt}(-2\sin t)\mathbf{i}+\frac{d}{dt}(3\cos t)\mathbf{j}$
$=(-2\cos t)\mathbf{i}-(3\sin t)\mathbf{j}$

(c) $\mathbf{v}\left(\frac{\pi}{2}\right)=\langle-2,0\rangle$; speed $=\sqrt{(-2)^2+0^2}=2$,
direction $=\frac{1}{2}\langle-2,0\rangle=\langle-1,0\rangle$

(d) Velocity $=2\langle-1,0\rangle$

2. (a)



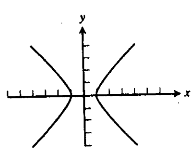
[−4.5, 4.5] by [−3, 3]

(b) $\mathbf{v}(t)=\frac{d}{dt}(\cos 2t)\mathbf{i}+\frac{d}{dt}(2\sin t)\mathbf{j}$
$=(-2\sin 2t)\mathbf{i}+(2\cos t)\mathbf{j}$
$\mathbf{a}(t)=\frac{d}{dt}(-2\sin 2t)\mathbf{i}+\frac{d}{dt}(2\cos t)\mathbf{j}$
$=(-4\cos 2t)\mathbf{i}-(2\sin t)\mathbf{j}$

(c) $\mathbf{v}(0)=\langle 0,2\rangle$; speed $=\sqrt{0^2+2^2}=2$,
direction $=\frac{1}{2}\langle 0,2\rangle=\langle 0,1\rangle$

(d) Velocity $=2\langle 0,1\rangle$

3. (a)



(b) $\mathbf{v}(t) = \frac{d}{dt}(\sec t)\mathbf{i} + \frac{d}{dt}(\tan t)\mathbf{i} = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j}$

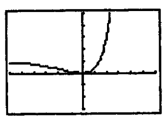
$\mathbf{a}(t) = \frac{d}{dt}(\sec t \tan t)\mathbf{i} + \frac{d}{dt}(\sec^2 t)\mathbf{j}$

$= (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2 \sec^2 t \tan t)\mathbf{j}$

(c) $\mathbf{v}\left(\frac{\pi}{6}\right) = \left\langle\frac{2}{3}, \frac{4}{3}\right\rangle$; speed $= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{2\sqrt{5}}{3}$, direction $= \frac{3}{2\sqrt{5}}\left\langle\frac{2}{3}, \frac{4}{3}\right\rangle = \left\langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

(d) Velocity $= \frac{2\sqrt{5}}{3}\left\langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

4. (a)



[−6, 6] by [−3, 5]

(b) $\mathbf{v}(t) = \frac{d}{dt}(2 \ln(t+1))\mathbf{i} + \frac{d}{dt}(t^2)\mathbf{j}$

$= \left(\frac{2}{t+1}\right)\mathbf{i} + (2t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}\left(\frac{2}{t+1}\right)\mathbf{i} + \frac{d}{dt}(2t)\mathbf{j} = \left(-\frac{2}{(t+1)^2}\right)\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{v}(1) = \langle 1, 2\rangle$; speed $= \sqrt{1^2 + 2^2} = \sqrt{5}$, direction $= \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \left\langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

(d) Velocity $= \sqrt{5}\left\langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

5. $\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(\sin t - \sin t \cos t) + (\sin t \cos t) = 0$ implies $\sin t = 0$, which is true for $t = 0$, π, or 2π.

6. $\mathbf{v}(t) = (\cos t)\mathbf{i} + \mathbf{j}$, and $\mathbf{a}(t) = (-\sin t)\mathbf{i}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $-\sin t \cos t = 0$, which is true for $t = \frac{k\pi}{2}$, k any nonnegative integer.

7. $\mathbf{v}(t) = (-3 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-3 \cos t)\mathbf{i} + (-4 \sin t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(9 \sin t \cos t) - (16 \sin t \cos t) = 0$, which is true when $\sin t = 0$ or $\cos t = 0$, i.e., for $t = \frac{k\pi}{2}$, k any nonnegative integer.

8. $\mathbf{v}(t) = (-5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-5 \cos t)\mathbf{i} + (-5 \sin t)\mathbf{j}$. Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(25 \sin t \cos t) + (-25 \sin t \cos t) = 0$, which is true for all values of t.

9. $\mathbf{v}(t) = (-2 \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and $\mathbf{a}(t) = (-2 \cos t)\mathbf{i} + (-\sin t)\mathbf{j}$. So $\mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{j}$, and $\mathbf{a}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\right)\mathbf{j}$. Then $|\mathbf{v}| = |\mathbf{a}| = \sqrt{\frac{5}{2}}$. $\mathbf{v} \cdot \mathbf{a} = \frac{3}{2}$ and $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}||\mathbf{a}|}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$.

10. $\mathbf{v}(t) = 3\mathbf{i} + (2t)\mathbf{j}$, and $\mathbf{a}(t) = 2\mathbf{j}$. So $\mathbf{v}(0) = 3\mathbf{i}$, and $\mathbf{a}(0) = 2\mathbf{j}$. These are perpendicular, i.e., the angle between them measures 90°.

11. (a) Both components are continuous at $t = 3$, so the limit is $3\mathbf{i} + \left(\dfrac{3^2-9}{3^2+3(3)}\right)\mathbf{j} = 3\mathbf{i}$.

    (b) Continuous so long as $t^2 + 3t \neq 0$, i.e., $t \neq 0, -3$

    (c) Discontinuous when $t^2 + 3t = 0$, i.e., $t = 0$ or -3

12. (a) Use L'Hôpital's Rule for the $\mathbf{i}$-component:

$$\lim_{t\to 0}\left(\frac{\sin 2t}{t}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j} = \lim_{t\to 0}\left(\frac{2\cos 2t}{1}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j} = 2\mathbf{i} + 0\mathbf{j} = 2\mathbf{i}.$$

    (b) Continuous so long as $t \neq 0$ and $t + 1 > 0$, i.e., $(-1,0)\cup(0,\infty)$.

    (c) Discontinuous when $t = 0$ or $t + 1 \leq 0$, i.e., $(-\infty,-1)\cup\{0\}$.

13. $\mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j}$, $\mathbf{r}(0) = -\mathbf{j}$ and $\mathbf{v}(0) = \mathbf{i}$. So the slope is zero (the velocity vector is horizontal).
    (a) The horizontal line through $(0,-1)$: $y = -1$.
    (b) The vertical line through $(0,-1)$: $x = 0$.

14. $\mathbf{v}(t) = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$.

$\mathbf{r}\left(\frac{\pi}{4}\right) = (\sqrt{2}-3)\mathbf{i} + \left(\dfrac{3}{\sqrt{2}}+1\right)\mathbf{j}$ and $\mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(\dfrac{3}{\sqrt{2}}\right)\mathbf{j}$. So the slope is $\dfrac{3/\sqrt{2}}{-\sqrt{2}} = -\frac{3}{2}$.

    (a) $y - \left(\dfrac{3}{\sqrt{2}}+1\right) = -\frac{3}{2}\left[x - (\sqrt{2}-3)\right]$ or $y = -\frac{3}{2}x + \dfrac{6\sqrt{2}-7}{2}$

    (b) $y - \left(\dfrac{3}{\sqrt{2}}+1\right) = \frac{2}{3}\left[x - (\sqrt{2}-3)\right]$ or $y = \frac{2}{3}x + \dfrac{5\sqrt{2}+18}{6}$

15. $$\left(\int_1^2 (6-6t)\,dt\right)\mathbf{i} + \left(\int_1^2 3\sqrt{t}\,dt\right)\mathbf{j} = \left[6t - 3t^2\right]_1^2\mathbf{i} + \left[2t^{3/2}\right]_1^2\mathbf{j} = -3\mathbf{i} + (4\sqrt{2}-2)\mathbf{j}$$

16. $$\left(\int_{-\pi/4}^{\pi/4} \sin t\,dt\right)\mathbf{i} + \left(\int_{-\pi/4}^{\pi/4} (1+\cos t)\,dt\right)\mathbf{j} = [-\cos t]_{-\pi/4}^{\pi/4}\mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4}\mathbf{j} = \left(\sqrt{2}+\frac{\pi}{2}\right)\mathbf{j}$$

17. $$\left(\int \sec t \tan t\,dt\right)\mathbf{i} + \left(\int \tan t\,dt\right)\mathbf{j} = (\sec t + C_1)\mathbf{i} + (\ln|\sec t| + C_2)\mathbf{j} = (\sec t)\mathbf{i} + (\ln|\sec t|)\mathbf{j} + \mathbf{C}$$

18. $$\left(\int \frac{1}{t}\,dt\right)\mathbf{i} + \left(\int \frac{1}{5-t}\,dt\right)\mathbf{j} = (\ln|t| + C_1)\mathbf{i} + (-\ln|5-t| + C_2)\mathbf{j} = (\ln|t|)\mathbf{i} - (\ln|5-t|)\mathbf{j} + \mathbf{C}$$

19. $\mathbf{r}(t) = (t+1)^{3/2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{C}$, and $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + \mathbf{C} = \mathbf{0}$, so $\mathbf{C} = -(\mathbf{i}-\mathbf{j}) = -\mathbf{i}+\mathbf{j}$
    $\mathbf{r}(t) = \left((t+1)^{3/2} - 1\right)\mathbf{i} - (e^{-t} - 1)\mathbf{j}$

20. $\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}$, and $\mathbf{r}(0) = \mathbf{C} = \mathbf{i} + \mathbf{j}$, so $\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j}$.

21. $\frac{d\mathbf{r}}{dt} = (-32t)\mathbf{j} + \mathbf{C}_1$ and $\mathbf{r}(t) = (-16t^2)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$. $\mathbf{r}(0) = \mathbf{C}_2 = 100\mathbf{i}$ and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$. So $\mathbf{r}(t) = (-16t^2)\mathbf{j} + (8\mathbf{i} + 8\mathbf{j})t + 100\mathbf{i} = (8t + 100)\mathbf{i} + (-16t^2 + 8t)\mathbf{j}$.

22. $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} + \mathbf{C}_1$, and $\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$, $\mathbf{r}(0) = \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j}$, and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = \mathbf{0}$, so $\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + (10\mathbf{i} + 10\mathbf{j}) = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j}$

23. $\mathbf{v}(t) = (\sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$; i.e., $\frac{dx}{dt} = \sin t$, and $\frac{dy}{dt} = 1 - \cos t$

$$\text{Distance} = \int_0^{2\pi/3} \sqrt{(\sin t)^2 + (1 - \cos t)^2}\, dt = \int_0^{2\pi/3} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi/3} 2\sin\left(\frac{t}{2}\right) dt = \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi/3} = 2$$

24. (a) $\mathbf{r}(0) = \left(\frac{1}{4}e^0 - 0\right)\mathbf{i} + (e^0)\mathbf{j} = \frac{1}{4}\mathbf{i} + \mathbf{j}$.

$\mathbf{r}(2) = \left(\frac{1}{4}e^8 - 2\right)\mathbf{i} + (e^4)\mathbf{j}$

Initial $= \left(\frac{1}{4}, 1\right)$, terminal $= \left(\frac{1}{4}e^8 - 2, e^4\right)$

(b) $\mathbf{v}(t) = (e^{4t} - 1)\mathbf{i} + (2e^{2t})\mathbf{j}$; $\frac{dx}{dt} = e^{4t} - 1$, and $\frac{dy}{dt} = 2e^{2t}$.

$$\text{Length} = \int_0^2 \sqrt{(e^{4t} - 1)^2 + (2e^{2t})^2}\, dt = \int_0^2 \sqrt{(e^{4t} + 1)^2}\, dt = \int_0^2 (e^{4t} + 1)\, dt = \left[\frac{1}{4}e^{4t} + t\right]_0^2 = \frac{e^8 + 7}{4} \approx 746.989$$

25. (a) $\mathbf{v}(t) = (\cos t)\mathbf{i} - (2\sin 2t)\mathbf{j}$

(b) $\mathbf{v}(t) = \mathbf{0}$ when both $\cos t = 0$ and $\sin 2t = 0$. $\cos t = 0$ at $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$; $\sin 2t = 0$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. So $\mathbf{v}(t) = \mathbf{0}$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

(c) $x = \sin t$, $y = \cos 2t$. Relate the two using the identity $\cos 2u = 1 - 2\sin^2 u$: $y = 1 - 2x^2$, where as x ranges over all possible values, $-1 \le x \le 1$. When t increases from 0 to 2π, the particle starts at $(0, 1)$, goes to $(1, -1)$, then goes to $(-1, -1)$, and then goes to $(0, 1)$, tracing the curve twice.

26. (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 12}{6t^2 - 6t} = \frac{t^2 - 4}{2t^2 - 2t}$

(b) Horizontal tangents: $t^2 - 4 = 0$ for $t = \pm 2$.
Vertical tangents: $2t^2 - 2t = 0$ for $t = 0, 1$.
Plugging the t-values into $x = 2t^3 - 3t^2$ and $y = t^3 - 12t$ produces the x- and y-coordinates of the critical points.
$t = -2$; horizontal tangent at $(-28, 16)$
$t = 0$: vertical tangent at $(0, 0)$
$t = 1$: vertical tangent at $(-1, -11)$
$t = 2$: horizontal tangent at $(4, -16)$

27. $\mathbf{a}(t) = 3\mathbf{i} - \mathbf{j}$, so $\mathbf{v}(t) = (3t)\mathbf{i} - t\mathbf{j} + \mathbf{C}_1$ and $\mathbf{r}(t) = \left(\frac{3}{2}t^2\right)\mathbf{i} - \left(\frac{1}{2}t^2\right)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$. $\mathbf{r}(0) = \mathbf{C}_2 = \mathbf{i} + 2\mathbf{j}$, and since $\mathbf{v}(0)$ must point directly from $(1,2)$ toward $(4,1)$ with magnitude 2,

$$\mathbf{v}(0) = \mathbf{C}_1 = 2\left(\frac{(4-1)\mathbf{i} + (1-2)\mathbf{j}}{\sqrt{(4-1)^2 + (1-2)^2}}\right) = \frac{6}{\sqrt{10}}\mathbf{i} - \frac{2}{\sqrt{10}}\mathbf{j} = \frac{3\sqrt{10}}{5}\mathbf{i} - \frac{\sqrt{10}}{5}\mathbf{j}$$

So $\mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{3\sqrt{10}}{5}t + 1\right)\mathbf{i} + \left(-\frac{1}{2}t^2 - \frac{\sqrt{10}}{5}t + 2\right)\mathbf{j}$.

28. (a) $\frac{dx}{dt} = 1 - \frac{2}{t^2} = 0$ when $t = \sqrt{2}$. That corresponds to point $\left(\sqrt{2} + \frac{2}{\sqrt{2}}, 3\left(\sqrt{2}\right)^2\right) = \left(2\sqrt{2}, 6\right)$.

(b) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{6t}{1 - 2/t^2}$, which for $t = 1$ equals -6.

(c) When $y = 12$, $t = 2$. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{(1 - 2/t^2)6 - (4/t^3)6t}{(1 - 2/t^2)^3}$, which for $t = 2$ equals -24.

29. (a) $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

(b) $\mathbf{v}(t) = -(2 \sin 2t)\mathbf{i} + (2 \cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4 \cos 2t)\mathbf{i} - (4 \sin 2t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = 8 \sin 2t \cos 2t - 8 \cos 2t \sin 2t = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

(c) $\mathbf{v}(t) = -\sin\left(t - \frac{\pi}{2}\right)\mathbf{i} + \cos\left(t - \frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos\left(t - \frac{\pi}{2}\right)\mathbf{i} - \sin\left(t - \frac{\pi}{2}\right)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{\sin^2\left(t - \frac{\pi}{2}\right) + \cos^2\left(t - \frac{\pi}{2}\right)} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = \sin\left(t - \frac{\pi}{2}\right)\cos\left(t - \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$

(d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;

(ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;

(iii) clockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$

(e) $\mathbf{v}(t) = -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4t^2\cos(t^2) + 2\sin(t^2))\mathbf{i} + (2\cos(t^2) - 4t^2\sin(t^2))\mathbf{j}$;

(i) $|\mathbf{v}(t)| = \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} = 2t \Rightarrow$ variable speed

(ii) $\mathbf{v}\cdot\mathbf{a} = 2t\cos(t^2)(2\cos(t^2) - 4t^2\sin(t^2)) + 2t\sin(t^2)(2\sin(t^2) + 4t^2\cos(t^2))$

$= 4t\left((\sin(t^2))^2 + (\cos(t^2))^2\right) = 4t \Rightarrow$ orthogonal only at $t = 0$

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = 1\mathbf{i} + 0\mathbf{j}$

30. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y\frac{dy}{dx} = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2}{2\cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

31. (a) The $\mathbf{j}$-component is zero at $t = 0$ and $t = 160$: 160 seconds.

(b) $-\frac{3}{64}(40)(40 - 160) = 225$ m

(c) $\frac{d}{dt}\left[-\frac{3}{64}t(t - 160)\right] = -\frac{3}{32}t + \frac{15}{2}$, which for $t = 40$ equals $\frac{15}{4}$ meters per second.

(d) $v(t) = -\frac{3}{32}t + \frac{15}{2}$ equals 0 at $t = 80$ seconds (and is negative after that time).

32. (a) Solve $t - 3 = \frac{3t}{2} - 4$: $t = 2$. Then check that $(t-3)^2 = \frac{3t}{2} - 2$ for $t = 2$: it does.

(b) First particle: $\mathbf{v}_1(t) = \mathbf{i} + 2(t-3)\mathbf{j}$, so $\mathbf{v}_1(2) = \mathbf{i} - 2\mathbf{j}$ and the direction unit vector $\mathbf{v}_1$ is $\left\langle\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$.

Second particle: $\mathbf{v}_2(t) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$, which is constant, and the direction unit vector is $\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

33. (a) Referring to the figure, look at the circular arc from the point where $t = 0$ to the point "m." On one hand, this arc has length given by $r_0\theta$, but it also has length given by vt. Setting those two quantities equal gives the result.

(b) $\mathbf{v}(t) = \left(-v\sin\frac{vt}{r_0}\right)\mathbf{i} + \left(v\cos\frac{vt}{r_0}\right)\mathbf{j}$, and $\mathbf{a}(t) = \left(-\frac{v^2}{r_0}\cos\frac{vt}{r_0}\right)\mathbf{i} + \left(-\frac{v^2}{r_0}\sin\frac{vt}{r_0}\right)\mathbf{j} = -\frac{v^2}{r_0}\left[\left(\cos\frac{vt}{r_0}\right)\mathbf{i} + \left(\sin\frac{vt}{r_0}\right)\mathbf{j}\right]$

(c) From part (b) above, $\mathbf{a}(t) = -\left(\frac{v}{r_0}\right)^2\mathbf{r}(t)$. So, by Newton's second law, $\mathbf{F} = -m\left(\frac{v}{r_0}\right)^2\mathbf{r}$. Substituting for $\mathbf{F}$ in the law of gravitation gives the result.

(d) Set $\frac{vT}{r_0} = 2\pi$ and solve for vT.

(e) Substitute $\frac{2\pi r_0}{T}$ for v in $v^2 = \frac{GM}{r_0}$ and solve for T^2.

$$\left(\frac{2\pi r_0}{T}\right)^2 = \frac{GM}{r_0} \Rightarrow \frac{4\pi^2 r_0^2}{T^2} = \frac{GM}{r_0} \Rightarrow \frac{1}{T^2} = \frac{GM}{4\pi^2 r_0^3} \Rightarrow T^2 = \frac{4\pi^2}{GM}r_0^3$$

34. (a) The velocity of the boat at (x, y) relative to land is the sum of the velocity due to the rower and the velocity of the river, or $\mathbf{v} = \left[-\frac{1}{250}(y-50)^2 + 10\right]\mathbf{i} - 20\mathbf{j}$. Now, $\frac{dy}{dt} = -20 \Rightarrow y = -20t + c$; $y(0) = 100$

$\Rightarrow c = 100 \Rightarrow y = -20t + 100 \Rightarrow \mathbf{v} = \left[-\frac{1}{250}(-20t+50)^2 + 10\right]\mathbf{i} - 20\mathbf{j} = \left(-\frac{8}{5}t^2 + 8t\right)\mathbf{i} - 20\mathbf{j}$

$\Rightarrow \mathbf{r}(t) = \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} - 20t\mathbf{j} + \mathbf{C}_1$; $\mathbf{r}(0) = 0\mathbf{i} + 100\mathbf{j} \Rightarrow 100\mathbf{j} = \mathbf{C}_1 \Rightarrow \mathbf{r}(t)$
$= \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} + (100 - 20t)\mathbf{j}$

(b) The boat reaches the shore when $y = 0 \Rightarrow 0 = -20t + 100$ from part (a) $\Rightarrow t = 5$
$\Rightarrow \mathbf{r}(5) = \left(-\frac{8}{15}\cdot 125 + 4\cdot 25\right)\mathbf{i} + (100 - 20\cdot 5)\mathbf{j} = \left(-\frac{200}{3} + 100\right)\mathbf{i} = \frac{100}{3}\mathbf{i}$; the distance downstream is therefore $\frac{100}{3}$ m

35. (a) Apply Corollary 2 to each component separately. If the components all differ by scalar constants, the difference vector is a constant vector.
(b) Follows immediately from (a) since any two anti-derivatives of $\mathbf{r}(t)$ must have identical derivatives, namely $\mathbf{r}(t)$.

36. $\frac{d}{dv}|\mathbf{v}|^2 = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}'\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}\cdot\mathbf{v}' = 0$. Therefore, $|\mathbf{v}|$ is constant.

37. Let $\mathbf{u} = \mathbf{C} = \langle C_1, C_2\rangle$. $\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{C}}{dt} = \left\langle \frac{dC_1}{dt}, \frac{dC_2}{dt}\right\rangle = \langle 0, 0\rangle$.

38. (a) Suppose $\mathbf{u} = \langle u_1(t), u_2(t)\rangle$.

$$\frac{d}{dt}(c\mathbf{u}) = \frac{d}{dt}\langle cu_1(t), cu_2(t)\rangle = \left\langle \frac{d}{dt}(cu_1(t)), \frac{d}{dt}(cu_2(t))\right\rangle = \left\langle c\frac{du_1}{dt}, c\frac{du_2}{dt}\right\rangle = c\left\langle \frac{du_1}{dt}, \frac{du_2}{dt}\right\rangle = c\frac{d\mathbf{u}}{dt}$$

(b) $\frac{d}{dt}(f\mathbf{u}) = \frac{d}{dt}\langle fu_1, fu_2\rangle = \langle fu_1' + f'u_1, fu_2' + f'u_2\rangle = \langle fu_1', fu_2'\rangle + \langle f'u_1, f'u_2\rangle = f\mathbf{u}' + f'\mathbf{u}$

39. $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$
(a) $\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d}{dt}(\langle u_1 + v_1, u_2 + v_2\rangle) = \left\langle \frac{d}{dt}(u_1+v_1), \frac{d}{dt}(u_2+v_2)\right\rangle = \langle u_1' + v_1', u_2' + v_2'\rangle$
$= \langle u_1', u_2'\rangle + \langle v_1', v_2'\rangle = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$
(b) $\frac{d}{dt}(\mathbf{u}-\mathbf{v}) = \frac{d}{dt}(\langle u_1 - v_1, u_2 - v_2\rangle) = \left\langle \frac{d}{dt}(u_1-v_1), \frac{d}{dt}(u_2-v_2)\right\rangle = \langle u_1' - v_1', u_2' - v_2'\rangle$
$= \langle u_1', u_2'\rangle - \langle v_1', v_2'\rangle = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

40. Since $\mathbf{u}$ is a differentiable function of s, we can write $\mathbf{u}(s) = g(s)\mathbf{i} + h(s)\mathbf{j} = g(f(t))\mathbf{i} + h(f(t))\mathbf{j}$, where $g(s)$ and $h(s)$ are differentiable functions of s. Therefore, $\frac{d}{dt}[\mathbf{u}(f(t))] = \frac{d}{dt}[g(f(t))\mathbf{i} + h(f(t))\mathbf{j}] = \frac{d}{dt}[g(f(t))]\mathbf{i} + \frac{d}{dt}[h(f(t))]\mathbf{j}$
$= g'(f(t))f'(t)\mathbf{i} + h'(f(t))f'(t)\mathbf{j}$ (by the Chain Rule for scalar functions) $= f'(t)[g'(s)\mathbf{i} + h'(s)\mathbf{j}] = f'(t)\mathbf{u}'(s)$
$= f'(t)\mathbf{u}'(f(t))$.

41. $f(t)$ and $g(t)$ differentiable at $c \Rightarrow f(t)$ and $g(t)$ continuous at $c \Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is continuous at c.

42. (a) Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

$$\int_a^b k\mathbf{r}(t)\,dt = \int_a^b \langle kx(t), ky(t)\rangle\,dt = \left\langle \int_a^b kx(t)\,dt, \int_a^b ky(t)\,dt \right\rangle = \left\langle k\int_a^b x(t)\,dt, k\int_a^b y(t)\,dt \right\rangle$$

$$= k\left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt \right\rangle = k\int_a^b \langle x(t), y(t)\rangle\,dt = k\int_a^b \mathbf{r}(t)\,dt$$

(b) Let $\mathbf{r}_1(t) = \langle x_1(t), y_1(t)\rangle$ and $\mathbf{r}_2(t) = \langle x_2(t), y_2(t)\rangle$.

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t))\,dt = \int_a^b (\langle x_1(t), y_1(t)\rangle \pm \langle x_2(t), y_2(t)\rangle)\,dt = \int_a^b \langle x_1(t) \pm x_2(t), y_1(t) \pm y_2(t)\rangle\,dt$$

$$= \left\langle \int_a^b (x_1(t) \pm x_2(t))\,dt, \int_a^b (y_1(t) \pm y_2(t))\,dt \right\rangle = \left\langle \int_a^b x_1(t)\,dt \pm \int_a^b x_2(t)\,dt, \int_a^b y_1(t)\,dt \pm \int_a^b y_2(t)\,dt \right\rangle$$

$$= \left\langle \int_a^b x_1(t)\,dt, \int_a^b y_1(t)\,dt \right\rangle \pm \left\langle \int_a^b x_2(t)\,dt, \int_a^b y_2(t)\,dt \right\rangle = \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$$

(c) Let $\mathbf{C} = \langle C_1, C_2\rangle$, $\mathbf{r}(t) = \langle x(t), y(t)\rangle$

$$\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \int_a^b (C_1x(t) + C_2y(t)\,dt = C_1\int_a^b x(t)\,dt + C_2\int_a^b y(t)\,dt = \langle C_1, C_2\rangle \cdot \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt \right\rangle$$

$$= \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt$$

43. (a) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. Then

$$\frac{d}{dt}\int_a^t \mathbf{r}(q)\,dq = \frac{d}{dt}\int_a^t [f(q)\mathbf{i} + g(q)\mathbf{j}]\,dq = \frac{d}{dt}\left[\left(\int_a^t f(q)\,dq\right)\mathbf{i} + \left(\int_a^t g(q)\,dq\right)\mathbf{j}\right]$$

$$= \left(\frac{d}{dt}\int_a^t f(1)\,dq\right)\mathbf{i} + \left(\frac{d}{dt}\int_a^t g(q)\,dq\right)\mathbf{j} = f(t)\mathbf{i} + g(t)\mathbf{j} = \mathbf{r}(t).$$

(b) Let $S(t) = \int_a^t \mathbf{r}(q)\,dq$. Then part (a) shows that $\mathbf{S}(t)$ is an antiderivative of $\mathbf{r}(t)$. Let $\mathbf{R}(t)$ be any antiderivative of $\mathbf{r}(t)$. Then according to 35(b), $\mathbf{S}(t) = \mathbf{R}(t) + \mathbf{C}$. Letting $t = a$, we have $\mathbf{0} = \mathbf{S}(a) = \mathbf{R}(a) + \mathbf{C}$. Therefore, $\mathbf{C} = -\mathbf{R}(a)$ and $\mathbf{S}(t) = \mathbf{R}(t) - \mathbf{R}(a)$. The result follows by letting $t = b$.

9.4 MODELING PROJECTILE MOTION

1. $x = (v_0 \cos\alpha)t \Rightarrow (21\text{ km})\left(\frac{1000\text{ m}}{1\text{ km}}\right) = (840\text{ m/s})(\cos 60^\circ)t \Rightarrow t = \frac{21{,}000\text{ m}}{(840\text{ m/s})(\cos 60^\circ)} = 50$ seconds

2. $R = \frac{v_0^2}{g} \sin 2\alpha$ and maximum R occurs when $\alpha = 45° \Rightarrow 24.5 \text{ km} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90°)$
$\Rightarrow v_0 = \sqrt{(9.8)(24{,}500) \text{ m}^2/\text{s}^2} = 490 \text{ m/s}$

3. (a) $t = \frac{2v_0 \sin \alpha}{g} = \frac{2(500 \text{ m/s})(\sin 45°)}{9.8 \text{ m/s}^2} = 72.2$ seconds; $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{(500 \text{ m/s})^2}{9.8 \text{ m/s}^2}(\sin 90°) = 25{,}510.2 \text{ m}$

(b) $x = (v_0 \cos \alpha)t \Rightarrow 5000 \text{ m} = (500 \text{ m/s})(\cos 45°)t \Rightarrow t = \frac{5000 \text{ m}}{(500 \text{ m/s})(\cos 45°)} \approx 14.14$ s; thus,
$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500 \text{ m/s})(\sin 45°)(14.14 \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(14.14 \text{ s})^2 \approx 4020 \text{ m}$

(c) $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{((500 \text{ m/s})(\sin 45°))^2}{2(9.8 \text{ m/s}^2)} = 6378 \text{ m}$

4. $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32 \text{ ft} + (32 \text{ ft/sec})(\sin 30°)t - \frac{1}{2}(32 \text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$;
the ball hits the ground when $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$ or $t = 2 \Rightarrow t = 2$ sec since $t > 0$; thus,
$x = (v_0 \cos \alpha)\,t \Rightarrow x = (32 \text{ ft/sec})(\cos 30°)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4 \text{ ft}$

5. $x = x_0 + (v_0 \cos \alpha)t = 0 + (44 \cos 45°)t = 22\sqrt{2}t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (44 \sin 45°)t - 16t^2$
$= 6.5 + 22\sqrt{2}t - 16t^2$; the shot lands when $y = 0 \Rightarrow t = \frac{22\sqrt{2} \pm \sqrt{968 + 416}}{32} \approx 2.135$ sec since $t > 0$; thus
$x = 22\sqrt{2}t \approx (22\sqrt{2})(2.134839) \approx 66.42 \text{ ft}$

6. $x = 0 + (44 \cos 40°)t = 33.706t$ and $y = 6.5 + (44 \sin 40°)t - 16t^2 \approx 6.5 + 28.283t - 16t^2$; $y = 0$
$\Rightarrow t \approx \frac{28.283 + \sqrt{(28.283)^2 + 416}}{32} \approx 1.9735$ sec since $t > 0$; thus $x = (33.706)(1.9735) \approx 66.51 \text{ ft} \Rightarrow$ the difference in distances is about $66.51 - 66.42 = 0.09$ ft or about 1 inch

7. (a) $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 10 \text{ m} = \left(\frac{v_0^2}{9.8 \text{ m/s}^2}\right)(\sin 90°) \Rightarrow v_0^2 = 98 \text{ m}^2\text{s}^2 \Rightarrow v_0 \approx 9.9 \text{ m/s}$;

(b) $6\text{m} \approx \frac{(9.9 \text{ m/s})^2}{9.8 \text{ m/s}^2}(\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87°$ or $143.12° \Rightarrow \alpha \approx 18.4°$ or $71.6°$

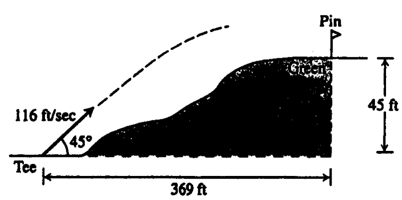
8. $v_0 = 5 \times 10^6$ m/s and $x = 40$ cm $= 0.4$ m; thus $x = (v_0 \cos \alpha)t \Rightarrow 0.4\text{m} = (5 \times 10^6 \text{ m/s})(\cos 0°)t$
$\Rightarrow t = 0.08 \times 10^{-6} \text{ s} = 8 \times 10^{-8}$ s; also, $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (5 \times 10^6 \text{ m/s})(\sin 0°)(8 \times 10^{-8} \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(8 \times 10^{-8} \text{ s})^2 = -3.136 \times 10^{-14}$ m or
-3.136×10^{-12} cm. Therefore, it drops 3.136×10^{-12} cm.

9. $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 3(248.8) \text{ ft} = \left(\frac{v_0^2}{32 \text{ ft/sec}^2}\right)(\sin 18°) \Rightarrow v_0^2 \approx 77{,}292.84 \text{ ft}^2/\text{sec}^2 \Rightarrow v_0 \approx 278.01 \text{ ft/sec} \approx 190 \text{ mph}$

10. $v_0 = \frac{80\sqrt{10}}{3}$ ft/sec and R = 200 ft $\Rightarrow 200 = \frac{\left(\frac{80\sqrt{10}}{3}\right)^2}{32}(\sin 2\alpha) \Rightarrow \sin 2\alpha = 0.9 \Rightarrow 2\alpha \approx 64.2^\circ \Rightarrow \alpha \approx 32.1^\circ$; $y_{max} = \frac{\left[\left(\frac{80\sqrt{10}}{3}\right)(\sin 32.1^\circ)\right]^2}{2(32)} \approx 31.4$ ft. In order to reach the cushion, the angle of elevation will need to be about 32.1°. At this angle, the circus performer will go 31.4 ft into the air at maximum height and will not strike the 75 ft high ceiling.

11. $x = (v_0 \cos \alpha)t \Rightarrow 135 \text{ ft} = (90 \text{ ft/sec})(\cos 30^\circ)t \Rightarrow t \approx 1.732 \text{ sec}$; $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow y \approx (90 \text{ ft/sec})(\sin 30^\circ)(1.732 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(1.732 \text{ sec})^2 \Rightarrow y \approx 29.94$ ft $\Rightarrow$ the golf ball will clip the leaves at the top

12. $v_0 = 116$ ft/sec, $\alpha = 45^\circ$, and $x = (v_0 \cos \alpha)t$ $\Rightarrow 369 = (116 \cos 45^\circ)t \Rightarrow t \approx 4.50$ sec; also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow y = (116 \sin 45^\circ)(4.50) - \frac{1}{2}(32)(4.50)^2$ ≈ 45.11 ft. It will take the ball 4.50 sec to travel 369 ft. At that time the ball will be 45.11 ft in the air and will hit the green just past the pin.



13. $x = (v_0 \cos \alpha)t \Rightarrow 315 \text{ ft} = (v_0 \cos 20^\circ)t \Rightarrow v_0 = \frac{315}{t \cos 20^\circ}$; also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow 34 \text{ ft} = \left(\frac{315}{t \cos 20^\circ}\right)(t \sin 20^\circ) - \frac{1}{2}(32)t^2 \Rightarrow 34 = 315 \tan 20^\circ - 16t^2 \Rightarrow t^2 \approx 5.04 \text{ sec}^2 \Rightarrow t \approx 2.25$ sec $\Rightarrow v_0 = \frac{315}{(2.25)(\cos 20^\circ)} \approx 149$ ft/sec

14. $R = \frac{v_0^2}{g}\sin 2\alpha = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g}[2 \cos(90^\circ - \alpha) \sin(90^\circ - \alpha)] = \frac{v_0^2}{g}[\sin 2(90^\circ - \alpha)]$

15. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 16{,}000 \text{ m} = \frac{(400 \text{ m/s})^2}{9.8 \text{ m/s}^2}\sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5^\circ$ or $2\alpha \approx 101.5^\circ \Rightarrow \alpha \approx 39.3^\circ$ or 50.7°

16. (a) $R = \frac{(2v_0)^2}{g}\sin 2\alpha = \frac{4v_0^2}{g}\sin 2\alpha = 4\left(\frac{v_0^2}{g}\sin \alpha\right)$ or 4 times the original range.

(b) Now, let the initial range be $R = \frac{v_0^2}{g}\sin 2\alpha$. Then we want the factor p so that pv_0 will double the range $\Rightarrow \frac{(pv_0)^2}{g}\sin 2\alpha = 2\left(\frac{v_0^2}{g}\sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately double the height.

17. $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40^\circ)t \approx 0.766\, v_0 t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40^\circ)t - 16t^2$ $\approx 6.5 + 0.643\, v_0 t - 16t^2$; now the shot went 73.833 ft $\Rightarrow 73.833 = 0.766\, v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$ sec; the shot lands when $y = 0 \Rightarrow 0 = 6.5 + (0.643)(96.383) - 16\left(\frac{96.383}{v_0}\right)^2 \Rightarrow 0 \approx 68.474 - \frac{148{,}634}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148{,}634}{68.474}}$ ≈ 46.6 ft/sec, the shot's initial speed

18. $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} \Rightarrow \frac{3}{4} y_{max} = \frac{3(v_0 \sin \alpha)^2}{8g}$ and $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin \alpha)^2}{8g} = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow 3(v_0 \sin \alpha)^2 = (8gv_0 \sin \alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin \alpha)t + 3(v_0 \sin \alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin \alpha = 0$ or $2gt - v_0 \sin \alpha = 0 \Rightarrow t = \frac{3v_0 \sin \alpha}{2g}$ or $t = \frac{v_0 \sin \alpha}{2g}$. Since the time it takes to reach y_{max} is $t_{max} = \frac{v_0 \sin \alpha}{g}$, then the time it takes the projectile to reach $\frac{3}{4}$ of y_{max} is the shorter time $t = \frac{v_0 \sin \alpha}{2g}$ or half the time it takes to reach the maximum height.

19. $\frac{d\mathbf{r}}{dt} = \int (-g\mathbf{j})\, dt = -gt\mathbf{j} + \mathbf{C}_1$ and $\frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}$ $\Rightarrow \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}$; $\mathbf{r} = \int [(v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}]\, dt$ $= (v_0 t \cos \alpha)\mathbf{i} + \left(v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2$ and $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0) \cos \alpha]\mathbf{i} + \left[v_0(0) \sin \alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2$ $= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t \cos \alpha)\mathbf{i} + \left(y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t \cos \alpha$ and $y = y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2$

20. From Example 3(b) in the text, $v_0 \sin \alpha = \sqrt{(68)(64)} \Rightarrow v_0 \sin 57^\circ \approx 65.97 \Rightarrow v_0 \approx 79$ ft/sec

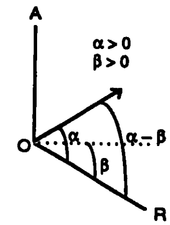
21. The horizontal distance from Rebollo to the center of the cauldron is 90 ft $\Rightarrow$ the horizontal distance to the nearest rim is $x = 90 - \frac{1}{2}(12) = 84 \Rightarrow 84 = x_0 + (v_0 \cos \alpha)t \approx 0 + \left(\frac{90g}{v_0 \sin \alpha}\right)t \Rightarrow 84 = \frac{(90)(32)}{\sqrt{(68)(64)}}t$ $\Rightarrow t = 1.92$ sec. The vertical distance at this time is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\approx 6 + \sqrt{(68)(64)}\,(1.92) - 16(1.92)^2 \approx 73.7$ ft $\Rightarrow$ the arrow clears the rim by 3.7 ft

22. The projectile rises straight up and then falls straight down, returning to the firing point.

23. Flight time = 1 sec and the measure of the angle of elevation is about 64° (using a protractor) so that $t = \frac{2v_0 \sin \alpha}{g} \Rightarrow 1 = \frac{2v_0 \sin 64^\circ}{32} \Rightarrow v_0 \approx 17.80$ ft/sec. Then $y_{max} = \frac{(17.80 \sin 64^\circ)^2}{2(32)} \approx 4.00$ ft and $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow R = \frac{(17.80)^2}{32} \sin 128^\circ \approx 7.80$ ft $\Rightarrow$ the engine traveled about 7.80 ft in 1 sec $\Rightarrow$ the engine velocity was about 7.80 ft/sec

24. When marble A is located R units downrange, we have $x = (v_0 \cos \alpha)t \Rightarrow R = (v_0 \cos \alpha)t \Rightarrow t = \frac{R}{v_0 \cos \alpha}$. At that time the height of marble A is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = (v_0 \sin \alpha)\left(\frac{R}{v_0 \cos \alpha}\right) - \frac{1}{2}g\left(\frac{R}{v_0 \cos \alpha}\right)^2$ $\Rightarrow y = R \tan \alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2 \cos^2 \alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0 \cos \alpha}$ seconds is $h = R \tan \alpha - \frac{1}{2}gt^2 = R \tan \alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2 \cos^2 \alpha}\right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0.

25. (a) At the time t when the projectile hits the line OR we have $\tan \beta = \frac{y}{x}$; $x = [v_0 \cos(\alpha - \beta)]t$ and $y = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2}gt^2 < 0$ since R is below level ground. Therefore let $|y| = \frac{1}{2}gt^2 - [v_0 \sin(\alpha - \beta)]t > 0$



so that $\tan \beta = \frac{\left[\frac{1}{2}gt^2 - (v_0 \sin(\alpha - \beta))t\right]}{[v_0 \cos(\alpha - \beta)]t} = \frac{\left[\frac{1}{2}gt - v_0 \sin(\alpha - \beta)\right]}{v_0 \cos(\alpha - \beta)}$

$\Rightarrow v_0 \cos(\alpha - \beta) \tan \beta = \frac{1}{2}gt - v_0 \sin(\alpha - \beta)$

$\Rightarrow t = \frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g}$, which is the time when the projectile hits the downhill slope. Therefore,

$x = [v_0 \cos(\alpha - \beta)]\left[\frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g}\right]$

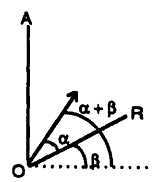
$= \frac{2v_0^2}{g}\left[\cos^2(\alpha - \beta) \tan \beta + \sin(\alpha - \beta) \cos(\alpha - \beta)\right]$. If x is maximized, then OR is maximized:

$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[-\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta)] = 0 \Rightarrow -\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta) = 0$

$\Rightarrow \tan \beta = \cot 2(\alpha - \beta) \Rightarrow 2(\alpha - \beta) = 90^\circ - \beta \Rightarrow \alpha - \beta = \frac{1}{2}(90^\circ - \beta) \Rightarrow \alpha = \frac{1}{2}(90^\circ + \beta) = \frac{1}{2}$ of $\angle AOR$.

(b) At the time t when the projectile hits OR we have $\tan \beta = \frac{y}{x}$;

$x = [v_0 \cos(\alpha + \beta)]t$ and $y = [v_0 \sin(\alpha + \beta)]t - \frac{1}{2}gt^2$



$\Rightarrow \tan \beta = \frac{[v_0 \sin(\alpha + \beta)]t - \frac{1}{2}gt^2}{[v_0 \cos(\alpha + \beta)]t} = \frac{\left[v_0 \sin(\alpha + \beta) - \frac{1}{2}gt\right]}{v_0 \cos(\alpha + \beta)}$

$\Rightarrow v_0 \cos(\alpha + \beta) \tan \beta = v_0 \sin(\alpha + \beta) - \frac{1}{2}gt$

$\Rightarrow t = \frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g}$, which is the time when the projectile hits the uphill slope. Therefore,

$x = [v_0 \cos(\alpha + \beta)]\left[\frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g}\right]$

$= \frac{2v_0^2}{g}\left[\sin(\alpha + \beta) \cos(\alpha + \beta) - \cos^2(\alpha + \beta) \tan \beta\right]$. If x is maximized, then OR is maximized:

$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[\cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta] = 0 \Rightarrow \cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta = 0$

$\Rightarrow \cot 2(\alpha+\beta)+\tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) = -\tan\beta = \tan(-\beta) \Rightarrow 2(\alpha+\beta) = 90^\circ - (-\beta)$

$= 90^\circ + \beta \Rightarrow \alpha = \frac{1}{2}(90^\circ - \beta) = \frac{1}{2}$ of $\angle AOR$. Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

26. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where $x(t) = (145\cos 23^\circ - 14)t$ and $y(t) = 2.5 + (145\sin 23^\circ)t - 16t^2$.

(b) $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + 2.5 = \frac{(145\sin 23^\circ)^2}{64} + 2.5 \approx 52.655$ feet, which is reached at $t = \frac{v_0 \sin\alpha}{g}$ $= \frac{145\sin 23^\circ}{32} \approx 1.771$ seconds.

(c) For the time, solve $y = 2.5 + (145\sin 23^\circ)t - 16t^2 = 0$ for t, using the quadratic formula

$t = \frac{145\sin 23^\circ + \sqrt{(145\sin 23^\circ)^2 + 160}}{32} \approx 3.585$ sec. Then the range at $t \approx 3.585$ is about

$x = (145\cos 23^\circ - 14)(3.585) \approx 428.262$ feet.

(d) For the time, solve $y = 2.5 + (145\sin 23^\circ)t - 16t^2 = 20$ for t, using the quadratic formula

$t = \frac{145\sin 23^\circ \pm \sqrt{(145\sin 23^\circ)^2 - 1120}}{32} \approx 0.342$ and 3.199 seconds. At those times the ball is about $x(0.342) = (145\cos 23^\circ - 14)(0.342) \approx 40.847$ feet from home plate and $x(3.199) = (145\cos 23^\circ - 14)(3.199)$ ≈ 382.208 feet from home plate.

(e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

27. (a) (Assuming that "x" is zero at the point of impact.)
$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where $x(t) = (35\cos 27^\circ)t$ and $y(t) = 4 + (35\sin 27^\circ)t - 16t^2$.

(b) $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + 4 = \frac{(35\sin 27^\circ)^2}{64} + 4 \approx 7.945$ feet, which is reached at $t = \frac{v_0 \sin\alpha}{g}$ $= \frac{35\sin 27^\circ}{32} \approx 0.497$ seconds.

(c) For the time, solve $y = 4 + (35\sin 27^\circ)t - 16t^2 = 0$ for t, using the quadratic formula

$t = \frac{35\sin 27^\circ + \sqrt{(-35\sin 27^\circ)^2 + 256}}{32} \approx 1.201$ seconds. Then the range is about

$x(1.201) = (35\cos 27^\circ)(1.201) \approx 37.453$ feet.

(d) For the time, solve $y = 4 + (35\sin 27^\circ)t - 16t^2 = 7$ for t, using the quadratic formula

$t = \frac{35\sin 27^\circ \pm \sqrt{(-35\sin 27^\circ)^2 - 192}}{32} \approx 0.254$ and 0.740 seconds. At those times the ball is about $x(0.254) = (35\cos 27^\circ)(0.254) \approx 7.906$ feet and $x(0.740) = (35\cos 27^\circ)(0.740) \approx 23.064$ feet from the impact point, or about $37.460 - 7.906 \approx 29.554$ feet and $37.460 - 23.064 \approx 14.396$ feet from the landing spot.

(e) Yes. It changes things because the ball won't clear the net ($y_{max} \approx 7.945$ ft).

28. The maximum height is $y = \frac{(v_0 \sin\alpha)^2}{2g}$ and this occurs for $x = \frac{v_0^2}{2g}\sin 2\alpha = \frac{v_0^2 \sin\alpha\cos\alpha}{g}$. These equations describe parametrically the points on a curve in the xy-plane associated with the maximum heights on the parabolic trajectories in terms of the parameter (launch angle) α. Eliminating the parameter α, we have

$$x^2 = \frac{v_0^4 \sin^2\alpha \cos^2\alpha}{g^2} = \frac{\left(v_0^4 \sin^2\alpha\right)\left(1-\sin^2\alpha\right)}{g^2} = \frac{v_0^4 \sin^2\alpha}{g^2} - \frac{v_0^4 \sin^4\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0$$

$$\Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{16g^2} \Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{16g^2}, \text{ where } x \geq 0.$$

29. $\frac{d^2\mathbf{r}}{dt^2} + k\frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$ and $\mathbf{Q}(t) = -g\mathbf{j} \Rightarrow \int P(t)\,dt = kt \Rightarrow v(t) = e^{\int P(t)\,dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt}$
$= \frac{1}{v(t)}\int v(t)\mathbf{Q}(t)\,dt = -ge^{-kt}\int \mathbf{j}e^{kt}\,dt = -ge^{-kt}\left[\frac{e^{kt}}{k}\mathbf{j} + \mathbf{C}_1\right] = -\frac{g}{k}\mathbf{j} + \mathbf{C}e^{-kt}$, where $\mathbf{C} = -g\mathbf{C}_1$; apply the initial condition: $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} = -\frac{g}{k}\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0\cos\alpha)\mathbf{i} + \left(\frac{g}{k} + v_0\sin\alpha\right)\mathbf{j}$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = \left(v_0e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j},\ \mathbf{r} = \int\left\{\left(v_0e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j}\right\}dt$$

$= \left(-\frac{v_0}{k}e^{-kt}\cos\alpha\right)\mathbf{i} + \left[-\frac{gt}{k} - \frac{e^{-kt}}{k}\left(\frac{g}{k} + v_0\sin\alpha\right)\right]\mathbf{j} + \mathbf{C}_2$; apply the initial condition:

$$\mathbf{r}(0) = \mathbf{0} = \left(-\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k^2} - \frac{v_0}{k}\sin\alpha\right)\mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left(\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(\frac{g}{k^2} + \frac{v_0}{k}\sin\alpha\right)\mathbf{j}$$

$$\Rightarrow \mathbf{r}(t) = \left(\frac{v_0}{k}\left(1-e^{-kt}\right)\cos\alpha\right)\mathbf{i} + \left[\frac{v_0}{k}\left(1-e^{-kt}\right)\sin\alpha + \frac{g}{k^2}\left(1-kt-e^{-kt}\right)\right]\mathbf{j}$$

30. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$x(t) = \left(\frac{152}{0.12}\right)(1-e^{-0.12t})(\cos 20°)$ and

$y(t) = 3 + \left(\frac{152}{0.12}\right)(1-e^{-0.12t})(\sin 20°) + \left(\frac{32}{0.12^2}\right)(1-0.12t-e^{-0.12t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.126 seconds. The range is about $x(3.126) = \left(\frac{152}{0.12}\right)(1-e^{-0.12(3.126)})(\cos 20°) \approx 372.323$ feet.

(d) Use a graphing calculator or CAS to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.513$ feet and $x(2.305) \approx 287.628$ feet from home plate.

(e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

31. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$x(t) = \left(\frac{1}{0.08}\right)(1-e^{-0.08t})(152\cos 20° - 17.6)$ and

$y(t) = 3 + \left(\frac{152}{0.08}\right)(1-e^{-0.08t})(\sin 20°) + \left(\frac{32}{0.08^2}\right)(1-0.08t-e^{-0.08t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.181 seconds. The range is about $x(3.181) = \left(\frac{1}{0.08}\right)(1-e^{-0.08(3.181)})(152\cos 20° - 17.6) \approx 351.734$ feet.

(d) Use a graphing calculator or CAS to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at at which times the ball is about $x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.

(e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds. Then define $x(w) = \left(\frac{1}{0.08}\right)(1-e^{-0.08(2.716)})(152\cos 20° + w)$, and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec.

This is the speed of a wind gust needed in the direction of the hit for the ball to clear the fence for a home run.

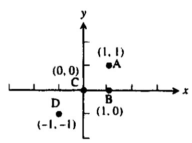
9.5 POLAR COORDINATES AND GRAPHS

For exercises 1 and 2, two pairs of polar coordinates label the same point if the r-coordinates are the same and the θ coordinates differ by an even multiple of π, or if the r-coordinates are opposites and the θ-coordinates differ by an odd multiple of π.

1. (a) and (e) are the same.
   (b) and (g) are the same.
   (c) and (h) are the same.
   (d) and (f) are the same.

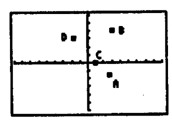
2. (a) and (f) are the same.
   (b) and (h) are the same.
   (c) and (g) are the same.
   (d) and (e) are the same.

3.



(a) $\left(\sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\right) = (1,1)$
(b) $(1\cos 0, 1\sin 0) = (1,0)$
(c) $\left(0\cos\frac{\pi}{2}, 0\sin\frac{\pi}{2}\right) = (0,0)$
(d) $\left(-\sqrt{2}\cos\frac{\pi}{4}, -\sqrt{2}\sin\frac{\pi}{4}\right) = (-1,-1)$

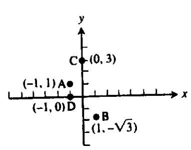
4.



[−9, 9] by [−6, 6]

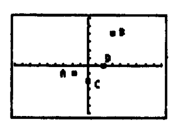
(a) $\left(-3\cos\frac{5\pi}{6}, -3\sin\frac{5\pi}{6}\right) = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$
(b) $\left(5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right), 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right) = (3,4)$
(c) $(-1\cos 7\pi, -1\sin 7\pi) = (1,0)$
(d) $\left(2\sqrt{3}\cos\frac{2\pi}{3}, 2\sqrt{3}\sin\frac{2\pi}{3}\right) = (-\sqrt{3},3)$

5.



(a) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\tan\theta = \frac{1}{-1} = -1$ with θ in quadrant II. The coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$. $\left(\sqrt{2}, -\frac{5\pi}{4}\right)$ also works, since r is the same and θ differs by 2π.

6.



[−9, 9] by [−6, 6]

(a) $r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$, $\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$ with θ in quadrant III. The coordinates are $\left(2, \frac{7\pi}{6}\right)$. $\left(-2, \frac{\pi}{6}\right)$ also works, since r has the opposite sign and θ differs by π.

(b) $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$, $\tan\theta = -\frac{\sqrt{3}}{1} = -\sqrt{3}$ with θ in quadrant IV. The coordinates are $\left(2, -\frac{\pi}{3}\right)$. $\left(-2, \frac{2\pi}{3}\right)$ also works, since r has the opposite sign and θ differs by π.

(c) $r = \sqrt{0^2 + 3^2} = 3$, $\tan\theta = \frac{3}{0}$ is undefined with θ on the positive y-axis. The coordinates are $\left(3, \frac{\pi}{2}\right)$. $\left(3, \frac{5\pi}{2}\right)$ also works, since r is the same and θ differs by 2π.

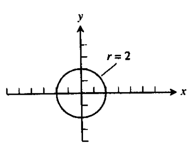
(d) $r = \sqrt{(-1)^2 + 0^2} = 1$, $\tan\theta = \frac{0}{-1} = 0$ with θ on the negative x-axis. The coordinates are $(1, \pi)$. $(-1, 0)$ also works, since r has the opposite sign and θ differs by π.

(b) $r = \sqrt{3^2 + 4^2} = 5$, $\tan\theta = \frac{4}{3}$ with θ in quadrant I. The coordinates are $\left(5, \tan^{-1}\frac{4}{3}\right)$. $\left(-5, \pi + \tan^{-1}\frac{4}{3}\right)$ also works, since r has the opposite sign and θ differs by π.

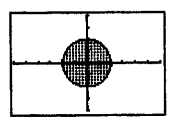
(c) $r = \sqrt{0 + (-2)^2} = 2$, $\tan\theta = -\frac{2}{0}$ is undefined with θ on the negative y-axis. The coordinates are $\left(2, \frac{3\pi}{2}\right)$. $\left(2, -\frac{\pi}{2}\right)$ also works, since r is the same and θ differs by 2π.

(d) $r = \sqrt{2^2 + 0^2} = 2$, $\tan\theta = \frac{0}{2} = 0$ with θ on the positive x-axis. The coordinates are $(2, 0)$. $(2, 2\pi)$ also works, since r is the same and θ differs by 2π.

7.

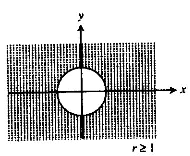


8.



[−6, 6] by [−4, 4]

9.



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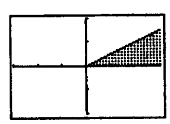
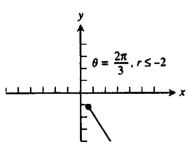
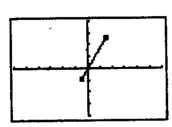


[−3, 3] by [−2, 2]

11.

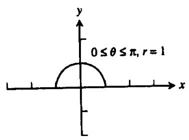


12.



[−6, 6] by [−4, 4]

13.



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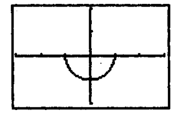
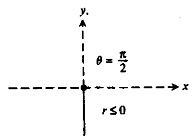
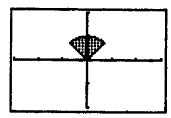


[−3, 3] by [−2, 2]

15.

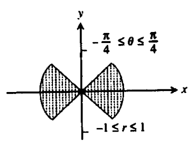


16.

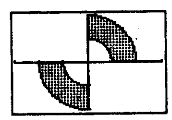


[−3, 3] by [−2, 2]

17.



18.



[−3, 3] by [−2, 2]

19. $y = r \sin \theta$, so the equation is $y = 0$, which is the x-axis.

20. $x = r \cos \theta$, so the equation is $x = 0$, which is the y-axis.

21. $r = 4 \csc \theta$
$r \sin \theta = 4$
$y = r \sin \theta$, so the equation is $y = 4$,
a horizontal line.

22. $r = -3 \sec \theta$
$r \cos \theta = -3$
$x = r \cos \theta$, so the equation is $x = -3$, a
vertical line.

23. $x = r \cos \theta$ and $y = r \sin \theta$, so the equation is $x + y = 1$, a line (slope $= -1$, y-intercept $= 1$).

24. $x^2 + y^2 = r^2$, so the equation is $x^2 + y^2 = 1$, a circle (center $= (0,0)$, radius $= 1$).

25. $x^2 + y^2 = r^2$ and $y = r \sin \theta$, so the equation is $x^2 + y^2 = 4y \Rightarrow x^2 + (y-2)^2 = 4$, a circle (center $= (0,2)$, radius $= 2$).

26. $r = \frac{5}{\sin\theta - 2\cos\theta} \Rightarrow r\sin\theta - 2r\cos\theta = 5$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y - 2x = 5$, a line (slope $= 2$, y-intercept $= 5$).

27. $r^2\sin 2\theta = 2 \Rightarrow 2r^2\sin\theta\cos\theta = 2 \Rightarrow (r\sin\theta)(r\cos\theta) = 1$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $xy = 1\left(\text{or, } y = \frac{1}{x}\right)$, a hyperbola.

28. $r = \cot\theta\csc\theta \Rightarrow r\sin\theta = \cot\theta$; $y = r\sin\theta$ and $\frac{x}{y} = \cot\theta$, so the equation is $y^2 = x$, a parabola.

29. $r = \csc\theta\, e^{r\cos\theta} \Rightarrow r\sin\theta = e^{r\cos\theta}$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y = e^x$, the exponential curve.

30. $\cos^2\theta = \sin^2\theta \Rightarrow (r\cos\theta)^2 = (r\sin\theta)^2$; $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $x^2 = y^2$ or $y = \pm x$, the union of two lines.

31. $r\sin\theta = \ln r + \ln\cos\theta \Rightarrow r\sin\theta = \ln(r\cos\theta) \Rightarrow y = \ln x$, the logarithmic curve.

32. $r^2 + 2r^2\cos\theta\sin\theta = 1 \Rightarrow r^2 + 2(r\cos\theta)(r\sin\theta) = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow (x+y)^2 = 1 \Rightarrow x + y = \pm 1$, the union of two lines.

33. $r^2 = -4r\sin\theta \Rightarrow x^2 + y^2 = -4x \Rightarrow x^2 + (y-4)^2 = 16$, a circle (center $= (0,4)$, radius $= 4$)

34. $r = 8\sin\theta \Rightarrow r^2 = 8r\sin\theta \Rightarrow x^2 + y^2 = 8y \Rightarrow (x+2)^2 + y^2 = 4$, a circle (center $= (-2,0)$, radius $= 2$).

35. $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2r\cos\theta + 2r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y \Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle (center $= (1,1)$, radius $= \sqrt{2}$)

36. $r\sin\left(\theta + \frac{\pi}{6}\right) = 2 \Rightarrow r\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right) = 2 \Rightarrow \frac{\sqrt{3}}{2}r\sin\theta + \frac{1}{2}r\cos\theta = 2 \Rightarrow \frac{\sqrt{3}}{2}y + \frac{1}{2}x = 2$ $\Rightarrow x + \sqrt{3}y = 4$, a line $\left(\text{slope} = -\frac{1}{\sqrt{3}}, \text{ y-intercept} = \frac{4}{\sqrt{3}}\right)$.

37. $x = 7 \Rightarrow r\cos\theta = 7$; The graph is a vertical line.

38. $y = 1 \Rightarrow r\sin\theta = 1$; The graph is a horizontal line.

39. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. More generally, $\theta = \frac{\pi}{4} + 2k\pi$ for any integer k. The graph is a slanted line.

40. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

41. $x^2 + y^2 = 4 \Rightarrow r^2 = 4$ or $r = 2$ (or $r = -2$)

42. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$

43. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{r^2\cos^2\theta}{9} + \frac{r^2\sin^2\theta}{4} = 1 \Rightarrow r^2(4\cos^2\theta + 9\sin^2\theta) = 36$

44. $xy = 2 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow r^2\cos\theta\sin\theta = 2 \Rightarrow r^2\, 2\cos\theta\sin\theta = 4 \Rightarrow r^2\sin 2\theta = 4$

45. $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow r\sin^2\theta = 4\cos\theta$

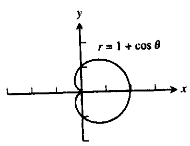
46. $x^2 + xy + y^2 = 1 \Rightarrow (r \cos \theta)^2 + (r \cos \theta)(r \sin \theta) + (r \sin \theta)^2 = 1 \Rightarrow r^2(1 + \cos \theta \sin \theta) = 1$

47. $x^2 + (y-2)^2 = 4 \Rightarrow r^2 \cos^2 \theta + (r \sin \theta - 2)^2 = 4 \Rightarrow r^2 \cos^2 \theta + r^2 \sin^2 \theta - 4r \sin \theta + 4 = 4$
$\Rightarrow r^2 - 4r \sin \theta = 0 \Rightarrow r = 4 \sin \theta$. The graph is a circle centered at $(0,2)$ with radius 2.

48. $(x-3)^2 + (y+1)^2 = 4 \Rightarrow (r \cos \theta - 3)^2 + (r \sin \theta + 1)^2 = 4$
$\Rightarrow r^2 \cos^2 \theta - 6r \cos\theta + 9 + r^2 \sin^2 \theta + 2r \sin \theta + 1 = 4 \Rightarrow r^2 - 6r \cos \theta + 2r \sin \theta + 6 = 0$
$\Rightarrow r = \dfrac{6 \cos \theta - 2 \sin \theta \pm \sqrt{(6 \cos \theta - 2 \sin \theta)^2 - 24}}{2} \Rightarrow r = 3 \cos \theta - \sin \theta \pm \sqrt{(3 \cos \theta - \sin \theta)^2 - 6}$

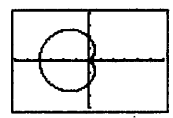
In Exercises 49-58, find the minimum θ-interval by trying different intervals on a graphing calculator.

49. (a)



(b) Length of interval $= 2\pi$

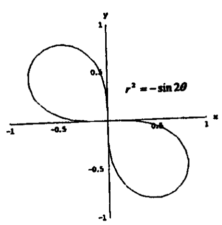
50. (a)



[−6, 6] by [−4, 4]

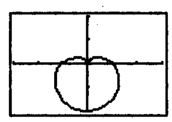
(b) Length of interval $= 2\pi$

51. (a)



(b) Length of interval $= \frac{\pi}{2}$

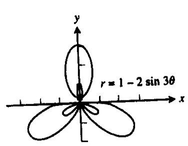
52. (a)



[−3, 3] by [−2, 2]

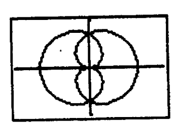
(b) Length of interval $= 2\pi$

53. (a)



(b) Length of interval $= 2\pi$

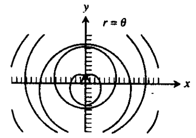
54. (a)



[−1.5, 1.5] by [−1, 1]

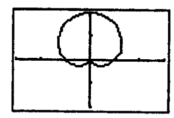
(b) Length of interval $= 4\pi$

55. (a)



(b) Required interval = $(-\infty, \infty)$

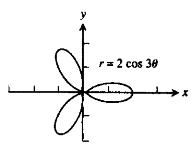
56. (a)



[−3, 3] by [−2, 2]

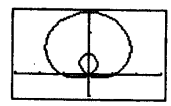
(b) Length of interval = 2π

57. (a)



(b) Length of interval = π

58. (a)



[−3, 3] by [−1, 3]

(b) Length of interval = 2π

59. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if (r, θ) is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the x-axis. And since any curve with x-axis and origin symmetry also has y-axis symmetry, the curve is symmetric about the y-axis.

60. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. The curve does not have x-axis or y-axis symmetry.

61. If (r, θ) is a solution, so is $(r, \pi - \theta)$. Therefore, the curve is symmetric about the y-axis. The curve does not have x-axis or origin symmetry.

62. If (r, θ) is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if (r, θ) is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the x-axis. And since any curve with x-axis and origin symmetry also has y-axis symmetry, the curve is symmetric about the y-axis.

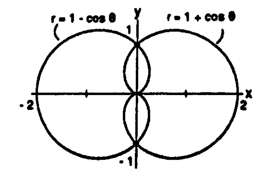
63. (a) Because $r = a \sec \theta$ is equivalent to $r \cos \theta = a$, which is equivalent to the Cartesian equation $x = a$.
(b) $r = a \csc \theta$ is equivalent to $y = a$.

64. (a) Let $r = f(\theta)$ be symmetric about the x-axis and the y-axis. Then (r, θ) on the graph $\Rightarrow$ $(r, -\theta)$ is on the graph because of symmetry about the x-axis. Then $(-r, -(-\theta)) = (-r, \theta)$ is on the graph because of symmetry about the y-axis. Therefore $r = f(\theta)$ is symmetric about the origin.
(b) Let $r = f(\theta)$ be symmetric about the x-axis and the origin. Then (r, θ) on the graph $\Rightarrow$ $(r, -\theta)$ is on the graph because of symmetry about the x-axis. Then $(-r, -\theta)$ is on the graph because of symmetry about the origin. Therefore $r = f(\theta)$ is symmetric about the y-axis.
(c) Let $r = f(\theta)$ be symmetric about the y-axis and the origin. Then (r, θ) on the graph $\Rightarrow$ $(-r, -\theta)$ is on the graph because of symmetry about the y-axis. Then $(-(-r), -\theta) = (r, -\theta)$ is on the graph because of symmetry about the origin. Therefore $r = f(\theta)$ is symmetric about the x-axis.

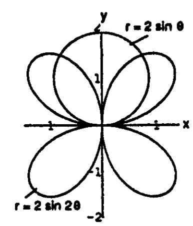
65. $\left(2, \frac{3\pi}{4}\right)$ is the same point as $\left(-2, -\frac{\pi}{4}\right)$; $r = 2 \sin 2\left(-\frac{\pi}{4}\right) = 2 \sin\left(-\frac{\pi}{2}\right) = -2 \Rightarrow \left(-2, -\frac{\pi}{4}\right)$ is on the graph $\Rightarrow \left(2, \frac{3\pi}{4}\right)$ is on the graph

66. $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is the same point as $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$; $r = -\sin\left(\frac{\left(\frac{\pi}{2}\right)}{3}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, \frac{\pi}{2}\right)$ is on the graph $\Rightarrow \left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is on the graph

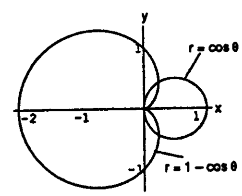
67. $1 + \cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow r = 1$; points of intersection are $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$. The point of intersection $(0,0)$ is found by graphing.



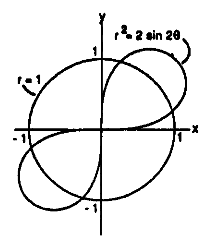
68. $2 \sin\theta = 2 \sin 2\theta \Rightarrow \sin\theta = \sin 2\theta \Rightarrow \sin\theta = 2 \sin\theta \cos\theta \Rightarrow \sin\theta - 2 \sin\theta \cos\theta = 0 \Rightarrow (\sin\theta)(1 - 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0, \frac{\pi}{3}$, or $-\frac{\pi}{3}$; $\theta = 0 \Rightarrow r = 0$, $\theta = \frac{\pi}{3} \Rightarrow r = \sqrt{3}$, and $\theta = -\frac{\pi}{3} \Rightarrow r = -\sqrt{3}$; points of intersection are $(0,0)$, $\left(\sqrt{3}, \frac{\pi}{3}\right)$, and $\left(-\sqrt{3}, -\frac{\pi}{3}\right)$



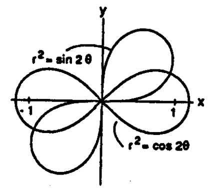
69. $\cos\theta = 1 - \cos\theta \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow r = \frac{1}{2}$; points of intersection are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$. The point $(0,0)$ is found by graphing.



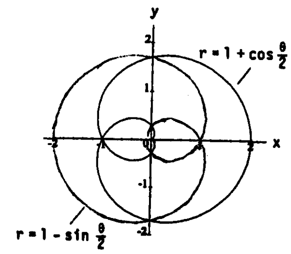
70. $1 = 2 \sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$, $\left(1, \frac{13\pi}{12}\right)$, and $\left(1, \frac{17\pi}{12}\right)$. No other points are found by graphing.



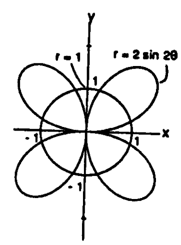
71. $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$ are generated completely for $0 \le \theta \le \frac{\pi}{2}$. Then $\sin 2\theta = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{4}$ is the only solution on that interval $\Rightarrow \theta = \frac{\pi}{8} \Rightarrow r^2 = \sin 2\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}}$ $\Rightarrow r = \pm\frac{1}{\sqrt[4]{2}}$; points of intersection are $\left(\pm\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$. The point of intersection $(0,0)$ is found by graphing.



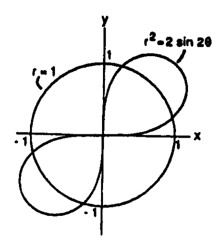
72. $1 - \sin\frac{\theta}{2} = 1 + \cos\frac{\theta}{2} \Rightarrow -\sin\frac{\theta}{2} = \cos\frac{\theta}{2} \Rightarrow \frac{\theta}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}$ $\Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$; $\theta = \frac{3\pi}{2} \Rightarrow r = 1 + \cos\frac{3\pi}{4} = 1 - \frac{\sqrt{2}}{2}$; $\theta = \frac{7\pi}{2} \Rightarrow r = 1 + \cos\frac{7\pi}{4} = 1 + \frac{\sqrt{2}}{2}$; points of intersection are $\left(1 - \frac{\sqrt{2}}{2}, \frac{3\pi}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \frac{7\pi}{2}\right)$. The three points of intersection $(0,0)$ and $\left(1 \pm \frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$ are found by graphing and symmetry.



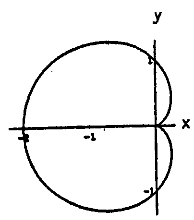
73. $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ $\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right), \left(1, \frac{5\pi}{12}\right), \left(1, \frac{13\pi}{12}\right)$, and $\left(1, \frac{17\pi}{12}\right)$. The points of intersection $\left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{19\pi}{12}\right)$ and $\left(1, \frac{23\pi}{12}\right)$ are found by graphing and symmetry.



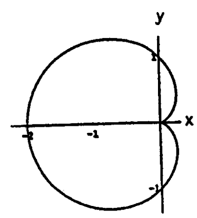
74. $r^2 = 2\sin 2\theta$ is completely generated on $0 \le \theta \le \frac{\pi}{2}$ so that $1 = 2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$ and $\left(1, \frac{5\pi}{12}\right)$. The points of intersection $\left(-1, \frac{\pi}{12}\right)$ and $\left(-1, \frac{5\pi}{12}\right)$ are found by graphing.



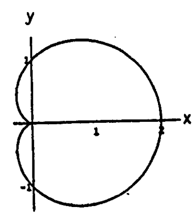
75. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = 1 - \cos\theta \Leftrightarrow -1 - \cos(\theta + \pi)$ $= -1 - (\cos\theta \cos\pi - \sin\theta \sin\pi) = -1 + \cos\theta = -(1 - \cos\theta) = -r$; therefore (r, θ) is on the graph of $r = 1 - \cos\theta \Leftrightarrow (-r, \theta + \pi)$ is on the graph of $r = -1 - \cos\theta \Rightarrow$ the answer is (a).



r = 1 - cos θ



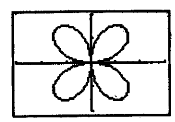
r = -1 - cos θ



r = 1 + cosθ

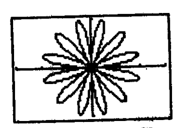
76. (a) The graph is the same for $n = 2$ and $n = -2$, and in general, it's the same for $n = 2k$ and $n = -2k$. The graphs for $n = 2$, 4, and 6 are roses with 4, 8, and 12 "petals" respectively. The graphs for $n = \pm 2$ and $n = \pm 6$ are shown below.

$n = \pm 2$



[−3, 3] by [−2, 2]

$n = \pm 6$

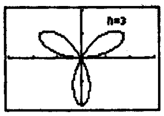


[−3, 3] by [−2, 2]

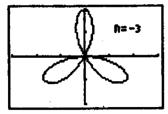
(b) 2π

(c) The graph is a rose with $2|n|$ "petals."

(d) The graphs are roses with 3, 5, and 7 "petals" respectively. The "center petal" points upward if $n = -3$, $+5$, or -7.
The graphs for $n = 3$ and $n = -3$ are shown below.



[−3, 3] by [−2, 2]

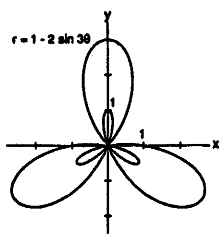


[−3, 3] by [−2, 2]

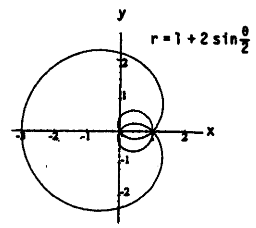
(e) π

(f) The graph is a rose with $|n|$ "petals."

77.



78.



79. (a) $r^2 = -4\cos\theta \Rightarrow \cos\theta = -\frac{r^2}{4}$; $r = 1 - \cos\theta \Rightarrow r = 1 - \left(-\frac{r^2}{4}\right) \Rightarrow 0 = r^2 - 4r + 4 \Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2$; therefore $\cos\theta = -\frac{2^2}{4} = -1 \Rightarrow \theta = \pi \Rightarrow (2, \pi)$ is a point of intersection

(b) $r = 0 \Rightarrow 0^2 = 4\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \left(0, \frac{\pi}{2}\right)$ or $\left(0, \frac{3\pi}{2}\right)$ is on the graph; $r = 0 \Rightarrow 0 = 1 - \cos\theta$

$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow (0,0)$ is on the graph. Since $(0,0) = \left(0, \frac{\pi}{2}\right)$ for polar coordinates, the graphs intersect at the origin.

80. (a) We have $x = r\cos\theta$ and $y = r\sin\theta$. By taking $t = \theta$, we have $r = f(t)$, so $x = f(t)\cos t$ and $y = f(t)\sin t$.

(b) $x = 3\cos t$, $y = 3\sin t$

(c) $x = (1 - \cos t)\cos t$, $y = (1 - \cos t)\sin t$

(d) $x = (3\sin 2t)\cos t$, $y = (3\sin 2t)\sin t$

81. $d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$

$$= \left[(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2\right]^{1/2}$$

$$= \left[r_2^2\cos^2\theta_2 - 2r_2r_1\cos\theta_2\cos\theta_1 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_2r_1\sin\theta_2\sin\theta_1 + r_1^2\sin^2\theta_1\right]^{1/2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

82. We wish to maximize $y = r \sin\theta = 2(1+\cos\theta)(\sin\theta) = 2\sin\theta + 2\sin\theta\cos\theta$. Then $\frac{dy}{d\theta} = 2\cos\theta + 2(\sin\theta)(-\sin\theta) + 2\cos\theta\cos\theta = 2\cos\theta - 2\sin^2\theta + 2\cos^2\theta = 2\cos\theta + 4\cos^2\theta - 2$; thus $\frac{dy}{d\theta} = 0 \Rightarrow 4\cos^2\theta + 2\cos\theta - 2 = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1 \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$. From the graph, we can see that the maximum occurs in the first quadrant so we choose $\theta = \frac{\pi}{3}$. Then $y = 2\sin\frac{\pi}{3} + 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$. The x-coordinate of this point is $x = r\cos\frac{\pi}{3}$ $= 2\left(1+\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}\right) = \frac{3}{2}$. Thus the maximum height is $h = \frac{3\sqrt{3}}{2}$ occurring at $x = \frac{3}{2}$.

9.6 CALCULUS OF POLAR CURVES

1. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\cos\theta\sin\theta + (-1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (-1+\sin\theta)\sin\theta} = \frac{2\sin\theta\cos\theta - \cos\theta}{\cos^2\theta - \sin^2\theta + \sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0} = -\frac{1}{1} = -1$, $\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{1}{1} = 1$

2. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{1}{0}$, which is undefined; $\left.\frac{dy}{dx}\right|_{\theta=\pm\pi/2} = \pm\frac{0}{1} = 0$; and $\left.\frac{dy}{dx}\right|_{\theta=\pi} = -\frac{1}{0}$, which is undefined.

3. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-3\cos\theta\sin\theta + (2-3\sin\theta)\cos\theta}{-3\cos\theta\cos\theta - (2-3\sin\theta)\sin\theta} = \frac{2\cos\theta - 6\sin\theta\cos\theta}{-2\sin\theta - 3(\cos^2\theta - \sin^2\theta)}$

$\left.\frac{dy}{dx}\right|_{(2,0)} = \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2}{-3} = -\frac{2}{3}$, $\left.\frac{dy}{dx}\right|_{(-1,\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0}{-1} = 0$, $\left.\frac{dy}{dx}\right|_{(2,\pi)} = \left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{2}{3}$,

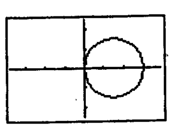
and $\left.\frac{dy}{dx}\right|_{(5,3\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{0}{-5} = 0$.

4. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{3\sin^2\theta + 3\cos\theta(1-\cos\theta)}{3\sin\theta\cos\theta - 3\sin\theta(1-\cos\theta)} = \frac{3\cos\theta - 3(\cos^2\theta - \sin^2\theta)}{6\sin\theta\cos\theta - 3\sin\theta}$

$\left.\frac{dy}{dx}\right|_{(1.5,\pi/3)} = \frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}$, which is undefined; $\left.\frac{dy}{dx}\right|_{(4.5,2\pi/3)} = \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}} = 0$;

$\left.\frac{dy}{dx}\right|_{(6,\pi)} = \frac{-1-1}{0-0}$, which is undefined; and $\left.\frac{dy}{dx}\right|_{(3,3\pi/2)} = \frac{0-(-1)}{0-(-1)} = 1$.

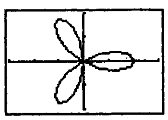
5.



[−3.8, 3.8] by [−2.5, 2.5]

The graph passes through the pole when $r = 3\cos\theta = 0$, which occurs when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{3\pi}{2}$. Since the θ-interval $0 \le \theta \le \pi$ produce the entire graph, we need only consider $\theta = \frac{\pi}{2}$. At this point, there appears to be a vertical tangent line with equation $\theta = \frac{\pi}{2}$ (or $x = 0$). Confirm analytically:
$x = (3\cos\theta)\cos\theta = 3\cos^2\theta$ and $y = (3\cos\theta)\sin\theta$;
$\frac{dy}{d\theta} = (-3\sin\theta)\sin\theta + (3\cos\theta)\cos\theta = 3(\cos^2\theta - \sin^2\theta)$ and $\frac{dx}{d\theta} = 6\cos\theta(-\sin\theta)$. At $\left(0, \frac{\pi}{2}\right)$, $\left.\frac{dx}{d\theta}\right|_{\theta=\pi/2} = 0$, and $\left.\frac{dy}{d\theta}\right|_{\theta=\pi/2} = 3(0^2 - 1^2) = -3$. So at $\left(0, \frac{\pi}{2}\right)$, $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \ne 0$, so $\frac{dy}{dx}$ is undefined and the tangent line is vertical.

6.



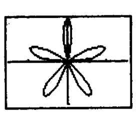
[−3, 3] by [−2, 2]

A trace of the graph suggests three tangent lines, one with positive slope for $\theta = \frac{\pi}{6}$, a vertical one for $\theta = \frac{\pi}{2}$, and one with negative slope for $\theta = \frac{5\pi}{6}$. Confirm analytically:
$\frac{dy}{d\theta} = -6\sin 3\theta \sin\theta + 2\cos 3\theta\cos\theta$ and $\frac{dx}{dt} = -6\sin 3\theta\cos\theta - 2\cos 3\theta\sin\theta$. $\left(0, \frac{\pi}{6}\right)$, $\left(0, \frac{\pi}{2}\right)$, and $\left(0, \frac{5\pi}{6}\right)$ are all solutions. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, and so $\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-6(1)(1/2) + 2(0)(\sqrt{3}/2)}{-6(1)(\sqrt{3}/2) - 2(0)(1/2)} = \frac{1}{\sqrt{3}}$;

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{-6(-1)(1) + 2(0)(0)}{-6(-1)(0) - 2(0)(1)}$, which is undefined; and $\left.\frac{dy}{dx}\right|_{\theta=5\pi/6} = \frac{-6(1)(1/2) + 2(0)(-\sqrt{3}/2)}{-6(1)(-\sqrt{3}/2) - 2(0)(1/2)} = -\frac{1}{\sqrt{3}}$.

The tangent lines have equations $\theta = \frac{\pi}{6}\left[y = \frac{1}{\sqrt{3}}x\right]$, $\theta = \frac{\pi}{2}[x = 0]$, and $\theta = \frac{5\pi}{6}\left[y = -\frac{1}{\sqrt{3}}x\right]$.

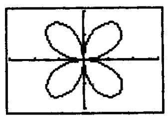
7.



[−1.5, 1.5] by [−1, 1]

The polar solutions are $\left(0, \frac{k\pi}{5}\right)$ for k = 0, 1, 2, 3, 4, and for a given k, the line $\theta = \frac{k\pi}{5}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{5}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan \frac{k\pi}{5}$. so the tangent lines are $\theta = 0$ $[y = 0]$, $\theta = \frac{\pi}{5}\left[y = \left(\tan \frac{\pi}{5}\right)x\right]$, $\theta = \frac{2\pi}{5}\left[y = \left(\tan \frac{2\pi}{5}\right)x\right]$, $\theta = \frac{3\pi}{5}\left[y = \left(\tan \frac{3\pi}{5}\right)x\right]$, and $\theta = \frac{4\pi}{5}\left[y = \left(\tan \frac{4\pi}{5}\right)x\right]$.

8.



[−3, 3] by [−2, 2]

The polar solutions are $\left(0, \frac{k\pi}{2}\right)$ for k = 0, 1, 2, 3, 4, and for a given k, the line $\theta = \frac{k\pi}{2}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{2}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan \frac{k\pi}{2}$. So the tangent lines are $\theta = 0$ $[y = 0]$ and $\theta = \frac{\pi}{2}$ $[x = 0]$. ($\theta = \pi$, $\theta = \frac{3\pi}{2}$ and $\theta = 2\pi$ are duplicate solutions.)

9. $\frac{dy}{d\theta} = \cos\theta \sin\theta + (-1 + \sin\theta)\cos\theta = \cos\theta(2\sin\theta - 1) = \sin 2\theta - \cos\theta$

$\frac{dx}{d\theta} = \cos^2\theta - (-1 + \sin\theta)\sin\theta = \cos^2\theta + \sin\theta - \sin^2\theta = -2\sin^2\theta + \sin\theta + 1$

$\frac{dy}{d\theta} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $(\cos\theta = 0)$ or when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $(2\sin\theta - 1 = 0)$. $\frac{dx}{d\theta} = 0$ when $\sin\theta = \frac{-1 \pm \sqrt{9}}{-4}$ $= -\frac{1}{2}$ or 1, i.e., when $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2}$. So there is a horizontal tangent line for $\theta = \frac{3\pi}{2}$, $r = -2$ [the line $y = -2\sin\frac{3\pi}{2} = 2$], for $\theta = \frac{\pi}{6}$, $r = -\frac{1}{2}$ [the line $y = -\frac{1}{2}\sin\frac{\pi}{6} = -\frac{1}{4}$] and for $\theta = \frac{5\pi}{6}$, $r = -\frac{1}{2}$ [again, the line $y = -\frac{1}{2}\sin\frac{5\pi}{6} = -\frac{1}{4}$]. There is a vertical tangent line for $\theta = \frac{7\pi}{6}$, $r = -\frac{3}{2}$ [the line $x = -\frac{3}{2}\cos\frac{7\pi}{6} = \frac{3\sqrt{3}}{4}$] and for $\theta = \frac{11\pi}{6}$, $r = -\frac{3}{2}$ [the line $x = -\frac{3}{2}\cos\frac{11\pi}{6} = -\frac{3\sqrt{3}}{4}$]. For $\theta = \frac{\pi}{2}$, $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but $\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = 2\cos 2\theta + \sin\theta = -1$ for $\theta = \frac{\pi}{2}$ and $\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -4\sin\theta\cos\theta + \cos\theta = 0$ for $\theta = \frac{\pi}{2}$, so by L'Hôpital's rule $\frac{dy}{dx}$ is undefined and the tangent line is vertical at $\theta = \frac{\pi}{2}$, $r = 0$ [the line $x = 0$]. This information can be summarized as follows.

Horizontal at: $\left(-\frac{1}{2}, \frac{\pi}{6}\right)$ $[y = -\frac{1}{4}]$, $\left(-\frac{1}{2}, \frac{5\pi}{6}\right)$ $[y = -\frac{1}{4}]$, $\left(-3, \frac{3\pi}{2}\right)$ $[y = 2]$

Vertical at: $\left(0, \frac{\pi}{2}\right)$ $[x = 0]$, $\left(-\frac{3}{2}, \frac{7\pi}{6}\right)$ $[x = \frac{3\sqrt{3}}{4}]$, $\left(-\frac{3}{5}, \frac{11\pi}{6}\right)$ $[x = -\frac{3\sqrt{3}}{4}]$

10. $\frac{dy}{d\theta} = -\sin^2\theta + (1+\cos\theta)\cos\theta = \cos^2\theta + \cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1$

$\frac{dx}{d\theta} = -\sin\theta\cos\theta - (1+\cos\theta)\sin\theta = -\sin\theta(1+2\cos\theta) = -\sin 2\theta - \sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-1\pm\sqrt{9}}{4} = -1$ or $\frac{1}{2}$, i.e., when $\theta = \pi$, $\frac{\pi}{3}$ or $\frac{5\pi}{3}$. $\frac{dx}{d\theta} = 0$ when $\theta = 0$, π, 2π (then $\sin\theta = 0$) or when $\theta = \frac{2\pi}{3}$, $\frac{4\pi}{3}$ (then $1+2\cos\theta = 0$). So there is a horizontal tangent line for $\theta = \frac{\pi}{3}$, $r = \frac{3}{2}$ [the line $y = \frac{3}{2}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}$] and for $\theta = \frac{5\pi}{3}$, $r = \frac{3}{2}$ [the line $y = \frac{3}{2}\sin\frac{5\pi}{3} = -\frac{3\sqrt{3}}{4}$]. There is a vertical tangent line for $\theta = 0$, $r = 2$ [the line $x = 2\cos 0 = 2$], for $\theta = \frac{2\pi}{3}$, $r = \frac{1}{2}$ [the line $x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}$] and for $\theta = \frac{4\pi}{3}$, $r = \frac{1}{2}$ [again, the line $x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}$]. For $\theta = \pi$, $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but $\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = -4\cos\theta\sin\theta - \sin\theta = 0$ for $\theta = \pi$, and $\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -2\cos 2\theta - \cos\theta = -1$ for $\theta = \pi$, so by L'Hôpital's rule $\frac{dy}{dx} = 0$ and the tangent line is horizontal at $\theta = \pi$, $r = 0$ [the line $y = 0$]. This information can be summarized as follows.

Horizontal at: $\left(\frac{3}{2},\frac{\pi}{3}\right)$ [$y = \frac{3\sqrt{3}}{4}$], $(0,\pi)$ [$y = 0$], $\left(\frac{3}{2},\frac{5\pi}{3}\right)$ [$y = -\frac{3\sqrt{3}}{4}$]

Vertical at: $(2,0)$ [$x = 2$], $\left(\frac{1}{2},\frac{2\pi}{3}\right)$ [$x = -\frac{1}{4}$], $\left(\frac{1}{2},\frac{4\pi}{3}\right)$ [$x = -\frac{1}{4}$], $(2,2\pi)$ [$x = 2$]

11. $y = 2\sin^2\theta \Rightarrow \frac{dy}{d\theta} = 4\sin\theta\cos\theta = 2\sin 2\theta$

$x = 2\sin\theta\cos\theta = \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2\cos 2\theta$

$\frac{dy}{d\theta} = 0$ when $\theta = 0$, $\frac{\pi}{2}$, π, and $\frac{dx}{d\theta} = 0$ when $\theta = \frac{\pi}{4}$, $\frac{3\pi}{4}$. They are never both zero. For $\theta = 0$, $\frac{\pi}{2}$, π the curve has horizontal asymptotes at $(0,0)$ [$y = 0\sin 0 = 0$], $\left(2,\frac{\pi}{2}\right)$ [$y = 2\sin\frac{\pi}{2} = 2$], and $(0,\pi)$ [$y = 0\sin\pi = 0$]. For $\theta = \frac{\pi}{4}$, $\frac{3\pi}{4}$ the curve has vertical asymptotes at $\left(\sqrt{2},\frac{\pi}{4}\right)$ [$x = \sqrt{2}\cos\frac{\pi}{4} = 1$] and $\left(\sqrt{2},\frac{3\pi}{4}\right)$ [$x = \sqrt{2}\cos\frac{3\pi}{4} = -1$]. This information can be summarized as follows.

Horizontal at: $(0,0)$ [$y = 0$], $\left(2,\frac{\pi}{2}\right)$ [$y = 2$], $(0,\pi)$ [$y = 0$]

Vertical at: $\left(\sqrt{2},\frac{\pi}{4}\right)$ [$x = 1$], $\left(\sqrt{2},\frac{3\pi}{4}\right)$ [$x = -1$]

12. $\frac{dy}{d\theta} = 4\sin^2\theta + (3-4\cos\theta)\cos\theta = 4(\sin^2\theta - \cos^2\theta) + 3\cos\theta = -8\cos^2\theta + 3\cos\theta + 4$

$\frac{dx}{d\theta} = 4\sin\theta\cos\theta - (3-4\cos\theta)\sin\theta = \sin\theta(8\cos\theta - 3) = 4\sin 2\theta - 3\sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-3\pm\sqrt{137}}{-16}$, i.e., when $\theta \approx 0.405$, 2.146, 4.137, or 5.878 (values solved for with a graphing calculator). $\frac{dx}{d\theta} = 0$ when $\theta = 0$, π or 2π (then $\sin\theta = 0$) or when $\theta = \cos^{-1}\left(\frac{3}{8}\right) \approx 1.186$ or $2\pi - \cos^{-1}\left(\frac{3}{8}\right) \approx 5.097$ (then $8\cos\theta - 3 = 0$). So there is a horizontal tangent line for $\theta \approx 0.405$, $r \approx -0.676$ [the line $y \approx -0.676\sin 0.405 \approx -0.267$], for $\theta \approx 2.146$, $r \approx 5.176$ [the line $y \approx 5.176\sin 2.146 \approx 4.343$], for $\theta \approx 4.137$, $r \approx 5.176$ [the line $y \approx 5.176\sin 4.137 \approx -4.343$], and for $\theta \approx 5.878$, $r \approx -0.676$ [the line

$y \approx -0.676 \sin 5.878 \approx 0.267$]. There is a vertical tangent for $\theta = 0$, $r = -1$ [the line $x = -1 \cos 0 = -1$], for $\theta = \pi$, $r = 7$ [the line $x = 7 \cos \pi = -7$], for $\theta = 2\pi$, $r = -1$ [again, the line $x = -1 \cos 2\pi = -1$], for $\theta = \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}$ [the line $x = \frac{9}{16}$], and for $\theta = 2\pi - \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}$ [again, the line $x = \frac{9}{16}$]. This information can be summarized as follows.

Horizontal at: $(-0.676, 0.405)$ [$y \approx -0.267$], $(5.176, 2.146)$ [$y \approx 4.343$], $(5.176, 4.137)$ [$y \approx -4.343$], $(-0.676, 5.878)$ [$y \approx 0.267$]

Vertical at: $(-1, 0)$ [$x = -1$], $(1.5, 1.186)$ [$x = \frac{9}{16}$], $(7, \pi)$ [$x = -7$], $(1.5, 5.097)$ [$x = \frac{9}{16}$], $(-1, 2\pi)$, [$x = -1$]

13. $A = \int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(16 + 16\cos\theta + 4\cos^2\theta\right) d\theta = \int_0^{2\pi} \left[8 + 8\cos\theta + 2\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$

$= \int_0^{2\pi} (9 + 8\cos\theta + \cos 2\theta)\, d\theta = \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 18\pi$

14. $A = \int_0^{2\pi} \frac{1}{2}[a(1+\cos\theta)]^2\, d\theta = \int_0^{2\pi} \frac{1}{2}a^2\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{2}a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{2}\pi a^2$

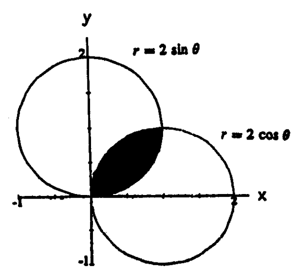
15. $A = 2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\, d\theta = \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$

16. $A = 2\int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(2a^2\cos 2\theta\right) d\theta = 2a^2 \int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4} = 2a^2$

17. $A = 2\int_0^{\pi/2} \frac{1}{2}(4\sin 2\theta)\, d\theta = \int_0^{\pi/2} 2\sin 2\theta\, d\theta = [-\cos 2\theta]_0^{\pi/2} = 2$

18. $A = (6)(2)\int_0^{\pi/6} \frac{1}{2}(2\sin 3\theta)\, d\theta = 12\int_0^{\pi/6} \sin 3\theta\, d\theta = 12\left[-\frac{\cos 3\theta}{3}\right]_0^{\pi/6} = 4$

19. $r = 2\cos\theta$ and $r = 2\sin\theta \Rightarrow 2\cos\theta = 2\sin\theta$ $\Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$; therefore



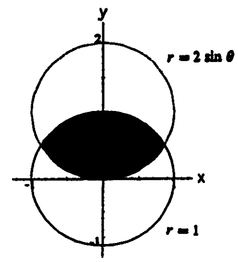
$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta = \int_0^{\pi/4} 4\sin^2\theta\, d\theta$

$= \int_0^{\pi/4} 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \int_0^{\pi/4} (2 - 2\cos 2\theta)\, d\theta$

$= [2\theta - \sin 2\theta]_0^{\pi/4} = \frac{\pi}{2} - 1$

20. $r = 1$ and $r = 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore $A = \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[(2\sin\theta)^2 - 1^2\right] d\theta$



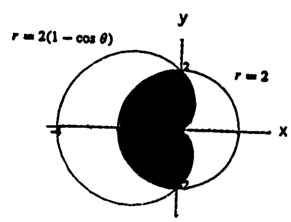
$= \pi - \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \frac{1}{2}\right) d\theta = \pi - \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - \frac{1}{2}\right) d\theta$

$= \pi - \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta\right) d\theta = \pi - \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$

$= \pi - \left(\frac{5\pi}{12} - \frac{1}{2}\sin\frac{5\pi}{3}\right) + \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{4\pi - 3\sqrt{3}}{6}$

21. $r = 2$ and $r = 2(1 - \cos\theta) \Rightarrow 2 = 2(1 - \cos\theta) \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pm\frac{\pi}{2}$; therefore $A = 2\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2\, d\theta$



$+\frac{1}{2}$ area of the circle $= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \left(\frac{1}{2}\pi\right)(2)^2$

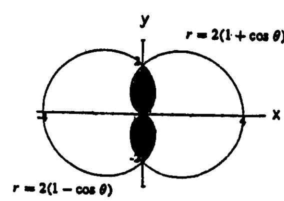
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + 2\pi$

$= \int_0^{\pi/2} (4 - 8\cos\theta + 2 + 2\cos 2\theta)\, d\theta + 2\pi$

$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi = 5\pi - 8$

22. $r = 2(1 - \cos\theta)$ and $r = 2(1 + \cos\theta) \Rightarrow 1 - \cos\theta = 1 + \cos\theta$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$; the graph also gives the point of intersection $(0, 0)$; therefore



$A = 2\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2\, d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2}[2(1 + \cos\theta)]^2\, d\theta$

$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$

$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

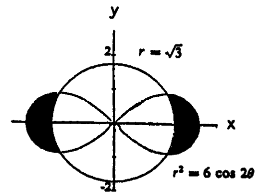
$= \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\, d\theta + \int_{\pi/2}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\, d\theta$

$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + \left[6\theta + 8\sin\theta + \sin 2\theta\right]_{\pi/2}^{\pi} = 6\pi - 16$

23. $r = \sqrt{3}$ and $r^2 = 6\cos 2\theta \Rightarrow 3 = 6\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph to find the area, so $A = 4\int_0^{\pi/6} \left[\frac{1}{2}(6\cos 2\theta) - \frac{1}{2}\left(\sqrt{3}\right)^2\right] d\theta$

$= 2\int_0^{\pi/6} (6\cos 2\theta - 3)\, d\theta = 2[3\sin 2\theta - 3\theta]_0^{\pi/6} = 3\sqrt{3} - \pi$



24. $r = 3a\cos\theta$ and $r = a(1+\cos\theta) \Rightarrow 3a\cos\theta = a(1+\cos\theta)$

$\Rightarrow 3\cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$; the graph also gives the point of intersection $(0,0)$; therefore

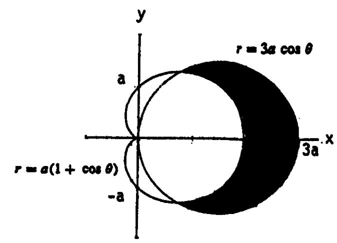
$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3a\cos\theta)^2 - a^2(1+\cos\theta)^2\right] d\theta$

$= \int_0^{\pi/3} \left(9a^2\cos^2\theta - a^2 - 2a^2\cos\theta - a^2\cos^2\theta\right) d\theta$

$= \int_0^{\pi/3} \left(8a^2\cos^2\theta - 2a^2\cos\theta - a^2\right) d\theta = \int_0^{\pi/3} \left[4a^2(1+\cos 2\theta) - 2a^2\cos\theta - a^2\right] d\theta$

$= \int_0^{\pi/3} \left(3a^2 + 4a^2\cos 2\theta - 2a^2\cos\theta\right) d\theta = \left[3a^2\theta + 2a^2\sin 2\theta - 2a^2\sin\theta\right]_0^{\pi/3} = \pi a^2 + 2a^2\left(\frac{1}{2}\right) - 2a^2\left(\frac{\sqrt{3}}{2}\right)$

$= a^2\left(\pi + 1 - \sqrt{3}\right)$



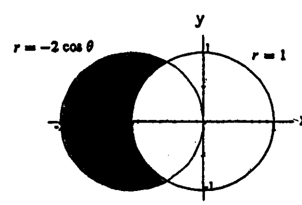
25. $r = 1$ and $r = -2\cos\theta \Rightarrow 1 = -2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$ in quadrant II; therefore

$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}\left[(-2\cos\theta)^2 - 1^2\right] d\theta = \int_{2\pi/3}^{\pi} \left(4\cos^2\theta - 1\right) d\theta$

$= \int_{2\pi/3}^{\pi} [2(1+\cos 2\theta) - 1]\, d\theta = \int_{2\pi/3}^{\pi} (1 + 2\cos 2\theta)\, d\theta$

$= \left[\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$



26. (a) $A = 2\int_0^{2\pi/3} \frac{1}{2}(2\cos\theta + 1)^2\, d\theta = \int_0^{2\pi/3} \left(4\cos^2\theta + 4\cos\theta + 1\right) d\theta = \int_0^{2\pi/3} [2(1 + \cos 2\theta) + 4\cos\theta + 1]\, d\theta$

$= \int_0^{2\pi/3} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta = \left[3\theta + \sin 2\theta + 4\sin\theta\right]_0^{2\pi/3} = 2\pi - \frac{\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} = 2\pi + \frac{3\sqrt{3}}{2}$

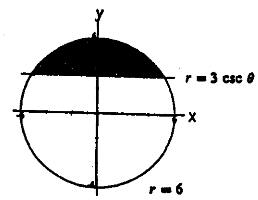
(b) $A = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$ (from 26(a) above and Example 4 in the text)

27. $r = 6$ and $r = 3\csc\theta \Rightarrow 6\sin\theta = 3 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

or $\frac{5\pi}{6}$; therefore $A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(6^2 - 9\csc^2\theta\right) d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(18 - \frac{9}{2}\csc^2\theta\right) d\theta = \left[18\theta + \frac{9}{2}\cot\theta\right]_{\pi/6}^{5\pi/6}$

$= \left(15\pi - \frac{9}{2}\sqrt{3}\right) - \left(3\pi + \frac{9}{2}\sqrt{3}\right) = 12\pi - 9\sqrt{3}$



28. $r^2 = 6\cos 2\theta$ and $r = \frac{3}{2}\sec\theta \Rightarrow \frac{9}{4}\sec^2\theta = 6\cos 2\theta \Rightarrow \frac{9}{24} = \cos^2\theta\cos 2\theta \Rightarrow \frac{3}{8} = \left(\cos^2\theta\right)\left(2\cos^2\theta - 1\right)$

$\Rightarrow \frac{3}{8} = 2\cos^4\theta - \cos^2\theta \Rightarrow 2\cos^4\theta - \cos^2\theta - \frac{3}{8} = 0 \Rightarrow 16\cos^4\theta - 8\cos^2\theta - 3 = 0 \Rightarrow \left(4\cos^2\theta - 1\right)\left(4\cos^2\theta - 3\right)$

$= 0 \Rightarrow \cos^2\theta = \frac{3}{4}$ or $\cos^2\theta = -\frac{1}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$ (the second equation has no real roots) $\Rightarrow \theta = \frac{\pi}{6}$ (in the first quadrant); thus $A = 2\int_0^{\pi/6} \frac{1}{2}\left(6\cos 2\theta - \frac{9}{4}\sec^2\theta\right) d\theta = \int_0^{\pi/6} \left(6\cos 2\theta - \frac{9}{4}\sec^2\theta\right) d\theta = \left[3\sin 2\theta - \frac{9}{4}\tan\theta\right]_0^{\pi/6}$

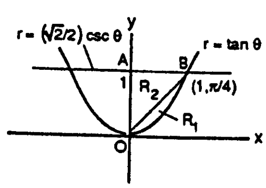
$= 3\left(\frac{\sqrt{3}}{2}\right) - \frac{9}{4\sqrt{3}} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$

29. (a) $r = \tan\theta$ and $r = \left(\frac{\sqrt{2}}{2}\right)\csc\theta \Rightarrow \tan\theta = \left(\frac{\sqrt{2}}{2}\right)\csc\theta$

$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta \Rightarrow 1 - \cos^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta$

$\Rightarrow \cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)\cos\theta - 1 = 0 \Rightarrow \cos\theta = -\sqrt{2}$ or

$\frac{\sqrt{2}}{2}$ (use the quadratic formula) $\Rightarrow \theta = \frac{\pi}{4}$ (the solution



in the first quadrant); therefore the area of R_1 is $A_1 = \int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4} \left(\sec^2\theta - 1\right) d\theta$

$= \frac{1}{2}[\tan\theta - \theta]_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2} - \frac{\pi}{8}$; $AO = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{2} = \frac{\sqrt{2}}{2}$ and $OB = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{4} = 1$

$\Rightarrow AB = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2} \Rightarrow$ the area of R_2 is $A_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$; therefore the area of the region shaded in the text is $2\left(\frac{1}{2}-\frac{\pi}{8}+\frac{1}{4}\right)=\frac{3}{2}-\frac{\pi}{4}$. Note: The area must be found this way since no common interval generates the region. For example, the interval $0 \le \theta \le \frac{\pi}{4}$ generates the arc OB of $r = \tan\theta$ but does not generate the segment AB of the line $r = \frac{\sqrt{2}}{2}\csc\theta$. Instead the interval generates the half-line from B to $+\infty$ on the line $r = \frac{\sqrt{2}}{2}\csc\theta$.

(b) $\lim\limits_{\theta\to\pi/2^-} \tan\theta = \infty$ and the line $x = 1$ is $r = \sec\theta$ in polar coordinates; then $\lim\limits_{\theta\to\pi/2^-} (\tan\theta - \sec\theta)$ $= \lim\limits_{\theta\to\pi/2^-}\left(\frac{\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}\right) = \lim\limits_{\theta\to\pi/2^-}\left(\frac{\sin\theta-1}{\cos\theta}\right) = \lim\limits_{\theta\to\pi/2^-}\left(\frac{\cos\theta}{-\sin\theta}\right) = 0 \Rightarrow r = \tan\theta$ approaches $r = \sec\theta$ as $\theta \to \frac{\pi}{2}^- \Rightarrow r = \sec\theta$ (or $x = 1$) is a vertical asymptote of $r = \tan\theta$. Similarly, $r = -\sec\theta$ (or $x = -1$) is a vertical asymptote of $r = \tan\theta$.

30. It is not because the circle is generated twice from $\theta = 0$ to 2π. The area of the cardioid is

$$A = 2\int_0^{\pi} \frac{1}{2}(\cos\theta+1)^2\, d\theta = \int_0^{\pi} \left(\cos^2\theta + 2\cos\theta + 1\right) d\theta = \int_0^{\pi}\left(\frac{1+\cos 2\theta}{2} + 2\cos\theta + 1\right) d\theta$$

$= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} = \frac{3\pi}{2}$. The area of the circle is $A = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \Rightarrow$ the area requested is actually $\frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$

31. $r = \theta^2,\ 0 \le \theta \le \sqrt{5} \Rightarrow \frac{dr}{d\theta} = 2\theta$; therefore Length $= \int_0^{\sqrt{5}} \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{\sqrt{5}} |\theta|\sqrt{\theta^2+4}\, d\theta =$ (since $\theta \ge 0$) $\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta$; $\left[u = \theta^2 + 4 \Rightarrow \frac{1}{2}\, du = \theta\, d\theta;\ \theta = 0 \Rightarrow u = 4,\right.$

$\left.\theta = \sqrt{5} \Rightarrow u = 9\right] \to \int_4^9 \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{19}{3}$

32. $r = \frac{e^\theta}{\sqrt{2}},\ 0 \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}$; therefore Length $= \int_0^{\pi} \sqrt{\left(\frac{e^\theta}{\sqrt{2}}\right)^2 + \left(\frac{e^\theta}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\, d\theta$

$= \int_0^{\pi} e^\theta\, d\theta = \left[e^\theta\right]_0^{\pi} = e^\pi - 1$

33. $r = 1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; therefore Length $= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$

$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{\frac{4(1+\cos\theta)}{2}}\, d\theta = 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}}\, d\theta = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8$

34. $r = a \sin^2 \frac{\theta}{2},\ 0 \le \theta \le \pi,\ a > 0 \Rightarrow \frac{dr}{d\theta} = a \sin \frac{\theta}{2} \cos \frac{\theta}{2}$; therefore Length $= \int_0^{\pi} \sqrt{\left(a \sin^2 \frac{\theta}{2}\right)^2 + \left(a \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2}\, d\theta$

$= \int_0^{\pi} \sqrt{a^2 \sin^4 \frac{\theta}{2} + a^2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}\, d\theta = \int_0^{\pi} a \left|\sin \frac{\theta}{2}\right| \sqrt{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}\, d\theta = (\text{since } 0 \le \theta \le \pi)\ a \int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$

$= \left[-2a \cos \frac{\theta}{2}\right]_0^{\pi} = 2a$

35. $r = \frac{6}{1 + \cos \theta},\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 + \cos \theta)^2}$; therefore Length $= \int_0^{\pi/2} \sqrt{\left(\frac{6}{1 + \cos \theta}\right)^2 + \left(\frac{6 \sin \theta}{(1 + \cos \theta)^2}\right)^2}\, d\theta$

$= \int_0^{\pi/2} \sqrt{\frac{36}{(1 + \cos \theta)^2} + \frac{36 \sin^2 \theta}{\left(1 + \cos^2 \theta\right)^4}}\, d\theta = 6 \int_0^{\pi/2} \left|\frac{1}{1 + \cos \theta}\right| \sqrt{1 + \frac{\sin^2 \theta}{(1 + \cos \theta)^2}}\, d\theta$

$= \left(\text{since } \frac{1}{1 + \cos \theta} > 0 \text{ on } 0 \le \theta \le \frac{\pi}{2}\right)\ 6 \int_0^{\pi/2} \left(\frac{1}{1 + \cos \theta}\right) \sqrt{\frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2}}\, d\theta$

$= 6 \int_0^{\pi/2} \left(\frac{1}{1 + \cos \theta}\right) \sqrt{\frac{2 + 2 \cos \theta}{(1 + \cos \theta)^2}}\, d\theta = 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{(1 + \cos \theta)^{3/2}} = 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\left(2 \cos^2 \frac{\theta}{2}\right)^{3/2}} = 6 \int_0^{\pi/2} \left|\sec^3 \frac{\theta}{2}\right| d\theta$

$= 6 \int_0^{\pi/2} \sec^3 \frac{\theta}{2}\, d\theta = 12 \int_0^{\pi/4} \sec^3 u\, du = (\text{use tables})\ 6\left(\left[\frac{\sec u \tan u}{2}\right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec u\, du\right)$

$= 6\left(\frac{1}{\sqrt{2}} + \left[\frac{1}{2} \ln |\sec u + \tan u|\right]_0^{\pi/4}\right) = 3\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

36. $r = \frac{2}{1 - \cos \theta},\ \frac{\pi}{2} \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{-2 \sin \theta}{(1 - \cos \theta)^2}$; therefore Length $= \int_{\pi/2}^{\pi} \sqrt{\left(\frac{2}{1 - \cos \theta}\right)^2 + \left(\frac{-2 \sin \theta}{(1 - \cos \theta)^2}\right)^2}\, d\theta$

$= \int_{\pi/2}^{\pi} \sqrt{\frac{4}{(1 - \cos \theta)^2}\left(1 + \frac{\sin^2 \theta}{\left(1 - \cos^2 \theta\right)^2}\right)}\, d\theta = 6 \int_{\pi/2}^{\pi} \left|\frac{2}{1 - \cos \theta}\right| \sqrt{\frac{(1 - \cos \theta)^2 + \sin^2 \theta}{(1 - \cos \theta)^2}}\, d\theta$

$= \left(\text{since } 1 - \cos \theta \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2 \int_{\pi/2}^{\pi} \left(\frac{1}{1 - \cos \theta}\right) \sqrt{\frac{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 - \cos \theta)^2}}\, d\theta$

$= 2 \int_{\pi/2}^{\pi} \left(\frac{1}{1 - \cos \theta}\right) \sqrt{\frac{2 - 2 \cos \theta}{(1 - \cos \theta)^2}}\, d\theta = 2\sqrt{2} \int_{\pi/2}^{\pi} \frac{d\theta}{(1 - \cos \theta)^{3/2}} = 2\sqrt{2} \int_{\pi/2}^{\pi} \frac{d\theta}{\left(2 \sin^2 \frac{\theta}{2}\right)^{3/2}} = \int_{\pi/2}^{\pi} \left|\csc^3 \frac{\theta}{2}\right| d\theta$

$= 6 \int_{\pi/2}^{\pi} \csc^3\left(\frac{\theta}{2}\right) d\theta = \left(\text{since } \csc \frac{\theta}{2} \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2 \int_{\pi/4}^{\pi/2} \csc^3 u\, du = (\text{use tables})$

$$6\left(\left[-\frac{\csc u \cot u}{2}\right]_{\pi/4}^{\pi/2}+\frac{1}{2}\int_{\pi/4}^{\pi/2} \csc u\, du\right)=2\left(\frac{1}{\sqrt{2}}-\left[\frac{1}{2}\ln|\csc u+\cot u|\right]_{\pi/4}^{\pi/2}\right)=2\left[\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right]$$
$$=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

37. $r=\cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta}=-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$; therefore Length $=\int_0^{\pi/4}\sqrt{\left(\cos^3\frac{\theta}{3}\right)^2+\left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2}\,d\theta$

$$=\int_0^{\pi/4}\sqrt{\cos^6\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)\cos^4\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\left(\cos^2\frac{\theta}{3}\right)\sqrt{\cos^2\left(\frac{\theta}{3}\right)+\sin^2\left(\frac{\theta}{3}\right)}\,d\theta=\int_0^{\pi/4}\cos^2\left(\frac{\theta}{3}\right)d\theta$$
$$=\int_0^{\pi/4}\frac{1+\cos\left(\frac{2\theta}{3}\right)}{2}\,d\theta=\frac{1}{2}\left[\theta+\frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4}=\frac{\pi}{8}+\frac{3}{8}$$

38. $r=\sqrt{1+\sin 2\theta}$, $0\le\theta\le\pi\sqrt{2} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}(1+\sin 2\theta)^{-1/2}(2\cos 2\theta)=(\cos 2\theta)(1+\sin 2\theta)^{-1/2}$; therefore

$$\text{Length}=\int_0^{\pi\sqrt{2}}\sqrt{(1+\sin 2\theta)+\frac{\cos^2 2\theta}{(1+\sin 2\theta)}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{1+2\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta}{1+\sin 2\theta}}\,d\theta$$
$$=\int_0^{\pi\sqrt{2}}\sqrt{\frac{2+2\sin 2\theta}{1+\sin 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{2}\,d\theta=\left[\sqrt{2}\,\theta\right]_0^{\pi\sqrt{2}}=2\pi$$

39. $r=\sqrt{1+\cos 2\theta} \Rightarrow \frac{dr}{d\theta}=\frac{1}{2}(1+\cos 2\theta)^{-1/2}(-2\sin 2\theta)$; therefore Length $=\int_0^{\pi\sqrt{2}}\sqrt{(1+\cos 2\theta)+\frac{\sin^2 2\theta}{(1+\cos 2\theta)}}\,d\theta$

$$=\int_0^{\pi\sqrt{2}}\sqrt{\frac{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}{1+\cos 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{\frac{2+2\cos 2\theta}{1+\cos 2\theta}}\,d\theta=\int_0^{\pi\sqrt{2}}\sqrt{2}\,d\theta=\left[\sqrt{2}\,\theta\right]_0^{\pi\sqrt{2}}=2\pi$$

40. (a) $r=a \Rightarrow \frac{dr}{d\theta}=0$; Length $=\int_0^{2\pi}\sqrt{a^2+0^2}\,d\theta=\int_0^{2\pi}|a|\,d\theta=[a\theta]_0^{2\pi}=2\pi a$

(b) $r=a\cos\theta \Rightarrow \frac{dr}{d\theta}=-a\sin\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(-a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$$=\int_0^{\pi}|a|\,d\theta=[a\theta]_0^{\pi}=\pi a$$

(c) $r=a\sin\theta \Rightarrow \frac{dr}{d\theta}=a\cos\theta$; Length $=\int_0^{\pi}\sqrt{(a\cos\theta)^2+(a\sin\theta)^2}\,d\theta=\int_0^{\pi}\sqrt{a^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$

$$= \int_0^{\pi} |a|\ d\theta = [a\theta]_0^{\pi} = \pi a$$

41. Let $r = f(\theta)$. Then $x = f(\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2$
$= [f'(\theta)]^2\cos^2\theta - 2f'(\theta)\,f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\sin^2\theta$; $y = f(\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$
$\Rightarrow \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2 = [f'(\theta)]^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\cos^2\theta$. Therefore
$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + [f(\theta)]^2(\cos^2\theta + \sin^2\theta) = [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$.
Thus, $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$.

42. (a) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a(1-\cos\theta)\ d\theta = \frac{a}{2\pi}[\theta - \sin\theta]_0^{2\pi} = a$

(b) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a\ d\theta = \frac{1}{2\pi}[a\theta]_0^{2\pi} = a$

(c) $r_{av} = \frac{1}{\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{\pi/2} a\cos\theta\ d\theta = \frac{1}{\pi}[a\sin\theta]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

43. $r = 2f(\theta)$, $\alpha \le \theta \le \beta \Rightarrow \frac{dr}{d\theta} = 2f'(\theta) \Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = [2f(\theta)]^2 + [2f'(\theta)]^2 \Rightarrow$ Length $= \int_{\alpha}^{\beta} \sqrt{4[f(\theta)]^2 + 4[f'(\theta)]^2}\ d\theta$
$= 2\int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\ d\theta$ which is twice the length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$.

44. (a) Let $r = 1.75 + \frac{0.06\theta}{2\pi}$.

(b) Since $\frac{dr}{d\theta} = \frac{b}{2\pi}$, this is just Equation 4 for the length of the curve.

(c) Using the integral function on a calculator or CAS, $\int_0^{80\pi} \sqrt{\left(1.75 + \frac{0.06\theta}{2\pi}\right)^2 + \left(\frac{0.06}{2\pi}\right)^2}\ d\theta$ evaluates to ≈ 741.420 cm ≈ 7.414 m.

(d) $\left(r^2 + \left(\frac{b}{2\pi}\right)^2\right)^{1/2} = r\left(1 + \left(\frac{b}{2\pi r}\right)^2\right)^{1/2} \approx r$ since $\left(\frac{b}{2\pi r}\right)^2$ is a very small quantity squared.

(e) $L \approx 741.420$ cm (from part (c)), $L_a = \int_0^{80\pi} \left(1.75 + \frac{0.06\theta}{2\pi}\right) d\theta = \left[1.750 + \frac{0.03\theta^2}{2\pi}\right]_0^{80\pi} = 236\pi \approx 741.416$ cm

45. (a) Use the approximation, L_a, from Exercise #45(e). If the reel has made n complete turns, then the angle is $2\pi n$. So from the integral, $L_a = \pi b n^2 + 2\pi r_0 n$. Solving for n gives $n = \left(\frac{r_0}{b}\right)\left(\sqrt{\frac{bL}{r_0^2\pi} + 1} - 1\right)$.

(b) The take up reel slows down as time progresses.

(c) Since L is proportional to time, the formula in part (a) shows that n will grow roughly as the square root of time.

CHAPTER 9 PRACTICE EXERCISES

1. (a) $3\langle -3,4\rangle - 4\langle 2,-5\rangle = \langle -9-8, 12+20\rangle = \langle -17, 32\rangle$
   (b) $\sqrt{17^2+32^2} = \sqrt{1313}$

2. (a) $\langle -3+2, 4-5\rangle = \langle -1,-1\rangle$
   (b) $\sqrt{1^2+1^2} = \sqrt{2}$

3. (a) $\langle -2(-3), -2(4)\rangle = \langle 6,-8\rangle$
   (b) $\sqrt{6^2+8^2} = 10$

4. (a) $\langle 5(2), 5(-5)\rangle = \langle 10,-25\rangle$
   (b) $\sqrt{10^2+25^2} = \sqrt{725} = 5\sqrt{29}$

5. $\frac{\pi}{6}$ radians below the negative x-axis: $\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$ [assuming counterclockwise].

6. $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i}-\mathbf{j}) = \left(\frac{8}{\sqrt{17}}\mathbf{i} - \frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{(3/5)^2+(4/5)^2}}\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right) = (-3\mathbf{i}-4\mathbf{j})$

9. length $= \left|\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}\right| = \sqrt{2+2} = 2$, $\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j} = 2\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$

10. length $= |-\mathbf{i}-\mathbf{j}| = \sqrt{1+1} = \sqrt{2}$, $-\mathbf{i}-\mathbf{j} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$

11. $\frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$; at the point $(0,2)$, $t = \frac{\pi}{2} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\pi/2} = -2\mathbf{i}$; length $= |-2\mathbf{i}| = 2$;
direction $= -\mathbf{i} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\pi/2} = 2(-\mathbf{i})$

12. $\frac{d\mathbf{r}}{dt} = \left[e^t(\cos t - \sin t)\right]\mathbf{i} + \left[e^t(\sin t + \cos t)\right]\mathbf{j} \Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\ln 2} = 2[\cos(\ln 2) - \sin(\ln 2)]\mathbf{i} + 2[\sin(\ln 2) + \cos(\ln 2)]\mathbf{j}$

$\Rightarrow$ length $= 2\sqrt{[\cos(\ln 2) - \sin(\ln 2)]^2 + [\sin(\ln 2) + \cos(\ln 2)]^2}$

$= 2\sqrt{[1 - 2\sin(\ln 2)\cos(\ln 2)] + [1 + 2\sin(\ln 2)\cos(\ln 2)]} = 2\sqrt{2}$;

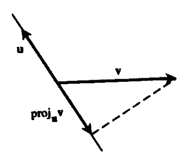
direction $= \frac{[\cos(\ln 2) - \sin(\ln 2)]}{\sqrt{2}}\mathbf{i} + \frac{[\sin(\ln 2) + \cos(\ln 2)]}{\sqrt{2}}\mathbf{j}$

$\Rightarrow \left.\frac{d\mathbf{r}}{dt}\right|_{t=\ln 2} = 2\sqrt{2}\left(\frac{[\cos(\ln 2) - \sin(\ln 2)]}{\sqrt{2}}\mathbf{i} + \frac{[\sin(\ln 2) + \cos(\ln 2)]}{\sqrt{2}}\mathbf{j}\right)$

13. $y = \tan x \Rightarrow [y']_{\pi/4} = [\sec^2 x]_{\pi/4} = 2 = \frac{2}{1} \Rightarrow \mathbf{T} = \mathbf{i} + 2\mathbf{j} \Rightarrow$ the unit tangents are $\pm\left(\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}\right)$ and the unit normals are $\pm\left(-\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right)$

14. $x^2 + y^2 = 25 \Rightarrow [y']_{(3,4)} = \left[-\frac{x}{y}\right]_{(3,4)} = \frac{-3}{4} \Rightarrow \mathbf{T} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow$ the unit tangents are $\pm\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ and the unit normals are $\pm\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$

15.



16. $\mathbf{a} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$, $\mathbf{b} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$, $\mathbf{c} = \mathbf{v} - \mathbf{b} = \mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}$

17. $|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 1(2) + 1(1) = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

$= \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) \approx 0.32$ rad, $|\mathbf{u}| \cos\theta = \sqrt{5}\left(\frac{3}{\sqrt{10}}\right) = \frac{3\sqrt{2}}{2}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = (|\mathbf{u}| \cos\theta)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)$

$= \left(\frac{3\sqrt{2}}{2}\right)\left(\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}\right) = \frac{3}{2}(\mathbf{i} + \mathbf{j})$

18. $|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = (1)(-1) + (1)(-3) = -4$,

$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{2\sqrt{5}}\right) \approx 2.68$ rad, $|\mathbf{u}| \cos\theta = (\sqrt{10})\left(\frac{-2}{\sqrt{5}}\right) = -2\sqrt{2}$,

$\text{proj}_{\mathbf{v}}\,\mathbf{u} = (|\mathbf{u}| \cos\theta)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = (-2\sqrt{2})\left(\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}\right) = -2(\mathbf{i} + \mathbf{j})$

19. Vector component of $\mathbf{u}$ parallel to $\mathbf{v}$: $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{(1)(2) + (-1)(1)}{2^2 + 1^2}(2\mathbf{i} - \mathbf{j}) = \frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}$

Vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$: $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{i} + \mathbf{j}) - \left(\frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}\right) = \frac{3}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}$

Therefore, $\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) = \left(\frac{2}{5}\mathbf{i} - \frac{1}{5}\mathbf{j}\right) + \left(\frac{3}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}\right)$.

20. Vector component of **u** parallel to **v**:

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{(-1)(1)+(1)(-2)}{1^2+2^2}(\mathbf{i}-2\mathbf{j}) = -\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}$$

Vector component of **u** orthogonal to **v**: $\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u} = (-\mathbf{i}+\mathbf{j}) - \left(-\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}\right) = -\frac{2}{5}\mathbf{i}-\frac{1}{5}\mathbf{j}$

Therefore, $\mathbf{u} = \text{proj}_{\mathbf{v}}\,\mathbf{u} + (\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u}) = \left(-\frac{3}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}\right)+\left(-\frac{2}{5}\mathbf{i}-\frac{1}{5}\mathbf{j}\right)$.

21. (a) $\mathbf{v}(t) = \frac{d}{dt}\left[(4\cos t)\mathbf{i}+(\sqrt{2}\sin t)\mathbf{j}\right]$

$= (-4\sin t)\mathbf{i}+(\sqrt{2}\cos t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}\left[(-4\sin t)\mathbf{i}+(\sqrt{2}\cos t)\mathbf{j}\right]$

$= (-4\cos t)\mathbf{i}+(-\sqrt{2}\sin t)\mathbf{j}$

(b) $\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(-4\sin\frac{\pi}{4}\right)^2+\left(\sqrt{2}\cos\frac{\pi}{4}\right)^2} = \sqrt{8+1} = 3$

(c) At $t=\frac{\pi}{4}$, $\mathbf{v} = -2\sqrt{2}\mathbf{i}+\mathbf{j}$, $\mathbf{a} = -2\sqrt{2}\mathbf{i}-\mathbf{j}$, and

$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} = \cos^{-1}\frac{8-1}{(3)(3)} = \cos^{-1}\frac{7}{9} \approx 38.94°$.

22. (a) $\mathbf{v}(t) = \frac{d}{dt}[(\sqrt{3}\sec t)\mathbf{i}+(\sqrt{3}\tan t)\mathbf{j}]$

$= (\sqrt{3}\sec t\tan t)\mathbf{i}+(\sqrt{3}\sec^2 t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}[(\sqrt{3}\sec t\tan t)\mathbf{i}+(\sqrt{3}\sec^2 t)\mathbf{j}]$

$= \sqrt{3}(\sec t\tan^2 t+\sec^3 t)\mathbf{i}+(2\sqrt{3}\sec^2 t\tan t)\mathbf{j}$

(b) $|\mathbf{v}(0)| = \sqrt{3\sec^2 0\tan^2 0+3\sec^4 0} = \sqrt{0+3} = \sqrt{3}$

(c) At $t=0$, $\mathbf{v} = \sqrt{3}\mathbf{j}$, $\mathbf{a} = \sqrt{3}\mathbf{i}$

$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} = \frac{0+0}{(\sqrt{3})(\sqrt{3})} = \cos^{-1}0 = 90°$.

23. $\mathbf{v}(t) = -\frac{t}{(1+t^2)^{3/2}}\mathbf{i}+\frac{1}{(1+t^2)^{3/2}}\mathbf{j}$

$\left|\frac{d\mathbf{r}}{dt}\right| = |\mathbf{v}(t)| = \sqrt{\left(-\frac{t}{(1+t^2)^{3/2}}\right)^2+\left(\frac{1}{(1+t^2)^{3/2}}\right)^2} = \frac{1}{1+t^2}$ which is at a maximum of 1 when $t=0$.

24. Minimizing $\left|\frac{d\mathbf{r}}{dt}\right|^2$ will minimize $\left|\frac{d\mathbf{r}}{dt}\right|$.

$\frac{d\mathbf{r}}{dt} = [e^t(\cos t-\sin t)]\mathbf{i}+[e^t(\sin t+\cos t]\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|^2 = [e^t(\cos t-\sin t)]^2+[e^t(\sin t+\cos t)]^2$

$= e^{2t}[(1-2\sin t\cos t)+(1+2\sin t\cos t)] = 2e^{2t}$. For $t \geq 0$, the minimum value of $2e^{2t}$ is 2 at $t=0$, and it has no maximum value. Therefore, the minimum speed is $\sqrt{2}$ and there is no maximum speed.

25. $\left(\int_0^t (3+6t)\,dt\right)\mathbf{i}+\left(\int_0^1 6\pi\cos\pi t\,dt\right)\mathbf{j}$

$=\left[3t+3t^2\right]_0^1\mathbf{i}+\left[6\sin\pi t\right]_0^1\mathbf{j}=6\mathbf{i}$

26. $\left(\int_e^{e^2} \frac{2\ln t}{t}\,dt\right)\mathbf{i}+\left(\int_e^{e^2} \frac{1}{t\ln t}\,dt\right)\mathbf{j}$

$=\left[\ln^2 t\right]_e^{e^2}\mathbf{i}+\left[\ln(\ln t)\right]_e^{e^2}\mathbf{j}=3\mathbf{i}+(\ln 2)\mathbf{j}$

27. $\mathbf{r}(t)=\int \frac{d\mathbf{r}}{dt}\,dt=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0)=\mathbf{i}+\mathbf{C}=\mathbf{j}$, so $\mathbf{C}=-\mathbf{i}+\mathbf{j}$, and

$\mathbf{r}(t)=(\cos t-1)\mathbf{i}+(\sin t+1)\mathbf{j}$

28. $\mathbf{r}(t)=\int \frac{d\mathbf{r}}{dt}\,dt=(\tan^{-1}t)\mathbf{i}+\sqrt{t^2+1}\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0)=\mathbf{j}+\mathbf{C}=\mathbf{i}+\mathbf{j}$, so $\mathbf{C}=\mathbf{i}$, and

$\mathbf{r}(t)=(\tan^{-1}t+1)\mathbf{i}+\sqrt{t^2+1}\mathbf{j}$

29. $\frac{d\mathbf{r}}{dt}=\int \frac{d^2\mathbf{r}}{dt^2}\,dt=2t\mathbf{j}+\mathbf{C}_1$, $\mathbf{r}(t)=\int \frac{d\mathbf{r}}{dt}\,dt=t^2\mathbf{j}+\mathbf{C}_1t+\mathbf{C}_2$

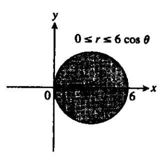
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0}=\mathbf{C}_1=\mathbf{0}$, so $\mathbf{r}(t)=t^2\mathbf{j}+\mathbf{C}_2$. And $\mathbf{r}(0)=\mathbf{C}_2=\mathbf{i}$, so $\mathbf{r}(t)=\mathbf{i}+t^2\mathbf{j}$

30. $\frac{d\mathbf{r}}{dt}=\int \frac{d^2\mathbf{r}}{dt^2}\,dt=(-2t)\mathbf{i}+(-2t)\mathbf{j}+\mathbf{C}_1$, $\mathbf{r}(t)=\int \frac{d\mathbf{r}}{dt}\,dt=-t^2\mathbf{j}-t^2\mathbf{j}+\mathbf{C}_1t+\mathbf{C}_2$

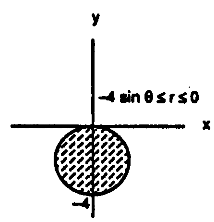
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=1}=-2\mathbf{i}-2\mathbf{j}+\mathbf{C}_1=4\mathbf{i}$, so $\mathbf{C}_1=6\mathbf{i}+2\mathbf{j}$ and $\mathbf{r}(t)=(-t^2+6t)\mathbf{i}+(-t^2+2t)\mathbf{j}+\mathbf{C}_2$

$\mathbf{r}(1)=5\mathbf{i}+\mathbf{j}+\mathbf{C}_2=3\mathbf{i}+3\mathbf{j}$, so $\mathbf{C}_2=-2\mathbf{i}+2\mathbf{j}$, and $\mathbf{r}(t)=(-t^2+6t-2)\mathbf{i}+(-t^2+2t+2)\mathbf{j}$

31.



32.



33. d

34. e

35. l

36. f

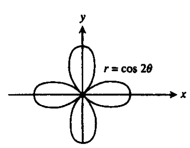
37. k

38. h

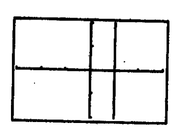
39. i

40. j

41. (a)



42. (a)

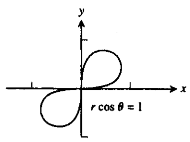


[−3, 3] by [−2, 2]

(b) 2π

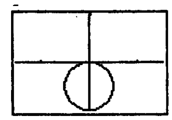
(b) π

43. (a)



(b) $\frac{\pi}{2}$

44. (a)



[−1.5, 1.5] by [−1, 1]

(b) π

45. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta \sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$

$\left(0, \frac{\pi}{4}\right)$, $\left(0, \frac{3\pi}{4}\right)$, $\left(0, \frac{5\pi}{4}\right)$ and $\left(0, \frac{7\pi}{4}\right)$ are polar solutions.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-2/\sqrt{2}}{-2\sqrt{2}} = 1$, $\left.\frac{dy}{dx}\right|_{\theta=3\pi/4} = \frac{2/\sqrt{2}}{-2\sqrt{2}} = -1$, $\left.\frac{dy}{dx}\right|_{\theta=5\pi/4} = \frac{2/\sqrt{2}}{2\sqrt{2}} = 1$, $\left.\frac{dy}{dx}\right|_{\theta=7\pi/4} = \frac{-2/\sqrt{2}}{2\sqrt{2}} = -1$.

The Cartesian equations are $y = \pm x$.

46. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + (1+\cos 2\theta)\cos\theta}{-2\sin 2\theta\cos\theta - (1+\cos 2\theta)\sin\theta} = \frac{-4\sin^2\theta\cos\theta + \cos\theta + 2\cos^3\theta - \cos\theta}{-4\cos^2\theta\sin\theta - \sin\theta - 2\cos^2\theta\sin\theta + \sin\theta}$

$= \frac{-4\sin^2\theta + 2\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{4\sin^2\theta - 2\cos^2\theta}{3\sin 2\theta}$.

$\left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{3\pi}{2}\right)$ are polar solutions.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{4}{0}$ is undefined, so the tangent lines are vertical with equation $x = 0$.

47. $\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\sin\theta\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta - \cos\left(\frac{\theta}{2}\right)\cos\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\cos\theta\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\theta - \sin\theta + \cos\left(\frac{\theta}{2}\right)\sin\theta$

Solve $\frac{dy}{d\theta} = 0$ for θ with a graphing calculator: the solutions are 0, ≈ 2.243, ≈ 4.892, ≈ 7.675, ≈ 10.323, and 4π. Using the middle four solutions to find $y = r\sin\theta$ reveals horizontal tangent lines at $y \approx \pm 0.443$ and $y \approx \pm 1.739$. Solve $\frac{dx}{d\theta} = 0$ for θ with a graphing calculator: the solutions are 0, ≈ 1.070, ≈ 3.531, 2π, ≈ 9.035, ≈ 11.497, and 4π. Using the middle five solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = 2$, $x \approx 0.067$, and $x \approx -1.104$. Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero ($\theta = 0, 4\pi$), close inspection of the plot shows that the tangent lines are horizontal, with equation $y = 0$. (This can be confirmed using L'Hôpital's rule.)

48. $\frac{dy}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\sin\theta] = -4\sin\theta\cos\theta + 2\cos\theta$

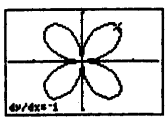
$\frac{dx}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\cos\theta] = -2\cos^2\theta - 2\sin\theta + 2\sin^2\theta = 4\sin^2\theta - 2\sin\theta - 2$

Solve $\frac{dy}{d\theta} = 0$ for θ: the solutions are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Using the first, third, and fourth solutions to find $y = r\sin\theta$ reveals horizontal tangent lines at $y = \frac{1}{2}$ and $y = -4$.

Solve $\frac{dx}{d\theta} = 0$ for θ (by first using the quadratic formula to find $\sin\theta$): the solutions are $\frac{\pi}{2}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Using the last two solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = \pm\frac{3\sqrt{3}}{2} \approx \pm 2.598$. Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero ($\theta = \frac{\pi}{2}$), inspection of the plot shows that the tangent line is vertical, with equation $x = 0$. (This can be confirmed using L'Hôpital's rule.)

49.



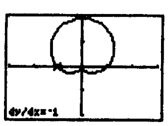
[−1.5, 1.5] by [−1, 1]

The tips have Cartesian coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. From the curve's symmetries, it is evident that the tangent lines at those points have slopes of -1, 1, -1, and 1, respectively. So the equations of the tangent lines are

$y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right)$ or $y = -x + \sqrt{2}$, $y - \frac{1}{\sqrt{2}} = x + \frac{1}{\sqrt{2}}$ or $y = x + \sqrt{2}$,

$y + \frac{1}{\sqrt{2}} = -\left(x + \frac{1}{\sqrt{2}}\right)$ or $y = -x - \sqrt{2}$, and $y + \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$ or $y = x - \sqrt{2}$.

50.



[−3, 3] by [−2, 2]

As the plot shows, the curve crosses the x-axis at (x, y)-coordinates $(-1, 0)$ and $(1, 0)$, with slope -1 and 1, respectively. (This can be confirmed analytically.) So the equations of the tangent lines are

$y - 0 = -(x + 1)$

$y = -x - 1$ and

$y - 0 = x - 1$

$y = x - 1$.

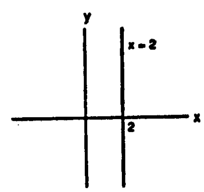
51. $r \cos\theta = r \sin\theta \Rightarrow x = y$, a line

52. $r = 3\cos\theta \Rightarrow r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x \Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4} \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$, a circle (center $= \left(\frac{3}{2}, 0\right)$, radius $= \frac{3}{2}$)

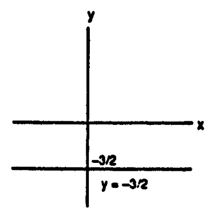
53. $r = 4\tan\theta\sec\theta \Rightarrow r\cos\theta = 4\,\frac{r\sin\theta}{r\cos\theta} \Rightarrow x = 4\frac{y}{x}$ or $x^2 = 4y$, a parabola

54. $r\cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3} \Rightarrow r\cos\theta\cos\left(\frac{\pi}{3}\right) - r\sin\theta\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3} \Rightarrow \frac{1}{2}r\cos\theta - \frac{\sqrt{3}}{2}r\sin\theta = 2\sqrt{3}$
$\Rightarrow \frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2\sqrt{3} \Rightarrow x - \sqrt{3}y = 4\sqrt{3}$ or $y = \frac{x}{\sqrt{3}} - 4$, a line

55. $r = 2\sec\theta \Rightarrow r = \frac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$



56. $r = -\frac{3}{2}\csc\theta \Rightarrow r\sin\theta = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}$



57. $x^2 + y^2 + 5y = 0$
$r^2 + 5r\sin\theta = 0$
$r = -5\sin\theta$

58. $x^2 + y^2 - 2y = 0$
$r^2 - 2r\sin\theta = 0$
$r = 2\sin\theta$

59. $x^2 + 4y^2 = 16$
$(r\cos\theta)^2 + 4(r\sin\theta)^2 = 16$
$r^2\cos^2\theta + 4r^2\sin^2\theta = 16$, or $r^2 = \frac{16}{\cos^2\theta + 4\sin^2\theta}$

60. $(x+2)^2 + (y-5)^2 = 16$
$(r\cos\theta + 2)^2 + (r\sin\theta - 5)^2 = 16$

61. $A = 2\int_0^{\pi} \frac{1}{2}r^2\,d\theta = \int_0^{\pi} (2 - \cos\theta)^2\,d\theta = \int_0^{\pi} \left(4 - 2\cos\theta + \cos^2\theta\right)d\theta = \int_0^{\pi} \left(4 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$

$$= \int_0^{\pi} \left(\frac{9}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \left[\frac{9}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{9}{2}\pi$$

62. $A = \int_0^{\pi/3} \frac{1}{2}\left(\sin^2 3\theta\right) d\theta = \int_0^{\pi/3} \frac{1}{2}\left(\frac{1-\cos 6\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi}{12}$

63. $r = 1 + \cos 2\theta$ and $r = 1 \Rightarrow 1 = 1 + \cos 2\theta \Rightarrow 0 = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$; therefore

$$A = 4\int_0^{\pi/4} \frac{1}{2}\left[(1+\cos 2\theta)^2 - 1^2\right] d\theta = 2\int_0^{\pi/4} \left(1 + 2\cos 2\theta + \cos^2 2\theta - 1\right) d\theta$$

$$= 2\int_0^{\pi/4} \left(2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta = 2\left[\sin 2\theta + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = 2\left(1 + \frac{\pi}{8} + 0\right) = 2 + \frac{\pi}{4}$$

64. The circle lies interior to the cardiod. Thus,

$A = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}[2(1+\sin\theta)]^2\, d\theta - \pi$ (the integral is the area of the cardiod minus the area of the circle)

$$= \int_{-\pi/2}^{\pi/2} 4\left(1 + 2\sin\theta + \sin^2\theta\right) d\theta - \pi = \int_{-\pi/2}^{\pi/2} (6 + 8\sin\theta - 2\cos 2\theta)\, d\theta - \pi = [6\theta - 8\cos\theta - \sin 2\theta]_{-\pi/2}^{\pi/2} - \pi$$

$$= [3\pi - (-3\pi)] - \pi = 5\pi$$

65. $r = -1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; Length $= \int_0^{2\pi} \sqrt{(-1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$$= \int_0^{2\pi} \sqrt{\frac{4(1-\cos\theta)}{2}}\, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = (-4)(-1) - (-4)(1) = 8$$

66. $r = 2\sin\theta + 2\cos\theta,\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = 2\cos\theta - 2\sin\theta$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (2\sin\theta + 2\cos\theta)^2 + (2\cos\theta - 2\sin\theta)^2$

$$= 8\left(\sin^2\theta + \cos^2\theta\right) = 8 \Rightarrow L = \int_0^{\pi/2} \sqrt{8}\, d\theta = \left[2\sqrt{2}\,\theta\right]_0^{\pi/2} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \pi\sqrt{2}$$

67. $r = 8\sin^3\left(\frac{\theta}{3}\right),\ 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = 8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[8\sin^3\left(\frac{\theta}{3}\right)\right]^2 + \left[8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2$

$$= 64\sin^4\left(\frac{\theta}{3}\right) \Rightarrow L = \int_0^{\pi/4} \sqrt{64\sin^4\left(\frac{\theta}{3}\right)}\, d\theta = \int_0^{\pi/4} 8\sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{\pi/4} 8\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right] d\theta$$

$$= \int_0^{\pi/4} \left[4 - 4\cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[4\theta - 6\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = 4\left(\frac{\pi}{4}\right) - 6\sin\left(\frac{\pi}{6}\right) - 0 = \pi - 3$$

68. $r = \sqrt{1+\cos 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1+\cos 2\theta)^{-1/2}(-2\sin 2\theta) = \frac{-\sin 2\theta}{\sqrt{1+\cos 2\theta}} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{1+\cos 2\theta}$

$\Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \cos 2\theta + \frac{\sin^2 2\theta}{1+\cos 2\theta} = \frac{(1+\cos 2\theta)^2 + \sin^2 2\theta}{1+\cos 2\theta} = \frac{1 + 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1+\cos 2\theta}$

$= \frac{2+2\cos 2\theta}{1+\cos 2\theta} = 2 \Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{2}\, d\theta = \sqrt{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2}\,\pi$

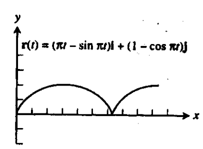
69. x degrees east of north is $(90 - x)$ degrees north of east.
Add the vectors:
$\langle 540 \cos 10^\circ, 540 \sin 10^\circ\rangle + \langle 55 \cos(-10^\circ), 55 \sin(-10^\circ)\rangle = \langle 595 \cos 10^\circ, 485 \sin 10^\circ\rangle \approx \langle 585.961, 84.219\rangle$.
Speed $\approx \sqrt{585.961^2 + 84.219^2} \approx 591.982$ mph.
Direction $\approx \tan^{-1}\left(\frac{585.961}{84.219}\right) \approx 81.821^\circ$ east of north

70. Add the vectors:
$\langle 120 \cos 20^\circ, 120 \sin 20^\circ\rangle + \langle 300 \cos(-5^\circ), 300 \sin(-5^\circ)\rangle \approx \langle 411.6220, 14.896\rangle$.
Direction $\approx \tan^{-1}\left(\frac{14.896}{411.622}\right) \approx 2.073^\circ$
Length $\approx \sqrt{411.622^2 + 14.896^2} \approx 411.891$ lbs

71. Taking the launch point as the origin, $y = (44 \sin 45^\circ)t - 16t^2$ equals -6.5 when $t \approx 2.135$ sec (as can be determined graphically or using the quadratic formula). Then $x \approx (44 \cos 45^\circ)(2.135) \approx 66.421$ horizontal feet from where it left the thrower's hand. Assuming it doesn't bounce or roll, it will still be there 3 seconds after it was thrown.

72. $y_{max} = \frac{(80 \sin 45^\circ)^2}{2(32)} + 7 = 57$ feet

73. (a)



(b) $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle \pi - \pi \cos \pi t, \pi \sin \pi t\rangle$ and $\mathbf{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t\rangle$

$\mathbf{v}(0) = \langle 0, 0\rangle$ $\quad$ $\mathbf{v}(1) = \langle 2\pi, 0\rangle$ $\quad$ $\mathbf{v}(2) = \langle 0, 0\rangle$ $\quad$ $\mathbf{v}(3) = \langle 2\pi, 0\rangle$

$\mathbf{a}(0) = \langle 0, \pi^2\rangle$ $\quad$ $\mathbf{a}(1) = \langle 0, -\pi^2\rangle$ $\quad$ $\mathbf{a}(2) = \langle 0, \pi^2\rangle$ $\quad$ $\mathbf{a}(3) = \langle 0, -\pi^2\rangle$

(c) Topmost point: 2π ft/sec; center of wheel: π ft/sec
Reasons: Since the wheel rolls half a circumference, or π feet every second, the center of the wheel will move π feet every second. Since the rim of the wheel is turning at a rate of π ft/sec about the center, the velocity of the topmost point relative to the center is π ft/sec, giving it a total velocity of 2π ft/sec.

74. $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}}$, where $\alpha = 45°$, $g = 32$, and $R =$ range

for 4325 yds = 12,975 ft; $v_0 \approx 644.360$ ft/sec
for 4752 yds = 14,256 ft; $v_0 \approx 675.420$ ft/sec

75. (a) $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}} = \sqrt{(109.5)(32)} \approx 59.195$ ft/sec

(b) The cork lands at $y = -4$, $x = 177.75$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2 + (\tan\alpha)x$ for v_0, with $\alpha = 45°$; $v_0 = \sqrt{-\frac{gx^2}{y-x}} \approx 74.584$ ft/sec

76. (a) The javelin lands at $y = -6.5$, $x = 262\frac{5}{12}$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2 + (\tan\alpha)x$ for v_0, with $\alpha = 40°$:

$$v_0 = \sqrt{-\frac{gx^2}{(2\cos^2 40°)(y - x\tan 40°)}} \approx 91.008 \text{ ft/sec}$$

(b) $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + 6.5 \approx \frac{(91.008 \sin 40°)^2}{64} + 6.5 \approx 59.970$ ft

77. We have $x = (v_0 t)\cos\alpha$ and $y + \frac{gt^2}{2} = (v_0 t)\sin\alpha$. Squaring and adding gives

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = (v_0 t)^2(\cos^2\alpha + \sin^2\alpha) = v_0^2 t^2.$$

78. (a) $\mathbf{r}(t) = (155\cos 18° - 11.7)t\mathbf{i} + (4 + 155\sin 18°t - 16t^2)\mathbf{j}$
$x(t) = (155\cos 18° - 11.7)t$
$y(t) = 4 + 155\sin 18°t - 16t^2$

(b) $y_{max} = \frac{(155\sin 18°)^2}{2(32)} + 4 \approx 39.847$ feet, reached at $t_{max} = \frac{155\sin 18°}{32} \approx 1.497$ sec

(c) $y(t) = 0$ when $t \approx 3.075$ sec (found using the quadratic formula), and then
$x \approx (155\cos 18° - 11.7)(3.075) \approx 417.307$ ft.

(d) Solve $y(t) = 25$ using the quadratic formula: $t = \frac{-155\sin 18° \pm \sqrt{155^2\sin^2 18° - 4(16)(21)}}{-32}$
≈ 0.534 and 2.460 seconds.

At those times, $x = (155\cos 18° - 11.7)t$ equals ≈ 72.406 and ≈ 333.867 feet from home plate.

(e) Yes, the batter has hit a home run. When the ball is 380 feet from home plate (at $t \approx 2.800$ seconds), it is approximately 12.673 feet off the ground and therefore clears the fence by at least two feet.

79. (a) $\mathbf{r}(t) = \left[(155\cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})\right]\mathbf{i}$

$$+\left[4 + \left(\frac{155\sin 18°}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t}\right]\mathbf{j}$$

$x(t) = (155\cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})$

$y(t) = 4 + \left(\frac{155 \sin 18^\circ}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t})$

(b) Plot $y(t)$ and use the maximum function fo find $y \approx 36.921$ feet at $t \approx 1.404$ seconds.

(c) Plot $y(t)$ and find that $y(t) = 0$ at $t \approx 2.959$ sec, then plug this into the expression for $x(t)$ to find $x(2.959) \approx 352.520$ ft.

(d) Plot $y(t)$ and find that $y(t) = 30$ at $t \approx 0.753$ and 2.068 seconds. At those times, $x \approx 98.799$ and 256.138 feet (from home plate).

(e) No, the batter has not hit a home run. If the drag coefficient k is less than ≈ 0.011, the hit will be a home run. (This result can be found by trying different k-values until the parametrically plotted curve has $y \geq 10$ for $x = 380$.)

80. (a) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

(b) $\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$

(c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, so by part (b), $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$.

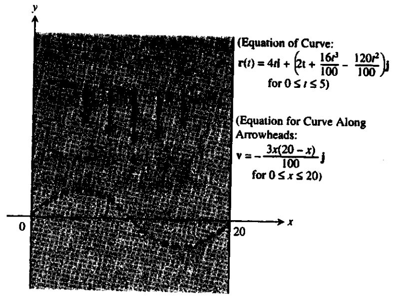
81. The widths between the successive turns are constant and are given by $2\pi a$.

CHAPTER 9 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) Let $a\mathbf{i} + b\mathbf{j}$ be the velocity of the boat. The velocity of the boat relative to an observer on the bank of the river is $\mathbf{v} = a\mathbf{i} + \left[b - \frac{3x(20-x)}{100}\right]\mathbf{j}$. The distance x of the boat as it crosses the river is related to time by $x = at \Rightarrow \mathbf{v} = a\mathbf{i} + \left[b - \frac{3at(20-at)}{100}\right]\mathbf{j} = a\mathbf{i} + \left(b + \frac{3a^2t^2 - 60at}{100}\right)\mathbf{j} \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3}{100} - \frac{30at^2}{100}\right)\mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} \Rightarrow \mathbf{C} = 0 \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j}$. The boat reaches the shore when $x = 20$ $\Rightarrow 20 = at \Rightarrow t = \frac{20}{a}$ and $y = 0 \Rightarrow 0 = b\left(\frac{20}{a}\right) + \frac{a^2\left(\frac{20}{a}\right)^3 - 30a\left(\frac{20}{a}\right)^2}{100} = \frac{20b}{a} + \frac{(20)^3 - 30(20)^2}{100a}$ $= \frac{2000b + 8000 - 12{,}000}{100a} \Rightarrow b = 2$; the speed of the boat is $\sqrt{20} = |\mathbf{v}| = \sqrt{a^2 + b^2} = \sqrt{a^2 + 4} \Rightarrow a^2 = 16$ $\Rightarrow a = 4$; thus, $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$ is the velocity of the boat

(b) $\mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j} = 4t\mathbf{i} + \left(2t + \frac{16t^3}{100} - \frac{120t^2}{100}\right)\mathbf{j}$ by part (a), where $0 \leq t \leq 5$

(c) $x = 4t$ and $y = 2t + \frac{16t^3}{100} - \frac{120t^2}{100}$
$= \frac{4}{25}t^3 - \frac{6}{5}t^2 + 2t = \frac{2}{25}t(2t^2 - 15t + 25)$
$= \frac{2}{25}t(2t-5)(t-5)$, which is the graph of the cubic displayed here



2. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = \frac{1}{2}\frac{d\mathbf{r}}{dt}\cdot\mathbf{r} + \frac{1}{2}\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) \Rightarrow \mathbf{r}\cdot\mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{r}\cdot\mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

3. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$
$\Rightarrow \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j}$
$= (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}||\mathbf{a}|}\right)$
$$= \cos^{-1}\left(\frac{-2e^{2t}\sin t\cos t + 2e^{2t}\sin t\cos t}{\sqrt{(e^t\cos t)^2 + (e^t\sin t)^2}\sqrt{(-2e^t\sin t)^2 + (2e^t\cos t)^2}}\right) = \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2} \text{ for all t}$$

4. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v}\cdot\mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle $x^2 + y^2 = 1$, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have $\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1,0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.

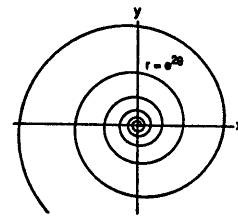
5. $9y = x^3 \Rightarrow 9\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$. Hence $\mathbf{v}\cdot\mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt} = \frac{1}{3}(3)^2(4) = 12$ at $(3,3)$. Also, $\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3}x\right)\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{3}x^2\right)\frac{d^2x}{dt^2}$. Hence $\mathbf{a}\cdot\mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and
$\mathbf{a}\cdot\mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26$ at the point $(x,y) = (3,3)$.

6. The two vectors $|\mathbf{v}|\mathbf{u}$ and $|\mathbf{u}|\mathbf{v}$ have the same magnitude, which is $|\mathbf{u}||\mathbf{v}|$. Therefore, using the result from Exercise 18, Section 9.2, the vector $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ bisects the angle between $|\mathbf{v}|\mathbf{u}$ and $|\mathbf{u}|\mathbf{v}$. The vector $\mathbf{w}$ also bisects the angle between $\mathbf{u}$ and $\mathbf{v}$ because $\mathbf{u}$ is in the same direction as $|\mathbf{v}|\mathbf{u}$ and $\mathbf{v}$ is in the same direction as $|\mathbf{u}|\mathbf{v}$.

7. (a) $x = e^{2t}\cos t$ and $y = e^{2t}\sin t \Rightarrow x^2 + y^2 = e^{4t}\cos^2 t + e^{4t}\sin^2 t = e^{4t}$. Also $\frac{y}{x} = \frac{e^{2t}\sin t}{e^{2t}\cos t} = \tan t$

$\Rightarrow t = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow x^2 + y^2 = e^{4\tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2 = x^2 + y^2$ and

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the polar equation is $r^2 = e^{4\theta}$ or $r = e^{2\theta}$ for $r > 0$



(b) $ds^2 = r^2\,d\theta^2 + dr^2$; $r = e^{2\theta} \Rightarrow dr = 2e^{2\theta}\,d\theta$

$\Rightarrow ds^2 = r^2\,d\theta^2 + \left(2e^{2\theta}\,d\theta\right)^2 = \left(e^{2\theta}\right)^2 d\theta^2 + 4e^{4\theta}\,d\theta^2$

$= 5e^{4\theta}\,d\theta^2 \Rightarrow ds = \sqrt{5}\,e^{2\theta}\,d\theta \Rightarrow L = \int_0^{2\pi} \sqrt{5}\,e^{2\theta}\,d\theta$

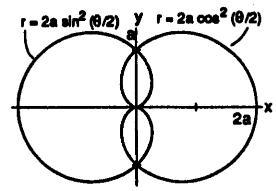
$= \left[\frac{\sqrt{5}\,e^{2\theta}}{2}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$

8. $r = 2\sin^3\left(\frac{\theta}{3}\right) \Rightarrow dr = 2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta \Rightarrow ds^2 = r^2\,d\theta^2 + dr^2 = \left[2\sin^3\left(\frac{\theta}{3}\right)\right]^2 d\theta^2 + \left[2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)d\theta\right]^2$

$= 4\sin^6\left(\frac{\theta}{3}\right)d\theta^2 + 4\sin^4\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right)d\theta^2 = \left[4\sin^4\left(\frac{\theta}{3}\right)\right]\left[\sin^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)\right]d\theta^2 = 4\sin^4\left(\frac{\theta}{3}\right)d\theta^2$

$\Rightarrow ds = 2\sin^2\left(\frac{\theta}{3}\right)d\theta$. Then $L = \int_0^{3\pi} 2\sin^2\left(\frac{\theta}{3}\right)d\theta = \int_0^{3\pi}\left[1 - \cos\left(\frac{2\theta}{3}\right)\right]d\theta = \left[\theta - \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi} = 3\pi$

9. The region in question is the figure eight in the middle. The arc of $r = 2a\sin^2\left(\frac{\theta}{2}\right)$ in the first quadrant gives $\frac{1}{4}$ of that region. Therefore the area is $A = 4\int_0^{\pi/2} \frac{1}{2}r^2\,d\theta$



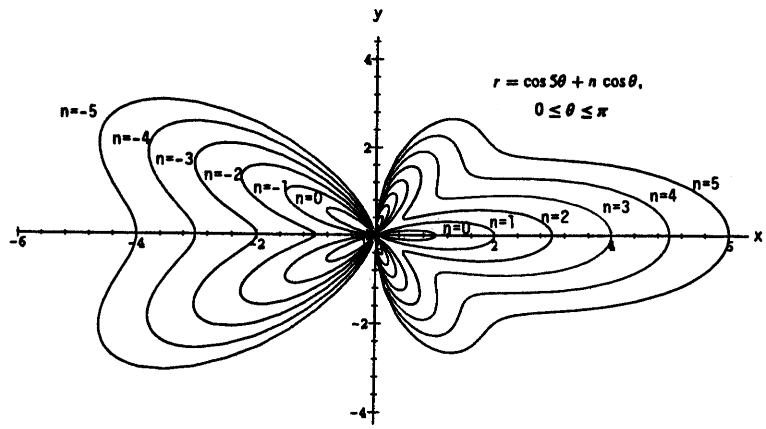
$= 4\int_0^{\pi/2} \frac{1}{2}\left[2a\sin^2\left(\frac{\theta}{2}\right)\right]^2 d\theta = 8a^2\int_0^{\pi/2}\sin^4\left(\frac{\theta}{2}\right)d\theta$

$= 8a^2\int_0^{\pi/2}\sin^2\left(\frac{\theta}{2}\right)\left[1 - \cos^2\left(\frac{\theta}{2}\right)\right]d\theta = 8a^2\int_0^{\pi/2}\left[\sin^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right]d\theta = 8a^2\int_0^{\pi/2}\left(\frac{1-\cos\theta}{2} - \frac{\sin^2\theta}{4}\right)d\theta$

$= 2a^2\int_0^{\pi/2}\left(2 - 2\cos\theta - \frac{1-\cos 2\theta}{2}\right)d\theta = a^2\int_0^{\pi/2}(3 - 4\cos\theta + \cos 2\theta)\,d\theta = a^2\left[3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= a^2\left(\frac{3\pi}{2} - 4\right)$

10.



$r = \cos 5\theta + n \cos\theta$,
$0 \le \theta \le \pi$
n=-5
n=-4
n=-3
n=-2
n=-1
n=0
n=0
n=1
n=2
n=3
n=4
n=5
x
y

NOTES: