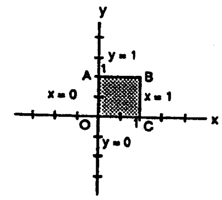
24. (i) On OA, $f(x,y) = f(0,y) = -24y^2$ on $0 \le y \le 1$; $f'(0,y) = -48y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0,0)$ is not an interior point of OA; $f(0,0) = 0$ and $f(0,1) = -24$



(ii) On AB, $f(x,y) = f(x,1) = 48x - 32x^3 - 24$ on $0 \le x \le 1$; $f'(x,1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ and $y = 1$, or $x = -\frac{1}{\sqrt{2}}$ and $y = 1$, but $\left(-\frac{1}{\sqrt{2}}, 1\right)$ is not in the interior of AB; $f\left(\frac{1}{\sqrt{2}}, 1\right) = 16\sqrt{2} - 24$ and $f(1,1) = -8$

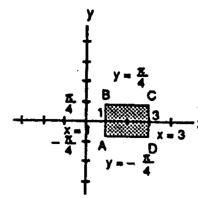
(iii) On BC, $f(x,y) = f(1,y) = 48y - 32 - 24y^2$ on $0 \le y \le 1$; $f'(1,y) = 48 - 48y = 0 \Rightarrow y = 1$ and $x = 1$, but $(1,1)$ is not an interior point of BC; $f(1,0) = -32$ and $f(1,1) = -8$

(iv) On OC, $f(x,y) = f(x,0) = -32x^3$ on $0 \le x \le 1$; $f'(x,0) = -96x^2 = 0 \Rightarrow x = 0$ and $y = 0$, but $(0,0)$ is not an interior point of OC; $f(0,0) = 0$ and $f(1,0) = -32$

(v) For interior points of the rectangular region, $f_x(x,y) = 48y - 96x^2 = 0$ and $f_y(x,y) = 48x - 48y = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{2}$, but $(0,0)$ is not an interior point of the region; $f\left(\frac{1}{2}, \frac{1}{2}\right) = 2$.

Therefore the absolute maximum is 2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the absolute minimum is -32 at $(1,0)$.

25. (i) On AB, $f(x,y) = f(1,y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(1,y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1,0) = 3$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$



(ii) On CD, $f(x,y) = f(3,y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(3,y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3,0) = 3$, $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$ and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iii) On BC, $f(x,y) = f\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, \frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f\left(2, \frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

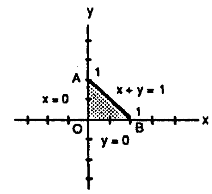
(iv) On AD, $f(x,y) = f\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, -\frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f\left(2, -\frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(v) For interior points of the region, $f_x(x,y) = (4 - 2x)\cos y = 0$ and $f_y(x,y) = -(4x - x^2)\sin y = 0 \Rightarrow x = 2$ and $y = 0$, which is an interior critical point with $f(2,0) = 4$. Therefore the absolute maximum is 4 at $(2,0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(3, -\frac{\pi}{4}\right)$, $\left(3, \frac{\pi}{4}\right)$, $\left(1, -\frac{\pi}{4}\right)$, and $\left(1, \frac{\pi}{4}\right)$.

26. (i) On OA, $f(x,y) = f(0,y) = 2y + 1$ on $0 \le y \le 1$;
$f'(0,y) = 2 \Rightarrow$ no interior critical points; $f(0,0) = 1$ and $f(0,1) = 3$

(ii) On OB, $f(x,y) = f(x,0) = 4x + 1$ on $0 \le x \le 1$;
$f'(x,0) = 4 \Rightarrow$ no interior critical points; $f(1,0) = 5$

(iii) On AB, $f(x,y) = f(x,-x+1) = 8x^2 - 6x + 3$ on $0 \le x \le 1$;
$f'(x,-x+1) = 16x - 6 = 0 \Rightarrow x = \frac{3}{8}$ and $y = \frac{5}{8}$;
$f\left(\frac{3}{8},\frac{5}{8}\right) = \frac{15}{8}$, $f(0,1) = 3$, and $f(1,0) = 5$



(iv) For interior points of the triangular region,
$f_x(x,y) = 4 - 8y = 0$ and $f_y(x,y) = -8x + 2 = 0$
$\Rightarrow y = \frac{1}{2}$ and $x = \frac{1}{4}$ which is an interior critical point with $f\left(\frac{1}{4},\frac{1}{2}\right) = 2$. Therefore the absolute maximum is 5 at $(1,0)$ and the absolute minimum is 1 at $(0,0)$.

27. Let $F(a,b) = \int_a^b (6 - x - x^2)\,dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ in the ab-plane, and $F(a,a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have: $\frac{\partial F}{\partial a} = -(6 - a - a^2) = 0 \Rightarrow a = -3, 2$ and $\frac{\partial F}{\partial b} = (6 - b - b^2) = 0 \Rightarrow b = -3, 2$. Since $a \le b$, there is only one interior critical point $(-3,2)$ and $F(-3,2) = \int_{-3}^{2} (6 - x - x^2)\,dx$ gives the area under the parabola $y = 6 - x - x^2$ that is above the x-axis. Therefore, $a = -3$ and $b = 2$.

28. Let $F(a,b) = \int_a^b (24 - 2x - x^2)^{1/3}\,dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ and on this line F is identically 0. For interior critical points we have: $\frac{\partial F}{\partial a} = -(24 - 2a - a^2)^{1/3} = 0 \Rightarrow a = -4, 6$ and $\frac{\partial F}{\partial b} = (24 - 2b - b^2)^{1/3} = 0 \Rightarrow b = -4, 6$. Since $a \le b$, there is only one critical point $(-4,6)$ and $F(-4,6) = \int_{-4}^{6} (24 - 2x - x^2)\,dx$ gives the area under the curve $y = (24 - 2x - x^2)^{1/3}$ that is above the x-axis. Therefore, $a = -4$ and $b = 6$.

29. $T_x(x,y) = 2x - 1 = 0$ and $T_y(x,y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T\left(\frac{1}{2},0\right) = -\frac{1}{4}$; on the boundary $x^2 + y^2 = 1$: $T(x,y) = -x^2 - x + 2$ for $-1 \le x \le 1 \Rightarrow T'(x,y) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$;
$T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T(-1,0) = 2$, and $T(1,0) = 0 \Rightarrow$ the hottest is $2\frac{1}{4}°$ at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$; the coldest is $-\frac{1}{4}°$ at $\left(\frac{1}{2},0\right)$.

30. $f_x(x,y) = y + 2 - \frac{2}{x} = 0$ and $f_y(x,y) = x - \frac{1}{y} = 0 \Rightarrow x = \frac{1}{2}$ and $y = 2$; $f_{xx}\left(\frac{1}{2}, 2\right) = \left.\frac{2}{x^2}\right|_{\left(\frac{1}{2},2\right)} = 8$, $f_{yy}\left(\frac{1}{2}, 2\right) = \left.\frac{1}{y^2}\right|_{\left(\frac{1}{2},2\right)} = \frac{1}{4}$, $f_{xy}\left(\frac{1}{2}, 2\right) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 1 > 0$ and $f_{xx} > 0 \Rightarrow$ a local minimum of $f\left(\frac{1}{2}, 2\right) = 2 - \ln \frac{1}{2} = 2 + \ln 2$

31. (a) $f_x(x,y) = 2x - 4y = 0$ and $f_y(x,y) = 2y - 4x = 0 \Rightarrow x = 0$ and $y = 0$; $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 2$, $f_{xy}(0,0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point at $(0,0)$
    (b) $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$; $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = 2$, $f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1,2)$
    (c) $f_x(x,y) = 9x^2 - 9 = 0$ and $f_y(x,y) = 2y + 4 = 0 \Rightarrow x = \pm 1$ and $y = -2$; $f_{xx}(1,-2) = 18x\big|_{(1,-2)} = 18$, $f_{yy}(1,-2) = 2$, $f_{xy}(1,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1,-2)$; $f_{xx}(-1,-2) = -18$, $f_{yy}(-1,-2) = 2$, $f_{xy}(-1,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point at $(-1,-2)$

32. (a) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)
    (b) Maximum of 1 at $(0,0)$ since $f(x,y) < 1$ for all other (x,y)
    (c) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
    (d) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
    (e) Neither since $f(x,y) < 0$ for $x < 0$ and $y > 0$, but $f(x,y) > 0$ for $x > 0$ and $y > 0$
    (f) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)

33. If $k = 0$, then $f(x,y) = x^2 + y^2 \Rightarrow f_x(x,y) = 2x = 0$ and $f_y(x,y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0,0)$ is the only critical point. If $k \neq 0$, $f_x(x,y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x,y) = kx + 2y = 0 \Rightarrow kx + 2\left(-\frac{2}{k}x\right) = 0 \Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow \left(k - \frac{4}{k}\right)x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = \left(-\frac{2}{k}\right)(0) = 0$ or $y = -x$; in any case $(0,0)$ is a critical point.

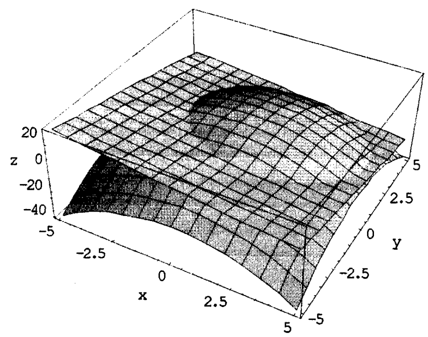
34. (See Exercise 33 above): $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, and $f_{xy}(x,y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0,0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0,0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.

35. (a) No; for example $f(x,y) = xy$ has a saddle point at $(a,b) = (0,0)$ where $f_x = f_y = 0$.
    (b) If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, then $f_{xx}(a,b)\, f_{yy}(a,b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a,b) by the second derivative test.

36. Suppose that f has a local minimum value at an interior point (a,b) of its domain. Then
    (i) $x = a$ is an interior point of the domain of the curve $z = f(x,b)$ in which the plane $y = b$ cuts the surface $z = f(x,y)$.
    (ii) The function $z = f(x,b)$ is a differentiable function of x at $x = a$ (the derivative is $f_x(a,b)$).
    (iii) The function $z = f(x,b)$ has a local minimum value at $x = a$.
    (iv) The value of the derivative of $z = f(x,b)$ at $x = a$ is therefore zero (Theorem 2, Section 3.1). Since this derivative is $f_x(a,b)$, we conclude that $f_x(a,b) = 0$.

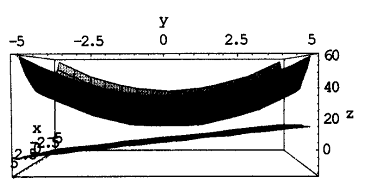
    A similar argument with the function $z = f(a,y)$ shows that $f_y(a,b) = 0$.

37.



As shown in the accompanying figure, the surface is a circular paraboloid that opens downward. Let (x, y, z) be points on the surface such that $z = 10 - x^2 - y^2$, and let (a, b, c) be points on the plane such that $z = -2b - 3c$. We want to maximize $d = (x-a)^2 + (y-b)^2 + (z-c)^2$. Substituting for z and a gives $d = (x + 2b + 3c)^2 + (y-b)^2 + (10 - x^2 - y^2 - c)^2$. Now we set the first partials equal to zero: $d_x = 2(x + 2b + 3c) - 4x(10 - x^2 - y^2 - c) = 0$, $d_y = 2(y-b) - 4y(10 - x^2 - y^2 - c) = 0$, $d_b = 4(x + 2b + 3c) - 2(y - b) = 0$, $d_c = 6(x + 2b + 3c) - 2(10 - x^2 - y^2 - c) = 0$. Solve the last two equations for $(y - b)$ and for $(10 - x^2 - y^2 - c)$ in terms of $(x + 2b + 3c)$ and substitute into the first two equations: $(y - b) = 2(x + 2b + 3c)$ and $(10 - x^2 - y^2 - c) = 3(x + 2b + 3c) \Rightarrow (2 - 12x)(x + 2b + 3c) = 0$ and $(4 - 12y)(x + 2b + 3c) = 0$. Therefore the critical values must occur where $x = -2b - 3c$, or where $x = 1/6$ and $y = 1/3$. For points on the paraboloid above the plane, the maximum value of d will occur at a single point, as shown in the accompanying figure. Therefore, the z-coordinate of the point on the surface above the plane that is farthest from the plane is $z = 10 - (1/6)^2 - (1/3)^2 = 355/36$, and the coordinates of the farthest point are $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$. Not that is $x = -2b - 3c = a$, then $y = b$ and $z = c$ giving the points where the surface and the plane intersect. These critical points are where d is at a minimum value of zero.

38.



As shown in the accompanying figure, the surface is a circular paraboloid that opens upward. Let (x, y, z) be points on the surface such that $z = x^2 + y^2 + 10$, and let (a, b, c) be points on the plane such that $c = a + 2b$. We want to minimize $d = (x-a)^2 + (y-b)^2 + (z-c)^2$. Substituting for z and c gives $d = (x-a)^2 + (y-b)^2 + (x^2 + y^2 + 10 - a - 2b)^2$. Now we set the first partials equal to zero: $d_x = 2(x-a) + 4x(x^2 + y^2 + 10 - a - 2b) = 0$, $d_y = 2(y-b) + 4y(x^2 + y^2 + 10 - a - 2b) = 0$,

$d_a = -2(x-a) - 2(x^2+y^2+10-a-2b) = 0$, $d_b = -2(y-b) - 4(x^2+y^2+10-a-2b) = 0$. Solve the last two equations for $(x-a)$ and $(y-b)$ in terms of $(x^2+y^2+10-a-2b)$ and substitute into the first two equations: $(x-a) = -(x^2+y^2+10-a-2b)$ and $(y-b) = -2(x^2+y^2+10-a-2b)$ $\Rightarrow (-2+4x)(x^2+y^2+10-a-2b) = 0$ and $(4-4y)(x^2+y^2+10-a-2b) = 0$. Therefore, the critical values occur where $x^2+y^2+10 = a+2b$, that is, where $z = c$, or where $x = 1/2$ and $y = 1$. The graph in the accompanying figure shows that the paraboloid and the plane do not intersect, therefore, there are no points where $z = c$, and the only critical points must have $x = 1/2$, $y = 1$, and $z = (1/2)^2 + 1^2 + 10 = 45/4$. The point on the surface nearest the plane is $\left(\frac{1}{4}, 1, \frac{45}{4}\right)$.

39. No, because the domain $x \ge 0$ and $y \ge 0$ is unbounded since x and y can be as large as we please. Absolute extrema are guaranteed for continuous functions defined over closed and bounded domains in the plane. Since the domain is unbounded, the continuous function $f(x,y) = x+y$ need not have an absolute maximum (although, in this case, it does have an absolute minimum value of $f(0,0) = 0$).

40. (a) (i) On $x = 0$, $f(x,y) = f(0,y) = y^2 - y + 1$ for $0 \le y \le 1$; $f'(0,y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ and $x = 0$; $f\left(0, \frac{1}{2}\right) = \frac{3}{4}$, $f(0,0) = 1$, and $f(0,1) = 1$

(ii) On $y = 1$, $f(x,y) = f(x,1) = x^2 + x + 1$ for $0 \le x \le 1$; $f'(x,1) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = 1$, but $\left(-\frac{1}{2}, 1\right)$ is outside the domain; $f(0,1) = 1$ and $f(1,1) = 3$

(iii) On $x = 1$, $f(x,y) = f(1,y) = y^2 + y + 1$ for $0 \le y \le 1$; $f'(1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ and $x = 1$, but $\left(1, -\frac{1}{2}\right)$ is outside the domain; $f(1,0) = 1$ and $f(1,1) = 3$

(iv) On $y = 0$, $f(x,y) = f(x,0) = x^2 - x + 1$ for $0 \le x \le 1$; $f'(x,0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$; $f\left(\frac{1}{2}, 0\right) = \frac{3}{4}$; $f(0,0) = 1$, and $f(1,0) = 1$

(v) On the interior of the square, $f_x(x,y) = 2x + 2y - 1 = 0$ and $f_y(x,y) = 2y + 2x - 1 = 0 \Rightarrow 2x + 2y = 1$ $\Rightarrow (x+y) = \frac{1}{2}$. Then $f(x,y) = x^2 + y^2 + 2xy - x - y + 1 = (x+y)^2 - (x+y) + 1 = \frac{3}{4}$ is the absolute minimum value when $2x + 2y = 1$.

(b) The absolute maximum is $f(1,1) = 3$ and the absolute minimum is $\frac{3}{4}$ along the line $x + y = \frac{1}{2}$ in the square $0 \le x \le 1$ and $0 \le y \le 1$, as found in part (a).

41. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -2\sin t + 2\cos t = 0 \Rightarrow \cos t = \sin t \Rightarrow x = y$

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$, we have $t = \frac{\pi}{4}$ and $x = y = \sqrt{2} \Rightarrow f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$. At the endpoints, $f(-2,0) = -2$ and $f(2,0) = 2$. Therefore the absolute minimum is $f(-2,0) = -2$ when $t = \pi$; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $f(0,2) = 2$ and $f(2,0) = 2$. Therefore the absolute minimum is $f(2,0) = 2$ and $f(0,2) = 2$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = -4\sin^2 t + 4\cos^2 t = 0 \Rightarrow \cos t = \pm\sin t \Rightarrow x = \pm y$.

(i) On the semicircle $x^2+y^2=4$, $y \ge 0$, we obtain $x=y=\sqrt{2}$ at $t=\frac{\pi}{4}$ and $x=-\sqrt{2}$, $y=\sqrt{2}$ at $t=\frac{3\pi}{4}$. Then $g(\sqrt{2},\sqrt{2})=2$ and $g(-\sqrt{2},\sqrt{2})=-2$. At the endpoints, $g(-2,0)=g(2,0)=0$. Therefore the absolute minimum is $g(-\sqrt{2},\sqrt{2})=-2$ when $t=\frac{3\pi}{4}$; the absolute maximum is $g(\sqrt{2},\sqrt{2})=2$ when $t=\frac{\pi}{4}$.

(ii) On the quartercircle $x^2+y^2=4$, $x \ge 0$ and $y \ge 0$, the endpoints give $g(0,2)=0$ and $g(2,0)=0$. Therefore the absolute minimum is $g(2,0)=0$ and $g(0,2)=0$ when $t=0, \frac{\pi}{2}$ respectively; the absolute maximum is $g(\sqrt{2},\sqrt{2})=2$ when $t=\frac{\pi}{4}$.

(c) $\frac{dh}{dt}=\frac{\partial h}{\partial x}\frac{dx}{dt}+\frac{\partial h}{\partial y}\frac{dy}{dt}=4x\frac{dx}{dt}+2y\frac{dy}{dt}=(8\cos t)(-2\sin t)+(4\sin t)(2\cos t)=-8\cos t\sin t=0$ $\Rightarrow t=0, \frac{\pi}{2}, \pi$ yielding the points $(2,0)$, $(0,2)$, and $(-2,0)$, respectively.

(i) On the semicircle $x^2+y^2=4$, $y \ge 0$ we have $h(2,0)=8$, $h(0,2)=4$, and $h(-2,0)=8$. Therefore, the absolute minimum is $h(0,2)=4$ when $t=\frac{\pi}{2}$; the absolute maximum is $h(2,0)=8$ and $h(-2,0)=8$ when $t=0, \pi$ respectively.

(ii) On the quartercircle $x^2+y^2=4$, $x \ge 0$ and $y \ge 0$ the absolute minimum is $h(0,2)=4$ when $t=\frac{\pi}{2}$; the absolute maximum is $h(2,0)=8$ when $t=0$.

42. (a) $\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=2\frac{dx}{dt}+3\frac{dy}{dt}=-6\sin t+6\cos t=0 \Rightarrow \sin t=\cos t \Rightarrow t=\frac{\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $\frac{x^2}{9}+\frac{y^2}{4}=1$, $y \ge 0$, $f(x,y)=2x+3y=6\cos t+6\sin t=6\left(\frac{\sqrt{2}}{2}\right)+6\left(\frac{\sqrt{2}}{2}\right)=6\sqrt{2}$ at $t=\frac{\pi}{4}$. At the endpoints, $f(-3,0)=-6$ and $f(3,0)=6$. The absolute minimum is $f(-3,0)=-6$ when $t=\pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)=6\sqrt{2}$ when $t=\frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $f(0,2)=6$ and $f(3,0)=6$. The absolute minimum is $f(3,0)=6$ and $f(0,2)=6$ when $t=0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)=6\sqrt{2}$ when $t=\frac{\pi}{4}$.

(b) $\frac{dg}{dt}=\frac{\partial g}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}=y\frac{dx}{dt}+x\frac{dy}{dt}=(2\sin t)(-3\sin t)+(3\cos t)(2\cos t)=6(\cos^2 t-\sin^2 t)=6\cos 2t=0$ $\Rightarrow t=\frac{\pi}{4}, \frac{3\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $g(x,y)=xy=6\sin t\cos t$. Then $g\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)=3$ when $t=\frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2},\sqrt{2}\right)=-3$ when $t=\frac{3\pi}{4}$. At the endpoints, $g(-3,0)=g(3,0)=0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2},\sqrt{2}\right)=-3$ when $t=\frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)=3$ when $t=\frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $g(0,2)=0$ and $g(3,0)=0$. The absolute minimum is $g(3,0)=0$ and $g(0,2)=0$ at $t=0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)=3$ when $t=\frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 6y\frac{dy}{dt} = (6 \cos t)(-3 \sin t) + (12 \sin t)(2 \cos t) = 6 \sin t \cos t = 0$

$\Rightarrow t = 0, \frac{\pi}{2}, \pi$ for $0 \le t \le \pi$, yielding the points $(3,0)$, $(0,2)$, and $(-3,0)$.

(i) On the semi-ellipse, $y \ge 0$ so that $h(3,0) = 9$, $h(0,2) = 12$, and $h(-3,0) = 9$. The absolute minimum is $h(3,0) = 9$ and $h(-3,0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

(ii) On the quarter ellipse, the absolute minimum is $h(3,0) = 9$ when $t = 0$; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

43. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

(i) $x = 2t$ and $y = t + 1 \Rightarrow \frac{df}{dt} = (t+1)(2) + (2t)(1) = 4t + 2 = 0 \Rightarrow t = -\frac{1}{2} \Rightarrow x = -1$ and $y = \frac{1}{2}$ with $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; there is no absolute maximum.

(ii) For the endpoints: $t = -1 \Rightarrow x = -2$ and $y = 0$ with $f(-2,0) = 0$; $t = 0 \Rightarrow x = 0$ and $y = 1$ with $f(0,1) = 0$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; the absolute maximum is $f(0,1) = 0$ and $f(-2,0) = 0$ when $t = -1, 0$ respectively.

(iii) There are no interior critical points. for the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 1$ with $f(0,1) = 0$; $t = 1 \Rightarrow x = 2$ and $y = 2$ with $f(2,2) = 4$. The absolute minimum is $f(0,1) = 0$ when $t = 0$; the absolute maximum is $f(2,2) = 4$ when $t = 1$.

44. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow \frac{df}{dt} = (2t)(1) + 2(2-2t)(-2) = 10t - 8 = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $f(0,2) = 4$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $f(1,0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0,2) = 4$ at the endpoint when $t = 0$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \left[\frac{-2x}{(x^2+y^2)^2}\right]\frac{dx}{dt} + \left[\frac{-2y}{(x^2+y^2)^2}\right]\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow x^2 + y^2 = 5t^2 - 8t + 4 \Rightarrow \frac{dg}{dt} = -\left(5t^2 - 8t + 4\right)^{-2}[(-2t)(1) + (-2)(2-2t)(-2)]$ $= -\left(5t^2 - 8t + 4\right)^{-2}(-10t + 8) = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $g(0,2) = \frac{1}{4}$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $g(1,0) = 1$. The absolute minimum is $g(0,2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

45. $w = \sum_{i=1}^{n} (mx_i + b - y_i)^2 \Rightarrow \frac{\partial w}{\partial m} = 2\sum_{i=1}^{n} x_i(mx_i + b - y_i) = 2m\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i = 0$

$\Rightarrow m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$ and $\frac{\partial w}{\partial b} = 2\sum_{i=1}^{n} (mx_i + b - y_i) = 2m\sum_{i=1}^{n} x_i + 2b\sum_{i=1}^{n} 1 - 2\sum_{i=1}^{n} y_i = 0$

$\Rightarrow m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$. These are two linear equations in the unknown variables m and b. Letting $a_{11} = \sum_{i=1}^{n} x_i^2$, $a_{12} = \sum_{i=1}^{n} x_i$, $b_1 = \sum_{i=1}^{n} x_i y_i$, $a_{21} = \sum_{i=1}^{n} x_i$, $a_{22} = n$, and $b_2 = \sum_{i=1}^{n} y_i$, the system of equations can be written as $a_{11}m + a_{12}b = b_1$ and $a_{21}m + a_{22}b = b_2$. Now solve the second equation for b in terms of m and substitute into the second equation and then solve for m.

$b = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}m \Rightarrow a_{11}m + a_{12}\left(\frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}m\right) = b_1 \Rightarrow \left(a_{11} - \frac{a_{12}a_{21}}{a_{22}}\right)m = b_1 - \frac{a_{12}b_2}{a_{22}}$

$$\Rightarrow \left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}}\right)m = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}} \Rightarrow m = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$\Rightarrow m = \dfrac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - n\left(\sum_{i=1}^{n} x_i y_i\right)}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\left(\sum_{i=1}^{n} x_i^2\right)}$, and from the first equation, $b = \frac{1}{a_{22}}(b_2 - ma_{21})$

$\Rightarrow b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)$.

Now apply the Second Derivative Test to show that the critical point is a minimum value of w.

$\frac{\partial^2 w}{\partial m^2} = 2\sum_{i=1}^{n} x_i^2$, $\frac{\partial^2 w}{\partial b^2} = 2n$, $\frac{\partial^2 w}{\partial m\,\partial b} = 2\sum_{i=1}^{n} x_i \Rightarrow \left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2$

$= \left(2\sum_{i=1}^{n} x_i^2\right)(2n) - \left(2\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]$. The second partial derivative, $\frac{\partial^2 w}{\partial m^2} = 2\sum_{i=1}^{n} x_i^2$, is greater than zero provided there is at least one $x \neq 0$. Furthermore, $n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 > 0$ provided not all of the x_i's are equal. This can be proved by mathematical induction as follows.

First, show that the statement is true for n = 3 (the minimum number of points for a least squares fit).

$$3\sum_{i=1}^{3} x_i^2 - \left(\sum_{i=1}^{n} x_i\right) = 3\left(x_1^2 + x_2^2 + x_3^2\right) - \left(x_1 + x_2 - x_3\right)^2 = 3\left(x_1^2 + x_2^2 + x_3^2\right)$$

$$- \left(x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\right)$$

$$= 2x_1^2 + 2x_2^2 - 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$

$$= \left(x_1^2 - 2x_1x_2 + x_2^2\right) + \left(x_1^2 - 2x_1x_2 + x_3^2\right) - \left(x_2^2 - 2x_2x_3 + x_3^2\right)$$

$$= \left(x_1 - x_2\right)^2 + \left(x_1 - x_3\right)^2 + \left(x_2 - x_3\right)^2$$

$$> 0$$

Assume true for $n = k \Rightarrow k\sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2 > 0$, and show that the relation is true for $n = k + 1$.

$$(k+1)\sum_{i=1}^{k+1} x_i^2 - \left(\sum_{i=1}^{k+1} x_i\right) = (k+1)\left(\sum_{i=1}^{k} x_i^2 + x_{i+1}^2\right) - \left(\sum_{i=1}^{k} x_i + x_{i+1}\right)^2$$

$$= k\sum_{i=1}^{k} x_i^2 + kx_{i+1}^2 + \sum_{i=1}^{k} x_i^2 + x_{i+1}^2 - \left(\left(\sum_{i=1}^{k} x_i\right)^2 + 2x_{i+1}\sum_{i=1}^{k} x_i + x_{i+1}^2\right)$$

$$= \left(k\sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2\right) + \left(kx_{i+1}^2 + \sum_{i=1}^{k} x_i^2 - 2x_{i+1}\sum_{i=1}^{k} x_i\right)$$

$$= \left(k\sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2\right) + \left(x_i^2 - 2x_ix_{k+1} + x_{k+1}^2\right) + \left(x_2^2 - 2x_2x_{k+1} + x_{k+1}^2\right) + \ldots + \left(x_k^2 - 2x_ix_{k+1} + x_{k+1}^2\right)$$

$$= \left(k\sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2\right) + \left(\left(x_1 - x_{k+1}\right)^2 + \left(x_2 - x_{k+1}\right)^2 + \ldots + \left(x_k - x_{k+1}\right)^2\right)$$

$$= \left(k\sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2\right) + \sum_{i=1}^{n} \left(x_i - x_{k+1}\right)^2$$

The first expression in parentheses is positive by our assumption, and the last sum is positive, provided there is at least one $x \neq x_{k+1}$ for $i = 1, 2, \ldots, k$, because it is a sum of squares with at least one term not equal to zero. Therefore, by mathematical induction,

$n \sum_{i=1}^{k} x_i^2 - \left(\sum_{i=1}^{k} x_i\right)^2 > 0$, for any integer n, such that $n \geq 3$.

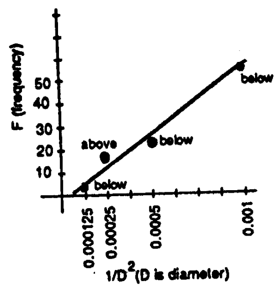
Finally, this proves that the critical point for the least squares problem is a minimum because $\frac{\partial^2 w}{\partial m^2} > 0$ and $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\, \partial b}\right)^2 > 0$.

46. $m = \frac{(0.001863)(91) - 4(0.065852)}{(0.001863)^2 - 4(0.000001323)} \approx 51{,}545$

and $b = \frac{1}{4}(91 - 51{,}545(0.001863)) \approx -1.26$

$\Rightarrow F = 51{,}545 \frac{1}{D^2} - 1.26$

k	$\left(\frac{1}{D^2}\right)_k$	F_k	$\left(\frac{1}{D^2}\right)_k^2$	$\left(\frac{1}{D^2}\right)_k F_k$
1	0.001	51	0.000001	0.051
2	0.0005	22	0.00000025	0.011
3	0.00024	14	0.0000000576	0.00336
4	0.000123	4	0.0000000153	0.000492
Σ	0.001863	91	0.000001323	0.065852



47-52. Example CAS commands:

<u>Maple</u>:

```
with(plots):
f:= (x,y) -> 2*x^4 + y^4 - 2*x^2 - 2*y^2 + 3;
plot3d(f(x,y), x=-1..1, y=-1..1, axes=BOXED);
contourplot(f(x,y), x=-3/2..3/2, y=-3/2..3/2, axes=NORMAL);
exp1:= diff(f(x,y),x) = 0;
exp2:= diff(f(x,y),y) = 0;
critical:= evalf(solve({exp1,exp2}, {x,y}));
diff(diff(f(x,y),x),x): fxx:= unapply(%,(x,y));
diff(diff(f(x,y),x),y): fxy:= unapply(%,(x,y));
diff(diff(f(x,y),y),y): fyy:= unapply(%,(x,y));
fxx(x,y)*fyy(x,y) - (fxy(x,y))^2: disc:= unapply(%,(x,y));
subs(critical[1],[fxx(x,y),disc(x,y)]);
subs(critical[2],[fxx(x,y),disc(x,y)]);
subs(critical[3],[fxx(x,y),disc(x,y)]);
subs(critical[4],[fxx(x,y),disc(x,y)]);
subs(critical[5],[fxx(x,y),disc(x,y)]);
subs(critical[6],[fxx(x,y),disc(x,y)]);
```

Mathematica:

```
Clear[x,y]
SetOptions[ContourPlot, PlotPoints -> 25,
 Contours -> 20, ContourShading -> False];
f[x_,y_] = 2 x^4 + y^4 - 2 x^2 - 2 y^2 + 3
{xa,xb} = {-3/2,3/2};
{ya,yb} = {-3/2,3/2};
Plot3D[ f[x,y], {x,xa,xb}, {y,ya,yb} ]
ContourPlot[ f[x,y], {x,xa,xb}, {y,ya,yb} ]
fx = D[f[x,y],x]
fy = D[f[x,y],y]
crit = Solve[{fx==0,fy==0}]
critpts = {x,y} /. crit
fxx = D[fx,x]
fxy = D[fx,y]
fyy = D[fy,y]
disc = fxx fyy - fxy^2
{{x,y},disc,fxx} /. crit
```

11.8 LAGRANGE MULTIPLIERS

1. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 4y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 4y\lambda$ $\Rightarrow x = 8x\lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4}$ or $x = 0$.
   CASE 1: If $x = 0$, then $y = 0$. But $(0,0)$ is not on the ellipse so $x \neq 0$.
   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4} \Rightarrow x = \pm\sqrt{2}y \Rightarrow \left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1 \Rightarrow y = \pm\frac{1}{2}$.
   Therefore f takes on its extreme values at $\left(\pm\sqrt{2}, \frac{1}{2}\right)$ and $\left(\pm\sqrt{2}, -\frac{1}{2}\right)$. The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{2}$.

2. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$ $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.
   CASE 1: If $x = 0$, then $y = 0$. But $(0,0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.
   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow y = 2x\left(\pm\frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm\sqrt{5} \Rightarrow y = \pm\sqrt{5}$.
   Therefore f takes on its extreme values at $\left(\pm\sqrt{5}, \sqrt{5}\right)$ and $\left(\pm\sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

3. $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$ $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow$ f takes on its extreme value at $(1,3)$ on the line. The extreme value is $f(1,3) = 49 - 1 - 9 = 39$.

4. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2xy\mathbf{i} + x^2\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow 2xy = \lambda$ and $x^2 = \lambda$ $\Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $2y = x$.

CASE 1: If $x = 0$, then $x + y = 3 \Rightarrow y = 3$.

CASE 2: If $x \neq 0$, then $2y = x$ so that $x + y = 3 \Rightarrow 2y + y = 3 \Rightarrow y = 1 \Rightarrow x = 2$.

Therefore f takes on its extreme values at $(0,3)$ and $(2,1)$. The extreme values of f are $f(0,3) = 0$ and $f(2,1) = 4$.

5. We optimize $f(x,y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x,y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = \lambda(y^2\mathbf{i} + 2xy\mathbf{j}) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.

   CASE 1: If $y = 0$, then $x = 0$. But $(0,0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.

   CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54$

   $\Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \pm\frac{1}{3} \Rightarrow x = \pm 3$. Since $xy^2 = 54$ we cannot have $x = -3$, so $x = 3$ and $y^2 = 18$

   $\Rightarrow x = 3$ and $y = \pm 3\sqrt{2}$.

   Therefore $(3, \pm 3\sqrt{2})$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

6. We optimize $f(x,y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint $g(x,y) = x^2y - 2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2xy\lambda$ and $2y = x^2\lambda$ $\Rightarrow \lambda = \frac{2y}{x^2}$, since $x = 0 \Rightarrow y = 0$ (but $g(0,0) \neq 0$). Thus $x \neq 0$ and $2x = 2xy\left(\frac{2y}{x^2}\right) \Rightarrow x^2 = 2y^2$ $\Rightarrow (2y^2)y - 2 = 0 \Rightarrow y = 1$ (since $y > 0$) $\Rightarrow x = \pm\sqrt{2}$. Therefore $(\pm\sqrt{2}, 1)$ are the points on the curve $x^2y = 2$ nearest the origin (since $x^2y = 2$ has points increasingly far away as x gets close to 0, no points are farthest away).

7. (a) $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4,4)$. Now, $xy = 16$, $x > 0$, $y > 0$ is a branch of a hyperbola in the first quadrant with the x-and y-axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.

   (b) $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow y = \lambda = x \Rightarrow y + y = 16 \Rightarrow y = 8$ $\Rightarrow x = 8 \Rightarrow f(8,8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and $y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x- and y-axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.

8. Let $f(x,y) = x^2 + y^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x + y)\mathbf{i} + (2y + x)\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda(2x + y)$ and $2y = \lambda(2y + x) \Rightarrow \frac{2y}{2y + x} = \lambda$

   $\Rightarrow 2x = \left(\frac{2y}{2y + x}\right)(2x + y) \Rightarrow x(2y + x) = y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.

   CASE 1: $y = x \Rightarrow x^2 + x(x) + x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = x$.

CASE 2: $y = -x \Rightarrow x^2 + x(-x) + (-x)^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = -x$. Thus $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3}$

$= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $f(1,-1) = 2 = f(-1,1)$.

Therefore the points $(1,-1)$ and $(-1,1)$ are the farthest away; $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are the closest points to the origin.

9. $V = \pi r^2 h \Rightarrow 16\pi = \pi r^2 h \Rightarrow 16 = r^2 h \Rightarrow g(r,h) = r^2 h - 16$; $S = 2\pi rh + 2\pi r^2 \Rightarrow \nabla S = (2\pi h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2rh\mathbf{i} + r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow (2\pi rh + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j} = \lambda(2rh\mathbf{i} + r^2\mathbf{j}) \Rightarrow 2\pi rh + 4\pi r = 2rh\lambda$ and $2\pi r = \lambda r^2 \Rightarrow r = 0$ or $\lambda = \frac{2\pi}{r}$. But $r = 0$ gives no physical can, so $r \neq 0 \Rightarrow \lambda = \frac{2\pi}{r} \Rightarrow 2\pi h + 4\pi r$ $= 2rh\left(\frac{2\pi}{r}\right) \Rightarrow 2r = h \Rightarrow 16 = r^2(2r) \Rightarrow r = 2 \Rightarrow h = 4$; thus $r = 2$ cm and $h = 4$ cm give the only extreme surface area of 24π cm^2. Since $r = 4$ cm and $h = 1$ cm $\Rightarrow V = 16\pi$ cm^3 and $S = 40\pi$ cm^2, which is a larger surface area, then 24π cm^2 must be the minimum surface area.

10. For a cylinder of radius r and height h we want to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$. Thus $\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow 2\pi h = 2\lambda r$ and $2\pi r = \frac{\lambda h}{2} \Rightarrow \frac{\pi h}{r} = \lambda$ and $2\pi r = \left(\frac{\pi h}{r}\right)\left(\frac{h}{2}\right) \Rightarrow 4r^2 = h^2 \Rightarrow h = 2r \Rightarrow r^2 + \frac{4r^2}{4} = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}$ $\Rightarrow h = a\sqrt{2} \Rightarrow S = 2\pi\left(\frac{a}{\sqrt{2}}\right)(a\sqrt{2}) = 2\pi a^2$.

11. $A = (2x)(2y) = 4xy$ subject to $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$; $\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$ and $\nabla g = \frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}$ so that ∇A $= \lambda \nabla g \Rightarrow 4y\mathbf{i} + 4x\mathbf{j} = \lambda\left(\frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y = \left(\frac{x}{8}\right)\lambda$ and $4x = \left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right)$ $\Rightarrow y = \pm\frac{3}{4}x \Rightarrow \frac{x^2}{16} + \frac{\left(\pm\frac{3}{4}x\right)^2}{9} = 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$. We use $x = 2\sqrt{2}$ since x represents distance. Then $y = \frac{3}{4}(2\sqrt{2}) = \frac{3\sqrt{2}}{2}$, so the length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$.

12. $P = 4x + 4y$ subject to $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; $\nabla P = 4\mathbf{i} + 4\mathbf{j}$ and $\nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$ so that $\nabla P = \lambda \nabla g$ $\Rightarrow 4 = \left(\frac{2x}{a^2}\right)\lambda$ and $4 = \left(\frac{2y}{b^2}\right)\lambda \Rightarrow \lambda = \frac{2a^2}{x}$ and $4 = \left(\frac{2y}{b^2}\right)\left(\frac{2a^2}{x}\right) \Rightarrow y = \left(\frac{b^2}{a^2}\right)x \Rightarrow \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}\right)^2 x^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2x^2}{a^4}$ $= 1 \Rightarrow (a^2 + b^2)x^2 = a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2+b^2}}$, since $x > 0 \Rightarrow y = \left(\frac{b^2}{a^2}\right)x = \frac{b^2}{\sqrt{a^2+b^2}} \Rightarrow$ width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$ and height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}} \Rightarrow$ perimeter is $P = 4x + 4y = \frac{4a^2+4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x-2)\mathbf{i} + (2y-4)\mathbf{j}$ so that $\nabla f = \lambda \nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \lambda[(2x-2)\mathbf{i} + (2y-4)\mathbf{j}]$ $\Rightarrow 2x = \lambda(2x-2)$ and $2y = \lambda(2y-4) \Rightarrow x = \frac{\lambda}{\lambda-1}$ and $y = \frac{2\lambda}{\lambda-1}$, $\lambda \neq 1 \Rightarrow y = 2x \Rightarrow x^2 - 2x + (2x)^2 - 4(2x)$ $= 0 \Rightarrow x = 0$ and $y = 0$, or $x = 2$ and $y = 4$. Therefore $f(0,0) = 0$ is the minimum value and $f(2,4) = 20$ is the

maximum value. (Note that $\lambda = 1$ gives $2x = 2x - 2$ or $0 = -2$, which is impossible.)

14. $\nabla f = 3\mathbf{i} - \mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3 = 2\lambda x$ and $-1 = 2\lambda y \Rightarrow \lambda = \frac{3}{2x}$ and $-1 = 2\left(\frac{3}{2x}\right)y$ $\Rightarrow y = -\frac{x}{3} \Rightarrow x^2 + \left(-\frac{x}{3}\right)^2 = 4 \Rightarrow 10x^2 = 36 \Rightarrow x = \pm\frac{6}{\sqrt{10}} \Rightarrow x = \frac{6}{\sqrt{10}}$ and $y = -\frac{2}{\sqrt{10}}$, or $x = -\frac{6}{\sqrt{10}}$ and $y = \frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \frac{20}{\sqrt{10}} + 6 = 2\sqrt{10} + 6 \approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) = -2\sqrt{10} + 6 \approx -0.325$ is the minimum value.

15. $\nabla T = (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 25 = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 8x - 4y = 2\lambda x$ and $-4x + 2y = 2\lambda y \Rightarrow y = \frac{-2x}{\lambda - 1}$, $\lambda \neq 1$ $\Rightarrow 8x - 4\left(\frac{-2x}{\lambda - 1}\right) = 2\lambda x \Rightarrow x = 0$, or $\lambda = 0$, or $\lambda = 5$.
CASE 1: $x = 0 \Rightarrow y = 0$; but $(0,0)$ is not on $x^2 + y^2 = 25$ so $x \neq 0$.
CASE 2: $\lambda = 0 \Rightarrow y = 2x \Rightarrow x^2 + (2x)^2 = 25 \Rightarrow x = \pm\sqrt{5}$ and $y = 2x$.
CASE 3: $\lambda = 5 \Rightarrow y = \frac{-2x}{4} = -\frac{x}{2} \Rightarrow x^2 + \left(-\frac{x}{2}\right)^2 = 25 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow x = 2\sqrt{5}$ and $y = -\sqrt{5}$, or $x = -2\sqrt{5}$ and $y = \sqrt{5}$.
Therefore $T\left(\sqrt{5}, 2\sqrt{5}\right) = 0° = T\left(-\sqrt{5}, -2\sqrt{5}\right)$ is the minimum value and $T\left(2\sqrt{5}, -\sqrt{5}\right) = 125°$ $= T\left(-2\sqrt{5}, \sqrt{5}\right)$ is the maximum value. (Note: $\lambda = 1 \Rightarrow x = 0$ from the equation $-4x + 2y = 2\lambda y$; but we found $x \neq 0$ in CASE 1.)

16. The surface area is given by $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V(r,h) = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. Thus $\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla V = \left(4\pi r^2 + 2\pi rh\right)\mathbf{i} + \pi r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla V = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ $= \lambda\left[\left(4\pi r^2 + 2\pi rh\right)\mathbf{i} + \pi r^2\mathbf{j}\right] \Rightarrow 8\pi r + 2\pi h = \lambda\left(4\pi r^2 + 2\pi rh\right)$ and $2\pi r = \lambda\pi r^2 \Rightarrow r = 0$ or $2 = r\lambda$. But $r \neq 0$ so $2 = r\lambda \Rightarrow \lambda = \frac{2}{r} \Rightarrow 4r + h = \frac{2}{r}\left(2r^2 + rh\right) \Rightarrow h = 0 \Rightarrow$ the tank is a sphere (there is no cylindrical part) and $\frac{4}{3}\pi r^3 = 8000 \Rightarrow r = 10\left(\frac{6}{\pi}\right)^{1/3}$.

17. Let $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ be the square of the distance from $(1,1,1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \Rightarrow 2(x-1) = \lambda$, $2(y-1) = 2\lambda$, $2(z-1) = 3\lambda$ $\Rightarrow 2(y-1) = 2[2(x-1)]$ and $2(z-1) = 3[2(x-1)] \Rightarrow x = \frac{y+1}{2} \Rightarrow z + 2 = 3\left(\frac{y+1}{2}\right)$ or $z = \frac{3y-1}{2}$; thus $\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) - 13 = 0 \Rightarrow y = 2 \Rightarrow x = \frac{3}{2}$ and $z = \frac{5}{2}$. Therefore the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is closest (since no point on the plane is farthest from the point $(1,1,1)$).

18. Let $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ be the square of the distance from $(1,-1,1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y+1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow x - 1 = \lambda x$, $y + 1 = \lambda y$ and $z - 1 = \lambda z \Rightarrow x = \frac{1}{1-\lambda}$, $y = -\frac{1}{1-\lambda}$, and $z = \frac{1}{1-\lambda}$ for $\lambda \neq 1 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$

$\Rightarrow \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}} \Rightarrow x = \frac{2}{\sqrt{3}}$, $y = -\frac{2}{\sqrt{3}}$, $z = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}$, $y = \frac{2}{\sqrt{3}}$, $z = -\frac{2}{\sqrt{3}}$. The largest value of f occurs where $x < 0$, $y > 0$, and $z < 0$ or at the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ on the sphere.

19. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 2x\lambda$, $2y = 2y\lambda$, and $2z = -2z\lambda \Rightarrow \lambda = -1$, $x = 0$, and $y = 0$, or $\lambda = 1$ and $z = 0$.

CASE 1: $\lambda = -1$, $x = 0$, and $y = 0 \Rightarrow -z^2 = 1$, which has no solution.

CASE 2: $\lambda = 1$ and $z = 0 \Rightarrow x^2 + y^2 = 1$

Therefore the points on the unit circle, $x^2 + y^2 = 1$, are the points on the surface $x^2 + y^2 - z^2 = 1$ closest to the origin.

20. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(y\mathbf{i} + x\mathbf{j} - \mathbf{k}) \Rightarrow 2x = \lambda y$, $2y = \lambda x$, and $2z = -\lambda$ $\Rightarrow x = \frac{\lambda y}{2} \Rightarrow 2y = \lambda\left(\frac{\lambda y}{2}\right) \Rightarrow y = 0$ or $\lambda = \pm 2$.

CASE 1: $y = 0 \Rightarrow x = 0 \Rightarrow -z + 1 = 0 \Rightarrow z = 1$.

CASE 2: $\lambda = 2 \Rightarrow x = y$ and $z = -1 \Rightarrow x^2 - (-1) + 1 = 0 \Rightarrow x^2 + 2 = 0$, so no solution.

CASE 3: $\lambda = -2 \Rightarrow x = -y$ and $z = 1 \Rightarrow (-y)y - 1 + 1 = 0 \Rightarrow y = 0$, again.

Therefore $(0,0,1)$ is the point on the surface closest to the origin since this point gives the only extreme value and there is no maximum distance from the surface to the origin.

21. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda$, $2y = -x\lambda$, and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.

CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.

CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right)$ $\Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. Thus, $x = 2$ and $y = -2$, or $x = -2$ and $y = 2$.

Therefore we get four points: $(2,-2,0)$, $(-2,2,0)$, $(0,0,2)$ and $(0,0,-2)$. But the points $(0,0,2)$ and $(0,0,-2)$ are closest to the origin since they are 2 units away and the others are $2\sqrt{2}$ units away.

22. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda yz$, $2y = \lambda yz$, and $2z = \lambda xy \Rightarrow 2x^2 = \lambda xyz$ and $2y^2 = \lambda yxz$ $\Rightarrow x^2 = y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x(\pm x)(\pm x) = 1 \Rightarrow x = \pm 1 \Rightarrow$ the points are $(1,1,1)$, $(1,-1,-1)$, $(-1,-1,1)$, and $(-1,1,-1)$.

23. $\nabla f = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = -\frac{1}{\lambda} = -2x$, and $z = \frac{5}{2\lambda} = 5x \Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow x = \pm 1$. Thus, $x = 1$, $y = -2$, $z = 5$ or $x = -1$, $y = 2$, $z = -5$. Therefore $f(1,-2,5) = 30$ is the maximum value and $f(-1,2,-5) = -30$ is the minimum value.

24. $\nabla f = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$,

$2 = 2y\lambda$, and $3 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda} = 2x$, and $z = \frac{3}{2\lambda} = 3x \Rightarrow x^2 + (2x)^2 + (3x)^2 = 25 \Rightarrow x = \pm\frac{5}{\sqrt{14}}$.

Thus, $x = \frac{5}{\sqrt{14}}$, $y = \frac{10}{\sqrt{14}}$, $z = \frac{15}{\sqrt{14}}$ or $x = -\frac{5}{\sqrt{14}}$, $y = -\frac{10}{\sqrt{14}}$, $z = -\frac{15}{\sqrt{14}}$. Therefore $f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$ $= 5\sqrt{14}$ is the maximum value and $f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.

25. $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = x + y + z - 9 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow 2x = \lambda$, $2y = \lambda$, and $2z = \lambda \Rightarrow x = y = z \Rightarrow x + x + x - 9 = 0$ $\Rightarrow x = 3$, $y = 3$, and $z = 3$.

26. $f(x,y,z) = xyz$ and $g(x,y,z) = x + y + z^2 - 16 = 0 \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + 2z\mathbf{k}) \Rightarrow yz = \lambda$, $xz = \lambda$, and $xy = 2z\lambda \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$. But $z > 0$ so that $y = x \Rightarrow x^2 = 2z\lambda$ and $xz = \lambda$. Then $x^2 = 2z(xz) \Rightarrow x = 0$ or $x = 2z^2$. But $x > 0$ so that $x = 2z^2 \Rightarrow y = 2z^2 \Rightarrow 2z^2 + 2z^2 + z^2 = 16 \Rightarrow z = \pm\frac{4}{\sqrt{5}}$. We use $z = \frac{4}{\sqrt{5}}$ since $z > 0$. Then $x = \frac{32}{5}$ and $y = \frac{32}{5}$ which yields $f\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.

27. $V = 8xyz$ and $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \nabla V = 8yz\mathbf{i} + 8xz\mathbf{j} + 8xy\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 4yz = \lambda x$, $4xz = \lambda y$, and $4xy = \lambda z \Rightarrow 4xyz = \lambda x^2$ and $4xyz = \lambda y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x$ $\Rightarrow x^2 + x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$ since $x > 0 \Rightarrow$ the dimensions of the box are $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ for maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

28. $V = xyz$ with x, y, z all positive and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; thus $V = xyz$ and $g(x,y,z) = bcx + acy + abz - abc = 0$ $\Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda bc$, $xz = \lambda ac$, and $xy = \lambda ab$ $\Rightarrow xyz = \lambda bcx$, $xyz = \lambda acy$, and $xyz = \lambda abz \Rightarrow \lambda \neq 0$. Also, $\lambda bcx = \lambda acy = \lambda abz \Rightarrow bx = ay$, $cy = bz$, and $cx = ax \Rightarrow y = \frac{b}{a}x$ and $z = \frac{c}{a}x$. Then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \frac{x}{a} + \frac{1}{b}\left(\frac{b}{a}x\right) + \frac{1}{c}\left(\frac{c}{a}x\right) = 1 \Rightarrow \frac{3x}{a} = 1 \Rightarrow x = \frac{a}{3}$ $\Rightarrow y = \left(\frac{b}{a}\right)\left(\frac{a}{3}\right) = \frac{b}{3}$ and $z = \left(\frac{c}{a}\right)\left(\frac{a}{3}\right) = \frac{c}{3} \Rightarrow V = xyz = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}$ is the maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

29. $\nabla T = 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ and $\nabla g = 8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ $= \lambda(8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}) \Rightarrow 16x = 8x\lambda$, $4z = 2y\lambda$, and $4y - 16 = 8z\lambda \Rightarrow \lambda = 2$ or $x = 0$.

CASE 1: $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y$. Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}$. Then
$4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x = \pm\frac{4}{3}$.

CASE 2: $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right) \Rightarrow y^2 - 4y = 4z^2 \Rightarrow 4(0)^2 + y^2 + \left(y^2 - 4y\right) - 16 = 0$
$\Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4$ or $y = -2$. Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4)$
$\Rightarrow z = 0$ and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) \Rightarrow z = \pm\sqrt{3}$.

The temperatures are $T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 642\frac{2}{3}^\circ$, $T(0,4,0) = 600^\circ$, $T\left(0,-2,\sqrt{3}\right) = \left(600 - 24\sqrt{3}\right)^\circ$, and $T\left(0,-2,-\sqrt{3}\right) = \left(600 + 24\sqrt{3}\right)^\circ \approx 641.6^\circ$. Therefore $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

30. $\nabla T = 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 400yz^2 = 2x\lambda$, $400xz^2 = 2y\lambda$, and $800xyz = 2z\lambda$. Solving this system yields the points $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, and $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$. The corresponding temperatures are $T(0, \pm 1, 0) = 0$, $T(\pm 1, 0, 0) = 0$, and $T\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = \pm 50$. Therefore 50 is the maximum temperature at $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$; -50 is the minimum temperature at $\left(\frac{1}{2}, -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

31. $\nabla U = (y+2)\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + \mathbf{j}$ so that $\nabla U = \lambda \nabla g \Rightarrow (y+2)\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + \mathbf{j}) \Rightarrow y + 2 = 2\lambda$ and $x = \lambda \Rightarrow y + 2 = 2x \Rightarrow y = 2x - 2 \Rightarrow 2x + (2x - 2) = 30 \Rightarrow x = 8$ and $y = 14$. Therefore $U(8, 14) = \$128$ is the maximum value of U under the constraint.

32. $\nabla M = (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla M = \lambda \nabla g \Rightarrow (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ $= \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 6 + z = 2x\lambda$, $-2y = 2y\lambda$, $x = 2z\lambda \Rightarrow \lambda = -1$ or $y = 0$.

CASE 1: $\lambda = -1 \Rightarrow 6 + z = -2x$ and $x = -2z \Rightarrow 6 + z = -2(-2z) \Rightarrow z = 2$ and $x = -4$. Then $(-4)^2 + y^2 + 2^2 - 36 = 0 \Rightarrow y = \pm 4$.

CASE 2: $y = 0$, $6 + z = 2x\lambda$, and $x = 2z\lambda \Rightarrow \lambda = \frac{x}{2z} \Rightarrow 6 + z = 2x\left(\frac{x}{2z}\right) \Rightarrow 6z + z^2 = x^2$ $\Rightarrow (6z + z^2) + 0^2 + z^2 = 36 \Rightarrow z = -6$ or $z = 3$. Now $z = -6 \Rightarrow x^2 = 0 \Rightarrow x = 0$; $z = 3$ $\Rightarrow x^2 + 27 \Rightarrow x = \pm 3\sqrt{3}$.

Therefore we have the points $(\pm 3\sqrt{3}, 0, 3)$, $(0, 0, -6)$, and $(-4, 2, \pm 4)$. Then $M(3\sqrt{3}, 0, 3)$ $= 27\sqrt{3} + 60 \approx 106.8$, $M(-3\sqrt{3}, 0, 3) = 60 - 27\sqrt{3} \approx 13.2$, $M(0, 0, -6) = 60$, and $M(-4, 4, 2) = 12$ $= M(-4, -4, 2)$. Therefore, the weakest field is at $(-4, \pm 4, 2)$.

33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}$, $\nabla g_2 = \mathbf{j} + \mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$ $\Rightarrow 2x = 2\lambda$, $2 = \mu - \lambda$, and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$ $\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then $x = \frac{2}{3}$ so that $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is the point that gives the maximum value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(-\frac{4}{3}\right)^2$ $= \frac{4}{3}$.

34. Let $g_1(x, y, z) = x + 2y + 3z - 6 = 0$ and $g_2(x, y, z) = x + 3y + 9z - 9 = 0 \Rightarrow \nabla g_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\nabla g_2 = \mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \Rightarrow 2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, and $2z = 3\lambda + 9\mu$. Then $0 = x + 2y + 3z - 6$ $= \frac{1}{2}(\lambda + \mu) + (2\lambda + 3\mu) + \left(\frac{9}{2}\lambda + \frac{27}{2}\mu\right) - 6 \Rightarrow 7\lambda + 17\mu = 6$; $0 = x + 3y + 9z - 9$ $\Rightarrow \frac{1}{2}(\lambda + \mu) + \left(3\lambda + \frac{9}{2}\mu\right) + \left(\frac{27}{2}\lambda + \frac{81}{2}\mu\right) - 9 \Rightarrow 34\lambda + 91\mu = 18$. Solving these two equations for λ and μ gives $\lambda = \frac{240}{59}$ and $\mu = -\frac{78}{59} \Rightarrow x = \frac{\lambda + \mu}{2} = \frac{81}{59}$, $y = \frac{2\lambda + 3\mu}{2} = \frac{123}{59}$, and $z = \frac{3\lambda + 9\mu}{2} = \frac{9}{59}$. The minimum value is

$f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{21{,}771}{59^2} = \frac{369}{59}$. (Note that there is no maximum value of f subject to the constraints because at least one of the variables x, y, or z can be made arbitrary and assume a value as large as we please.)

35. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x,y,z)$ subject to the constraints $g_1(x,y,z) = y + 2z - 12 = 0$ and $g_2(x,y,z) = x + y - 6 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = \mathbf{j} + 2\mathbf{k}$, and $\nabla g_2 = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x = \mu$, $2y = \lambda + \mu$, and $2z = 2\lambda$. Then $0 = y + 2z - 12 = \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) + 2\lambda - 12 \Rightarrow \frac{5}{2}\lambda + \frac{1}{2}\mu = 12 \Rightarrow 5\lambda + \mu = 24$; $0 = x + y - 6 = \frac{\mu}{2} + \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) - 6$ $\Rightarrow \frac{1}{2}\lambda + \mu = 6 \Rightarrow \lambda + 2\mu = 12$. Solving these two equations for λ and μ gives $\lambda = 4$ and $\mu = 4 \Rightarrow x = \frac{\mu}{2} = 2$, $y = \frac{\lambda + \mu}{2} = 4$, and $z = \lambda = 4$. The point $(2,4,4)$ on the line of intersection is closest to the origin. (There is no maximum distance from the origin since points on the line can be arbitrarily far away.)

36. The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ from Exercise 33 above.

37. Let $g_1(x,y,z) = z - 1 = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.

CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.

CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$ $\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2(\pm\sqrt{3})^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $(\pm\sqrt{6}, \pm\sqrt{3}, 1)$.

Now $f(0, \pm 3, 1) = 1$ and $f(\pm\sqrt{6}, \pm\sqrt{3}, 1) = 6(\pm\sqrt{3}) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$, and the minimum of f is $1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$.

38. (a) Let $g_1(x,y,z) = x + y + z - 40 = 0$ and $g_2(x,y,z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\Rightarrow yz = \lambda + \mu$, $xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.

CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.

CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20)$ $= 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10$, $y = 2t + 10$, $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = (-4t^2 + 100)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10$, $y = 10$, and $z = 20$.

39. Let $g_1(x,y,z) = y - x = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.

CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ yielding the points

$(\pm\sqrt{2}, \pm\sqrt{2}, 0)$.

CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$ $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.

Now, $f(0, 0, \pm 2) = 4$ and $f(\pm\sqrt{2}, \pm\sqrt{2}, 0) = 2$. Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$.

40. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x,y,z)$ subject to the constraints $g_1(x,y,z) = 2y + 4z - 5 = 0$ and $g_2(x,y,z) = 4x^2 + 4y^2 - z^2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = 2\mathbf{j} + 4\mathbf{k}$, and $\nabla g_2 = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(2\mathbf{j} + 4\mathbf{k}) + \mu(8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 8x\mu$, $2y = 2\lambda + 8y\mu$, and $2z = 4\lambda - 2z\mu \Rightarrow x = 0$ or $\mu = \frac{1}{4}$.

CASE 1: $x = 0 \Rightarrow 4(0)^2 + 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y \Rightarrow 2y + 4(2y) - 5 = 0 \Rightarrow y = \frac{1}{2}$, or $2y + 4(-2y) - 5 = 0$ $\Rightarrow y = -\frac{5}{6}$ yielding the points $\left(0, \frac{1}{2}, 1\right)$ and $\left(0, -\frac{5}{6}, \frac{5}{3}\right)$.

CASE 2: $\mu = \frac{1}{4} \Rightarrow y = \lambda + y \Rightarrow \lambda = 0 \Rightarrow 2z = 4(0) - 2z\left(\frac{1}{4}\right) \Rightarrow z = 0 \Rightarrow 2y + 4(0) = 5 \Rightarrow y = \frac{5}{2}$ and $(0)^2 = 4x^2 + 4\left(\frac{5}{2}\right)^2 \Rightarrow$ no solution.

Then $f\left(0, \frac{1}{2}, 1\right) = \frac{5}{4}$ and $f\left(0, -\frac{5}{6}, \frac{5}{3}\right) = 25\left(\frac{1}{36} + \frac{1}{9}\right) = \frac{125}{36} \Rightarrow$ the point $\left(0, \frac{1}{2}, 1\right)$ is closest to the origin.

41. $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = y\lambda$ and $1 = x\lambda \Rightarrow y = x$ $\Rightarrow y^2 = 16 \Rightarrow y = \pm 4 \Rightarrow (4, 4)$ and $(-4, -4)$ are candidates for the location of extreme values. But as $x \to \infty$, $y \to \infty$ and $f(x,y) \to \infty$; as $x \to -\infty$, $y \to 0$ and $f(x,y) \to -\infty$. Therefore no maximum or minimum value exists subject to the constraint.

42. Let $f(A,B,C) = \sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2 = C^2 + (B + C - 1)^2 + (A + B + C - 1)^2 + (A + C - 1)^2$. We want to minimize f. Then $f_A(A,B,C) = 4A + 2B + 4C$, $f_B(A,B,C) = 2A + 4B + 4C - 4$, and $f_C(A,B,C) = 4A + 4B + 8C - 2$. Set each partial derivative equal to 0 and solve the system to get $A = -\frac{1}{2}$, $B = \frac{3}{2}$, and $C = -\frac{1}{4}$ or the critical point of f is $\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$.

43. (a) Maximize $f(a,b,c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$ $\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.

CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a,b,c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a,b,c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ $\le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a + b + c}{3}$, as claimed.

44. Let $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$. Then we want $\nabla f = \lambda \nabla g \Rightarrow a_1 = \lambda(2x_1)$, $a_2 = \lambda(2x_2)$, $\ldots$, $a_n = \lambda(2x_n)$, $\lambda \ne 0 \Rightarrow x_i = \frac{a_i}{2\lambda} \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \ldots + \frac{a_n^2}{4\lambda^2} = 1$

$\Rightarrow 4\lambda^2 = \sum_{i=1}^{n} a_i^2 \Rightarrow 2\lambda = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \Rightarrow f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left(\frac{a_i}{2\lambda}\right) = \frac{1}{2\lambda} \sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ is the maximum value.

45-50. Example CAS commands:

Maple:
```
f:= (x,y,z) -> x*y + y*z;
g1:= (x,y,z) -> x^2 + y^2 -2;
g2:= (x,y,z) -> x^2 + z^2 -2;
lambda1:= `lambda1`: lambda2:= `lambda2`:
h:= (x,y,z) -> f(x,y,z) - lambda1*g1(x,y,z) -lambda2*g2(x,y,z);
expn1:= diff(h(x,y,z),x) = 0;
expn2:= diff(h(x,y,z),y) = 0;
expn3:= diff(h(x,y,z),z) = 0;
expn4:= diff(h(x,y,z), lambda1) = 0;
expn5:= diff(h(x,y,z), lambda2) = 0;
s:= evalf(solve({expn1,expn2,expn3,expn4,expn5}, {x,y,z,lambda1,lambda2}));
subs({x=-1.306562965,y=.541196100,z=.5411961001},f(x,y,z));
subs({x=1.306562965,y=-.541196100,z=-.5411961001},f(x,y,z));
subs({x=.541196100,y=-1.306562965,z=1.306562965},f(x,y,z));
subs({x=-.541196100,y=1.306562965,z=-1.306562965},f(x,y,z));
```

Mathematica:
```
Clear[w,x,y,z,l1,l2]
f[x_,y_,z_] = x y + y z
g1[x_,y_,z_] = x^2 + y^2 -2
g2[x_,y_,z_] = x^2 + z^2 -2
h = f[x,y,z] - l1 g1[x,y,z] - l2 g2[x,y,z]
hx = D[h,x]
hy = D[h,y]
hz = D[h,z]
hl1 = D[h,l1]
hl2 = D[h,l2]
crit = NSolve[{hx==0,hy==0,hz==0,hl1==0,hl2==0}]
{{x,y,z},f[x,y,z]} /. crit
```

11.9 PARTIAL DERIVATIVES WITH CONSTRAINED VARIABLES

1. $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$:

(a) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y}$

$= 2x\frac{\partial x}{\partial y} + 2y \Rightarrow 0 = 2x\frac{\partial x}{\partial y} + 2y \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = (2x)\left(-\frac{y}{x}\right) + (2y)(1) + (2z)(0) = -2y + 2y = 0$

(b) $\begin{pmatrix} x \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y(x,z) \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$

$$\Rightarrow 1 = 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = \frac{1}{2y} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = (2x)(0) + (2y)\left(\frac{1}{2y}\right) + (2z)(1) = 1 + 2z$$

(c) $$\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z};\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$$

$$\Rightarrow 1 = 2x\frac{\partial x}{\partial z} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = (2x)\left(\frac{1}{2x}\right) + (2y)(0) + (2z)(1) = 1 + 2z$$

2. $w = x^2 + y - z + \sin t$ and $x + y = t$:

(a) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y};\ \frac{\partial x}{\partial y} = 0,\ \frac{\partial z}{\partial y} = 0, \text{ and}$$

$$\frac{\partial t}{\partial y} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,t} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t = 1 + \cos(x + y)$$

(b) $$\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y};\ \frac{\partial z}{\partial y} = 0 \text{ and } \frac{\partial t}{\partial y} = 0$$

$$\Rightarrow \frac{\partial x}{\partial y} = \frac{\partial t}{\partial y} - \frac{\partial y}{\partial y} = -1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = (2x)(-1) + (1)(1) + (-1)(0) + (\cos t)(0) = 1 - 2x$$

(c) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z};\ \frac{\partial x}{\partial z} = 0 \text{ and } \frac{\partial y}{\partial z} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$$

(d) $$\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z};\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial t}{\partial z} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$$

(e) $\begin{pmatrix} x \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = x \\ y = t - x \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}; \frac{\partial x}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t$

(f) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}; \frac{\partial y}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = (2x)(1) + (1)(0) + (-1)(0) + (\cos t)(1) = \cos t + 2x = \cos t + 2(t - y)$

3. $U = f(P, V, T)$ and $PV = nRT$

(a) $\begin{pmatrix} P \\ V \end{pmatrix} \to \begin{pmatrix} P = P \\ V = V \\ T = \frac{PV}{nR} \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial P} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial P} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial P} = \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial V}\right)(0) + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

$= \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

(b) $\begin{pmatrix} V \\ T \end{pmatrix} \to \begin{pmatrix} P = \frac{nRT}{V} \\ V = V \\ T = T \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T} = \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \left(\frac{\partial U}{\partial V}\right)(0) + \frac{\partial U}{\partial T}$

$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T}$

4. $w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 1$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}; \frac{\partial y}{\partial x} = 0$ and

$(y \cos z)\frac{\partial z}{\partial x} + (\sin x)\frac{\partial z}{\partial x} + z \cos x = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-z \cos x}{y \cos z + \sin x}$. At $(0, 1, \pi)$, $\frac{\partial z}{\partial x} = \frac{-\pi}{-1} = \pi$

$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_y \Big|_{(0,1,\pi)} = (2x)(1) + (2y)(0) + (2z)(\pi)\big|_{(0,1,\pi)} = 2\pi^2$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} = (2x)\frac{\partial x}{\partial z} + (2y)(0) + (2z)(1)$

$= (2x)\frac{\partial x}{\partial z} + 2z$. Now $(\sin z)\frac{\partial y}{\partial z} + y\cos z + \sin x + (z\cos x)\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow y\cos z + \sin x + (z\cos x)\frac{\partial x}{\partial z} = 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{-y\cos z - \sin x}{z\cos x}$. At $(0,1,\pi)$, $\frac{\partial x}{\partial z} = \frac{1-0}{(\pi)(1)} = \frac{1}{\pi}$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_y\Big|_{(0,1,\pi)} = 2(0)\left(\frac{1}{\pi}\right) + 2\pi = 2\pi$

5. $w = x^2y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x,y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)(0) + (2x^2y + z)(1) + (y - 3z^2)\frac{\partial z}{\partial y} = 2x^2y + z + (y - 3z^2)\frac{\partial z}{\partial y}$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and

$\frac{\partial x}{\partial y} = 0 \Rightarrow 2y + (2z)\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$. At $(w,x,y,z) = (4,2,1,-1)$, $\frac{\partial z}{\partial y} = -\frac{1}{-1} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x\Big|_{(4,2,1,-1)}$

$= \left[(2)(2)^2(1) + (-1)\right] + \left[1 - 3(-1)^2\right](1) = 5$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)\frac{\partial x}{\partial y} + (2x^2y + z)(1) + (y - 3z^2)(0) = (2xy^2)\frac{\partial x}{\partial y} + 2x^2y + z$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and

$\frac{\partial z}{\partial y} = 0 \Rightarrow (2x)\frac{\partial x}{\partial y} + 2y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$. At $(w,x,y,z) = (4,2,1,-1)$, $\frac{\partial x}{\partial y} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z\Big|_{(4,2,1,-1)}$

$= (2)(2)(1)^2\left(-\frac{1}{2}\right) + (2)(2)^2(1) + (-1) = 5$

6. $y = uv \Rightarrow 1 = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$; $x = u^2 + v^2$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \left(-\frac{u}{v}\right)\frac{\partial u}{\partial y} \Rightarrow 1$

$= v\frac{\partial u}{\partial y} + u\left(-\frac{u}{v}\frac{\partial u}{\partial y}\right) = \left(\frac{v^2 - u^2}{v}\right)\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{v}{v^2 - u^2}$. At $(u,v) = (\sqrt{2},1)$, $\frac{\partial u}{\partial y} = \frac{1}{1^2 - (\sqrt{2})^2} = -1$

$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = -1$

7. $\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$; $x^2 + y^2 = r^2 \Rightarrow 2x + 2y\frac{\partial y}{\partial x} = 2r\frac{\partial r}{\partial x}$ and $\frac{\partial y}{\partial x} = 0 \Rightarrow 2x = 2r\frac{\partial r}{\partial x}$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2 + y^2}}$

8. If x, y, and z are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$

$= (2x)(1) + (-2y)(0) + (4)(0) + (1)\left(\frac{\partial t}{\partial x}\right) = 2x + \frac{\partial t}{\partial x}$. Thus $x + 2z + t = 25 \Rightarrow 1 + 0 + \frac{\partial t}{\partial x} = 0 \Rightarrow \frac{\partial t}{\partial x} = -1$

$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x - 1$. On the other hand, if x, y, and t are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,t}$ $= \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + 4\frac{\partial z}{\partial x} + (1)(0) = 2x + 4\frac{\partial z}{\partial x}$. Thus, $x + 2z + t = 25$ $\Rightarrow 1 + 2\frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,t} = 2x + 4\left(-\frac{1}{2}\right) = 2x - 2$.

9. If x is a differentiable function of y and z, then $f(x,y,z) = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$ $\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial x}$. Similarly, if y is a differentiable function of x and z, $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial f/\partial z}{\partial f/\partial y}$ and if z is a differentiable function of x and y, $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial z}$. Then $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y$ $= \left(-\frac{\partial f/\partial y}{\partial f/\partial x}\right)\left(-\frac{\partial f/\partial z}{\partial f/\partial y}\right)\left(-\frac{\partial f/\partial x}{\partial f/\partial z}\right) = -1$.

10. $z = x + f(u)$ and $u = xy \Rightarrow \frac{\partial z}{\partial x} = 1 + \frac{df}{du}\frac{\partial u}{\partial x} = 1 + y\frac{df}{du}$; also $\frac{\partial z}{\partial y} = 0 + \frac{df}{du}\frac{\partial u}{\partial y} = x\frac{df}{du}$ so that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ $= x\left(1 + y\frac{df}{du}\right) - y\left(x\frac{df}{du}\right) = x$

11. If x and y are independent, then $g(x,y,z) = 0 \Rightarrow \frac{\partial g}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ $\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}$, as claimed.

12. Let x and y be independent. Then $f(x,y,z,w) = 0$, $g(x,y,z,w) = 0$ and $\frac{\partial y}{\partial x} = 0$

$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0$ and

$\frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial w} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial x}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}},\ \text{as claimed.}$$

Likewise, $f(x,y,z,w) = 0$, $g(x,y,z,w) = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$ $= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0$ and (similarly) $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}=-\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial g}{\partial w}\frac{\partial w}{\partial y}=-\frac{\partial g}{\partial y} \end{cases} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\begin{vmatrix} \frac{\partial f}{\partial z} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w}-\frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w}-\frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}, \text{ as claimed.}$$

11.10 TAYLOR'S FORMULA FOR TWO VARIABLES

1. $f(x,y) = xe^y \Rightarrow f_x = e^y,\ f_y = xe^y,\ f_{xx} = 0,\ f_{xy} = e^y,\ f_{yy} = xe^y$

   $\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$

   $= 0 + x\cdot 1 + y\cdot 0 + \frac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right) = x + xy$ quadratic approximation;

   $f_{xxx} = 0,\ f_{xxy} = 0,\ f_{xyy} = e^y,\ f_{yyy} = xe^y$

   $\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$

   $= x + xy + \frac{1}{6}\left(x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 1 + y^3\cdot 0\right) = x + xy + \frac{1}{2}xy^2$, cubic approximation

2. $f(x,y) = e^x\cos y \Rightarrow f_x = e^x\cos y,\ f_y = -e^x\sin y,\ f_{xx} = e^x\cos y,\ f_{xy} = -e^x\sin y,\ f_{yy} = -e^x\cos y$

   $\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$

   $= 1 + x\cdot 1 + y\cdot 0 + \frac{1}{2}[x^2\cdot 1 + 2xy\cdot 0 + y^2\cdot(-1)] = 1 + x + \frac{1}{2}\left(x^2 - y^2\right)$, quadratic approximation;

   $f_{xxx} = e^x\cos y,\ f_{xxy} = -e^x\sin y,\ f_{xyy} = -e^x\cos y,\ f_{yyy} = e^x\sin y$

   $\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$

   $= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}[x^3\cdot 1 + 3x^2y\cdot 0 + 3xy^2\cdot(-1) + y^3\cdot 0]$

   $= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left(x^3 - 3xy^2\right)$, cubic approximation

3. $f(x,y) = y\sin x \Rightarrow f_x = y\cos x,\ f_y = \sin x,\ f_{xx} = -y\sin x,\ f_{xy} = \cos x,\ f_{yy} = 0$

   $\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$

   $= 0 + x\cdot 0 + y\cdot 0 + \frac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right) = xy$, quadratic approximation;

   $f_{xxx} = -y\cos x,\ f_{xxy} = -\sin x,\ f_{xyy} = 0,\ f_{yyy} = 0$

   $\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$

   $= xy + \frac{1}{6}\left(x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 0 + y^3\cdot 0\right) = xy$, cubic approximation

4. $f(x,y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y,\ f_y = -\sin x \sin y,\ f_{xx} = -\sin x \cos y,\ f_{xy} = -\cos x \sin y,$

$f_{yy} = -\sin x \cos y \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 0 + x\cdot 1 + y\cdot 0 + \frac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 0 + y^2\cdot 0\right) = x$, quadratic approximation;

$f_{xxx} = -\cos x \cos y,\ f_{xxy} = \sin x \sin y,\ f_{xyy} = -\cos x \cos y,\ f_{yyy} = \sin x \sin y$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= x + \frac{1}{6}\left[x^3\cdot(-1) + 3x^2 y\cdot 0 + 3xy^2\cdot(-1) + y^3\cdot 0\right] = x - \frac{1}{6}\left(x^3 + 3xy^2\right)$, cubic approximation

5. $f(x,y) = e^x \ln(1+y) \Rightarrow f_x = e^x \ln(1+y),\ f_y = \frac{e^x}{1+y},\ f_{xx} = e^x \ln(1+y),\ f_{xy} = \frac{e^x}{1+y},\ f_{yy} = -\frac{e^x}{(1+y)^2}$

$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 0 + x\cdot 0 + y\cdot 1 + \frac{1}{2}\left[x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot(-1)\right] = y + \frac{1}{2}\left(2xy - y^2\right)$, quadratic approximation;

$f_{xxx} = e^x \ln(1+y),\ f_{xxy} = \frac{e^x}{1+y},\ f_{xyy} = -\frac{e^x}{(1+y)^2},\ f_{yyy} = \frac{2e^x}{(1+y)^3}$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left[x^3\cdot 0 + 3x^2 y\cdot 1 + 3xy^2\cdot(-1) + y^3\cdot 2\right]$

$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left(3x^2 y - 3xy^2 + 2y^3\right)$, cubic approximation

6. $f(x,y) = \ln(2x+y+1) \Rightarrow f_x = \frac{2}{2x+y+1},\ f_y = \frac{1}{2x+y+1},\ f_{xx} = \frac{-4}{(2x+y+1)^2},\ f_{xy} = \frac{-2}{(2x+y+1)^2},$

$f_{yy} = \frac{-1}{(2x+y+1)^2} \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 0 + x\cdot 2 + y\cdot 1 + \frac{1}{2}\left[x^2\cdot(-4) + 2xy\cdot(-2) + y^2\cdot(-1)\right] = 2x + y + \frac{1}{2}\left(-4x^2 - 4xy - y^2\right)$

$= (2x+y) - \frac{1}{2}(2x+y)^2$, quadratic approximation;

$f_{xxx} = \frac{16}{(2x+y+1)^3},\ f_{xxy} = \frac{8}{(2x+y+1)^3},\ f_{xyy} = \frac{4}{(2x+y+1)^3},\ f_{yyy} = \frac{2}{(2x+y+1)^3}$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= (2x+y) - \frac{1}{2}(2x+y)^2 + \frac{1}{6}\left(x^3\cdot 16 + 3x^2 y\cdot 8 + 3xy^2\cdot 4 + y^3\cdot 2\right)$

$= (2x+y) - \frac{1}{2}(2x+y)^2 + \frac{1}{3}\left(8x^3 + 12x^2 y + 6xy^2 + y^2\right)$

$= (2x+y) - \frac{1}{2}(2x+y)^2 + \frac{1}{3}(2x+y)^3$, cubic approximation

7. $f(x,y) = \sin\left(x^2+y^2\right) \Rightarrow f_x = 2x\cos\left(x^2+y^2\right),\ f_y = 2y\cos\left(x^2+y^2\right),\ f_{xx} = 2\cos\left(x^2+y^2\right) - 4x^2\sin\left(x^2+y^2\right),$
$f_{xy} = -4xy\sin\left(x^2+y^2\right),\ f_{yy} = 2\cos\left(x^2+y^2\right) - 4y^2\sin\left(x^2+y^2\right)$

$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right.$

$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 2\right) = x^2 + y^2$, quadratic approximation;

$f_{xxx} = -12x \sin\left(x^2+y^2\right) - 8x^3 \cos\left(x^2+y^2\right)$, $f_{xxy} = -4y \sin\left(x^2+y^2\right) - 8x^2y \cos\left(x^2+y^2\right)$,
$f_{xyy} = -4x \sin\left(x^2+y^2\right) - 8xy^2 \cos\left(x^2+y^2\right)$, $f_{yyy} = -12y \sin\left(x^2+y^2\right) - 8y^3 \cos\left(x^2+y^2\right)$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= x^2 + y^2 + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = x^2 + y^2$, cubic approximation

8. $f(x,y) = \cos\left(x^2+y^2\right) \Rightarrow f_x = -2x \sin\left(x^2+y^2\right)$, $f_y = -2y \sin\left(x^2+y^2\right)$,
$f_{xx} = -2 \sin\left(x^2+y^2\right) - 4x^2 \cos\left(x^2+y^2\right)$, $f_{xy} = -4xy \cos\left(x^2+y^2\right)$, $f_{yy} = -2 \sin\left(x^2+y^2\right) - 4y^2 \cos\left(x^2+y^2\right)$

$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right] = 1$, quadratic approximation;

$f_{xxx} = -12x \cos\left(x^2+y^2\right) + 8x^3 \sin\left(x^2+y^2\right)$, $f_{xxy} = -4y \cos\left(x^2+y^2\right) + 8x^2y \sin\left(x^2+y^2\right)$,
$f_{xyy} = -4x \cos\left(x^2+y^2\right) + 8xy^2 \sin\left(x^2+y^2\right)$, $f_{yyy} = -12y \cos\left(x^2+y^2\right) + 8y^3 \sin\left(x^2+y^2\right)$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= 1 + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = 1$, cubic approximation

9. $f(x,y) = \frac{1}{1-x-y} \Rightarrow f_x = \frac{1}{(1-x-y)^2} = f_y$, $f_{xx} = \frac{2}{(1-x-y)^3} = f_{xy} = f_{yy}$

$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 2 + y^2 \cdot 2\right) = 1 + (x+y) + \left(x^2 + 2xy + y^2\right)$

$= 1 + (x+y) + (x+y)^2$, quadratic approximation; $f_{xxx} = \frac{6}{(1-x-y)^4} = f_{xxy} = f_{xyy} = f_{yyy}$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= 1 + (x+y) + (x+y)^2 + \frac{1}{6}\left(x^3 \cdot 6 + 3x^2y \cdot 6 + 3xy^2 \cdot 6 + y^3 \cdot 6\right)$

$= 1 + (x+y) + (x+y)^2 + \left(x^3 + 3x^2y + 3xy^2 + y^3\right) = 1 + (x+y) + (x+y)^2 + (x+y)^3$, cubic approximation

10. $f(x,y) = \frac{1}{1-x-y+xy} \Rightarrow f_x = \frac{1-y}{(1-x-y+xy)^2}$, $f_y = \frac{1-x}{(1-x-y+xy)^2}$, $f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3}$,

$f_{xy} = \frac{1-3x-3y+3xy}{(1-x-y+xy)^3}$, $f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3}$

$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 1 + y^2 \cdot 2\right) = 1 + x + y + x^2 + xy + y^2$, quadratic approximation;

$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4}$, $f_{xxy} = \frac{[-4(1-x-y+xy) + 6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4}$,

$f_{xyy} = \frac{[-4(1-x-y+xy)+6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4}$, $f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4}$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6}\left(x^3 \cdot 6 + 3x^2 y \cdot 2 + 3xy^2 \cdot 2 + y^3 \cdot 6\right)$

$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + xy^2 + y^3$, cubic approximation

11. $f(x,y) = \cos x \cos y \Rightarrow f_x = -\sin x \cos y$, $f_y = -\cos x \sin y$, $f_{xx} = -\cos x \cos y$, $f_{xy} = \sin x \sin y$,

$f_{yy} = -\cos x \cos y \Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot (-1) + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 - \frac{x^2}{2} - \frac{y^2}{2}$, quadratic approximation. Since all partial derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal to $1 \Rightarrow E(x,y) \le \frac{1}{6}\left[(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + 0.1)^3\right] \le 0.00134$.

12. $f(x,y) = e^x \sin y \Rightarrow f_x = e^x \sin y$, $f_y = e^x \cos y$, $f_{xx} = e^x \sin y$, $f_{xy} = e^x \cos y$, $f_{yy} = -e^x \sin y$

$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = y + xy$, quadratic approximation. Now, $f_{xxx} = e^x \sin y$,

$f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, and $f_{yyy} = -e^x \cos y$. Since $|x| \le 0.1$, $|e^x \sin y| \le |e^{0.1} \sin 0.1| \approx 0.11$ and

$|e^x \cos y| \le |e^{0.1} \cos 0.1| \approx 1.11$. Therefore,

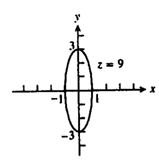
$E(x,y) \le \frac{1}{6}\left[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3\right] \le 0.000814$.

CHAPTER 11 PRACTICE EXERCISES

1. Domain: All points in the xy-plane

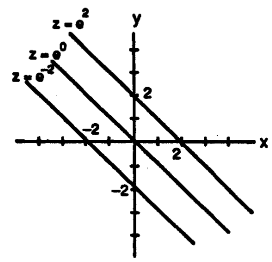
   Range: $z \ge 0$

   Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.



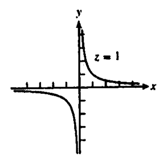
2. Domain: All points in the xy-plane
   Range: $0 < z < \infty$

   Level curves are the straight lines $x + y = \ln c$ with slope -1, and $c > 0$.



3. Domain: All (x,y) such that $x \neq 0$ and $y \neq 0$
   Range: $z \neq 0$

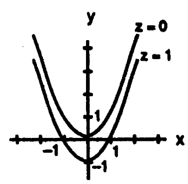
   Level curves are hyperbolas with the x- and y-axes as asymptotes.



4. Domain: All (x,y) so that $x^2 - y \geq 0$

   Range: $z \geq 0$

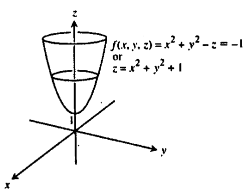
   Level curves are the parabolas $y = x^2 - c$, $c \geq 0$.



5. Domain: All points (x, y, z) in space

   Range: All real numbers

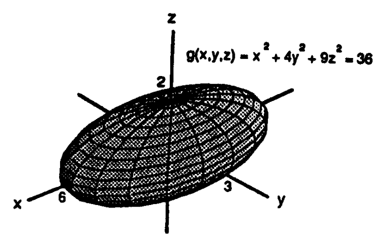
   Level surfaces are paraboloids of revolution with the z-axis as axis.



6. Domain: All points (x, y, z) in space

   Range: Nonnegative real numbers

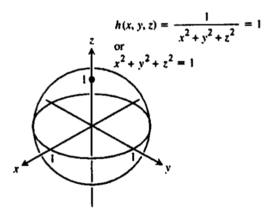
   Level surfaces are ellipsoids with center (0, 0, 0).



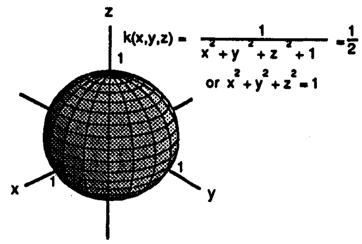
7. Domain: All (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$

   Range: Positive real numbers

   Level surfaces are spheres with center (0, 0, 0) and radius $r > 0$.



8. Domain: All points (x, y, z) in space
Range: $(0, 1]$



Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.

9. $\lim\limits_{(x,y)\to(\pi, \ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$

10. $\lim\limits_{(x,y)\to(0,0)} \frac{2+y}{x+\cos y} = \frac{2+0}{0+\cos 0} = 2$

11. $\lim\limits_{\substack{(x,y)\to(1,1) \\ x \neq \pm y}} \frac{x-y}{x^2-y^2} = \lim\limits_{\substack{(x,y)\to(1,1) \\ x \neq \pm y}} \frac{x-y}{(x-y)(x+y)} = \lim\limits_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

12. $\lim\limits_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = \lim\limits_{(x,y)\to(1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} = \lim\limits_{(x,y)\to(1,1)} (x^2y^2+xy+1) = 1^2 \cdot 1^2 + 1 \cdot 1 + 1 = 3$

13. $\lim\limits_{P\to(1,-1,e)} \ln|x+y+z| = \ln|1+(-1)+e| = \ln e = 1$

14. $\lim\limits_{P\to(1,-1,-1)} \tan^{-1}(x+y+z) = \tan^{-1}(1+(-1)+(-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$

15. Let $y = kx^2$, $k \neq 1$. Then $\lim\limits_{\substack{(x,y)\to(0,0) \\ y \neq x^2}} \frac{y}{x^2-y} = \lim\limits_{(x,kx^2)\to(0,0)} \frac{kx^2}{x^2-kx^2} = \frac{k}{1-k^2}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist.

16. Let $y = kx$, $k \neq 0$. Then $\lim\limits_{\substack{(x,y)\to(0,0) \\ xy \neq 0}} \frac{x^2+y^2}{xy} = \lim\limits_{(x,kx)\to(0,0)} \frac{x^2+(kx)^2}{x(kx)} = \frac{1+k^2}{k}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist.

17. Let $y = kx$. Then $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$ which gives different limits for different values of $k \Rightarrow$ the limit does not exist so $f(0,0)$ cannot be defined in a way that makes f continuous at the origin.

18. Along the x-axis, $y = 0$ and $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{|x+y|} = \lim\limits_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, so the limit fails to exist $\Rightarrow$ f is not continuous at $(0,0)$.

19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

20. $\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) + \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$,

$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) + \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+y}{x^2+y^2}$

21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}, \frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}, \frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

22. $h_x(x,y,z) = 2\pi \cos(2\pi x + y - 3z)$, $h_y(x,y,z) = \cos(2\pi x + y - 3z)$, $h_z(x,y,z) = -3\cos(2\pi x + y - 3z)$

23. $\frac{\partial P}{\partial n} = \frac{RT}{V}, \frac{\partial P}{\partial R} = \frac{nT}{V}, \frac{\partial P}{\partial T} = \frac{nR}{V}, \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

24. $f_r(r,\ell,T,w) = -\frac{1}{2r^2\ell}\sqrt{\frac{T}{\pi w}}$, $f_\ell(r,\ell,T,w) = -\frac{1}{2r\ell^2}\sqrt{\frac{T}{\pi w}}$, $f_T(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\left(\frac{1}{\sqrt{\pi w}}\right)\left(\frac{1}{2\sqrt{T}}\right)$

$= \frac{1}{4r\ell}\sqrt{\frac{1}{T\pi w}} = \frac{1}{4r\ell T}\sqrt{\frac{T}{\pi w}}$, $f_w(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}w^{-3/2}\right) = -\frac{1}{4r\ell w}\sqrt{\frac{T}{\pi w}}$

25. $\frac{\partial g}{\partial x} = \frac{1}{y}, \frac{\partial g}{\partial y} = 1 - \frac{x}{y^2} \Rightarrow \frac{\partial^2 g}{\partial x^2} = 0, \frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}, \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$

26. $g_x(x,y) = e^x + y\cos x$, $g_y(x,y) = \sin x \Rightarrow g_{xx}(x,y) = e^x - y\sin x$, $g_{yy}(x,y) = 0$, $g_{xy}(x,y) = g_{yx}(x,y) = \cos x$

27. $\frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2+1}, \frac{\partial f}{\partial y} = x \Rightarrow \frac{\partial^2 f}{\partial x^2} = -30x + \frac{2-2x^2}{(x^2+1)^2}, \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

28. $f_x(x,y) = -3y$, $f_y(x,y) = 2y - 3x - \sin y + 7e^y \Rightarrow f_{xx}(x,y) = 0$, $f_{yy}(x,y) = 2 - \cos y + 7e^y$, $f_{xy}(x,y) = f_{yx}(x,y)$
$= -3$

29. $\frac{\partial w}{\partial x} = y\cos(xy+\pi), \frac{\partial w}{\partial y} = x\cos(xy+\pi), \frac{dx}{dt} = e^t, \frac{dy}{dt} = \frac{1}{t+1}$
$\Rightarrow \frac{dw}{dt} = [y\cos(xy+\pi)]e^t + [x\cos(xy+\pi)]\left(\frac{1}{t+1}\right)$; $t = 0 \Rightarrow x = 1$ and $y = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0} = 0 \cdot 1 + [1 \cdot (-1)]\left(\frac{1}{0+1}\right) = -1$

30. $\frac{\partial w}{\partial x} = e^y, \frac{\partial w}{\partial y} = xe^y + \sin z, \frac{\partial w}{\partial z} = y\cos z + \sin z, \frac{dx}{dt} = t^{-1/2}, \frac{dy}{dt} = 1 + \frac{1}{t}, \frac{dz}{dt} = \pi$
$\Rightarrow \frac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)\pi$; $t = 1 \Rightarrow x = 2$, $y = 0$, and $z = \pi$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=1} = 1 \cdot 1 + (2 \cdot 1 + 0)(2) + (0 + 0)\pi = 5$

31. $\frac{\partial w}{\partial x} = 2\cos(2x - y), \frac{\partial w}{\partial y} = -\cos(2x - y), \frac{\partial x}{\partial r} = 1, \frac{\partial x}{\partial s} = \cos s, \frac{\partial y}{\partial r} = s, \frac{\partial y}{\partial s} = r$

$\Rightarrow \frac{\partial w}{\partial r} = [2\cos(2x-y)](1) + [-\cos(2x-y)](s)$; $r = \pi$ and $s = 0 \Rightarrow x = \pi$ and $y = 0$

$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{(\pi,0)} = (2\cos 2\pi) - (\cos 2\pi)(0) = 2$; $\frac{\partial w}{\partial s} = [2\cos(2x-y)](\cos s) + [-\cos(2x-y)](r)$

$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{(\pi,0)} = (2\cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$

32. $\frac{\partial w}{\partial u} = \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)$; $u = v = 0 \Rightarrow x = 2 \Rightarrow \left.\frac{\partial w}{\partial u}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5}$;

$\frac{\partial w}{\partial v} = \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v) \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0$

33. $\frac{\partial f}{\partial x} = y + z$, $\frac{\partial f}{\partial y} = x + z$, $\frac{\partial f}{\partial z} = y + x$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = -2\sin 2t$

$\Rightarrow \frac{df}{dt} = -(y+z)(\sin t) + (x+z)(\cos t) - 2(y+x)(\sin 2t)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, and $z = \cos 2$

$\Rightarrow \left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$

34. $\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = (5)\frac{dw}{ds}$ and $\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = (1)\frac{dw}{ds} = \frac{dw}{ds} \Rightarrow \frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 5\frac{dw}{ds} - 5\frac{dw}{ds} = 0$

35. $F(x,y) = 1 - x - y^2 - \sin xy \Rightarrow F_x = -1 - y\cos xy$ and $F_y = -2y - x\cos xy \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-1 - y\cos xy}{-2y - x\cos xy}$

$= \frac{1 + y\cos xy}{-2y - x\cos xy} \Rightarrow$ at $(x,y) = (0,1)$ we have $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1+1}{-2} = -1$

36. $F(x,y) = 2xy + e^{x+y} - 2 \Rightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$

$\Rightarrow$ at $(x,y) = (0, \ln 2)$ we have $\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -\frac{2\ln 2 + 2}{0 + 2} = -(\ln 2 + 1)$

37. $\nabla f = (-\sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j} \Rightarrow \nabla f|_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \Rightarrow |\nabla f| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$;

$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \frac{\sqrt{2}}{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\frac{\sqrt{2}}{2}$;

$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Rightarrow (D_{\mathbf{v}}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right) + \left(-\frac{1}{2}\right)\left(\frac{4}{5}\right) = -\frac{7}{10}$

38. $\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2e^{-2y}\mathbf{j} \Rightarrow \nabla f|_{(1,0)} = 2\mathbf{i} - 2\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

$\Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction

$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 2\sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{2}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$\Rightarrow (D_{\mathbf{v}}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0$

39. $\nabla f = \left(\frac{2}{2x+3y+6z}\right)\mathbf{i} + \left(\frac{3}{2x+3y+6z}\right)\mathbf{j} + \left(\frac{6}{2x+3y+6z}\right)\mathbf{k} \Rightarrow \nabla f|_{(-1,-1,1)} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 7$, $(D_{-\mathbf{u}}f)_{P_0} = -7$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow (D_{\mathbf{v}}f)_{P_0} = (D_{\mathbf{u}}f)_{P_0} = 7$

40. $\nabla f = (2x+3y)\mathbf{i} + (3x+2)\mathbf{j} + (1-2z)\mathbf{k} \Rightarrow \nabla f|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$;
$(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{5}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow (D_{\mathbf{v}}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$

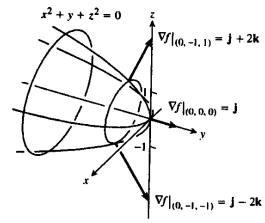
41. $\mathbf{r} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v}(t) = (-3\sin 3t)\mathbf{i} + (3\cos 3t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$; $f(x,y,z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $t = \frac{\pi}{3}$ yields the point on the helix $(-1, 0, \pi)$
$\Rightarrow \nabla f|_{(1,0,\pi)} = -\pi\mathbf{j} \Rightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{\pi}{\sqrt{2}}$

42. $f(x,y,z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1,1,1)$ we get $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the maximum value of $D_{\mathbf{u}}f|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

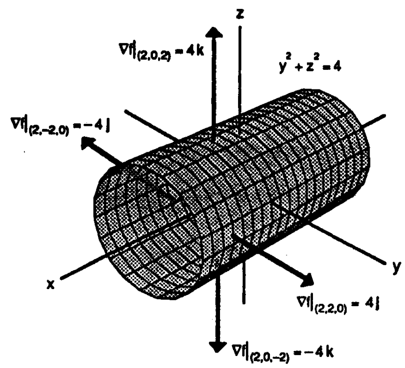
43. (a) Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1,2)$. The direction toward $(2,2)$ is determined by $\mathbf{v}_1 = (2-1)\mathbf{i} + (2-2)\mathbf{j} = \mathbf{i} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = 2 \Rightarrow a = 2$. The direction toward $(1,1)$ is determined by $\mathbf{v}_2 + (1-1)\mathbf{i} + (1-2)\mathbf{j} = -\mathbf{j} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = -2 \Rightarrow -b = -2 \Rightarrow b = 2$. Therefore $\nabla f = 2\mathbf{i} + 2\mathbf{j}$.
(b) The direction toward $(4,6)$ is determined by $\mathbf{v}_3 = (4-1)\mathbf{i} + (6-2)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
$\Rightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$.

44. (a) True (b) False (c) True (d) True

45. $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(0,-1,-1)} = \mathbf{j} - 2\mathbf{k}$,
$\nabla f|_{(0,0,0)} = \mathbf{j}$,
$\nabla f|_{(0,-1,1)} = \mathbf{j} + 2\mathbf{k}$



46. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(2,2,0)} = 4\mathbf{j}$,
$\nabla f|_{(2,-2,0)} = -4\mathbf{j}$,
$\nabla f|_{(2,0,2)} = 4\mathbf{k}$,
$\nabla f|_{(2,0,-2)} = -4\mathbf{k}$



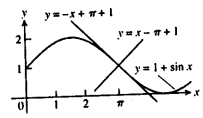
47. $\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \nabla f|_{(2,-1,1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow$ Tangent Plane: $4(x-2) - (y+1) - 5(z-1) = 0$ $\Rightarrow 4x - y - 5z = 4$; Normal Line: $x = 2 + 4t$, $y = -1 - t$, $z = 1 - 5t$

48. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f|_{(1,1,2)} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent Plane: $2(x-1) + 2(y-1) + (z-2) = 0$ $\Rightarrow 2x + 2y + z - 6 = 0$; Normal Line: $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 + t$

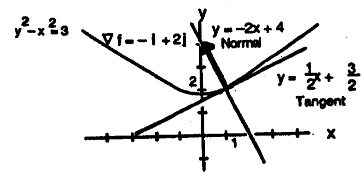
49. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(0,1,0)} = 0$ and $\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(0,1,0)} = 2$; thus the tangent plane is $2(y-1) - (z-0) = 0$ or $2y - z - 2 = 0$

50. $\frac{\partial z}{\partial x} = -2x\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$ and $\frac{\partial z}{\partial y} = -2y\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$; thus the tangent plane is $-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \left(z - \frac{1}{2}\right) = 0$ or $x + y + 2z - 3 = 0$

51. $\nabla f = (-\cos x)\mathbf{i} + \mathbf{j} \Rightarrow \nabla f|_{(\pi,1)} = \mathbf{i} + \mathbf{j} \Rightarrow$ the tangent line is $(x - \pi) + (y - 1) = 0 \Rightarrow x + y = \pi + 1$; the normal line is $y - 1 = 1(x - \pi) \Rightarrow y = x - \pi + 1$



52. $\nabla f = -x\mathbf{i} + y\mathbf{j} \Rightarrow \nabla f|_{(1,2)} = -\mathbf{i} + 2\mathbf{j} \Rightarrow$ the tangent line is $-(x - 1) + 2(y - 2) = 0 = \frac{1}{2}x + \frac{3}{2}$; the normal line is $y - 2 = -2(x - 1) \Rightarrow y = -2x + 4$



53. Let $f(x,y,z) = x^2 + 2y + 2z - 4$ and $g(x,y,z) = y - 1$. Then $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}|_{\left(1,1,\frac{1}{2}\right)} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow$ the line is $x = 1 - 2t$, $y = 1$, $z = \frac{1}{2} + 2t$

54. Let $f(x,y,z) = x + y^2 + z - 2$ and $g(x,y,z) = y - 1$. Then $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k}|_{\left(\frac{1}{2},1,\frac{1}{2}\right)} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow$ the line is $x = \frac{1}{2} - t$, $y = 1$, $z = \frac{1}{2} + t$

55. $f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}$, $f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \cos x \cos y|_{(\pi/4,\pi/4)} = \frac{1}{2}$, $f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\sin x \sin y|_{(\pi/4,\pi/4)} = -\frac{1}{2}$

$\Rightarrow L(x,y) = \frac{1}{2} + \frac{1}{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{2}y$; $f_{xx}(x,y) = -\sin x \cos y$, $f_{yy}(x,y) = -\sin x \cos y$, and

$f_{xy}(x,y) = -\cos x \sin y$. Thus an upper bound for E depends on the bound M used for $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$.

With $M = \frac{\sqrt{2}}{2}$ we have $|E(x,y)| \le \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\left|x - \frac{\pi}{4}\right| + \left|y - \frac{\pi}{4}\right|\right)^2 \le \frac{\sqrt{2}}{4}(0.2)^2 \le 0.0142$;

with $M = 1$, $|E(x,y)| \le \frac{1}{2}(1)\left(\left|x - \frac{\pi}{4}\right| + \left|y - \frac{\pi}{4}\right|\right)^2 = \frac{1}{2}(0.2)^2 = 0.02$.

56. $f(1,1) = 0$, $f_x(1,1) = y|_{(1,1)} = 1$, $f_y(1,1) = x - 6y|_{(1,1)} = -5 \Rightarrow L(x,y) = (x - 1) - 5(y - 1) = x - 5y + 4$;

$f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -6$, and $f_{xy}(x,y) = 1 \Rightarrow$ maximum of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ is 6 $\Rightarrow M = 6$

$\Rightarrow |E(x,y)| \le \frac{1}{2}(6)(|x - 1| + |y - 1|)^2 = \frac{1}{2}(6)(0.1 + 0.2)^2 = 0.27$

57. $f(1,0,0)=0$, $f_x(1,0,0)=y-3z|_{(1,0,0)}=0$, $f_y(1,0,0)=x+2z|_{(1,0,0)}=1$, $f_z(1,0,0)=2y-3x|_{(1,0,0)}=-3$
$\Rightarrow L(x,y,z)=0(x-1)+(y-0)-3(z-0)=y-3z$; $f(1,1,0)=1$, $f_x(1,1,0)=1$, $f_y(1,1,0)=1$, $f_z(1,1,0)=-1$
$\Rightarrow L(x,y,z)=1+(x-1)+(y-1)-1(z-0)=x+y-z-1$

58. $f\left(0,0,\frac{\pi}{4}\right)=1$, $f_x\left(0,0,\frac{\pi}{4}\right)=-\sqrt{2}\sin x\sin(y+z)\big|_{\left(0,0,\frac{\pi}{4}\right)}=0$, $f_y\left(0,0,\frac{\pi}{4}\right)=\sqrt{2}\cos x\cos(y+z)\big|_{\left(0,0,\frac{\pi}{4}\right)}=1$,
$f_z\left(0,0,\frac{\pi}{4}\right)=\sqrt{2}\cos x\cos(y+z)\big|_{\left(0,0,\frac{\pi}{4}\right)}=1 \Rightarrow L(x,y,z)=1+1(y-0)+1\left(z-\frac{\pi}{4}\right)=1+y+z-\frac{\pi}{4}$;
$f\left(\frac{\pi}{4},\frac{\pi}{4},0\right)=\frac{\sqrt{2}}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4},0\right)=-\frac{\sqrt{2}}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4},0\right)=\frac{\sqrt{2}}{2}$, $f_z\left(\frac{\pi}{4},\frac{\pi}{4},0\right)=\frac{\sqrt{2}}{2}$
$\Rightarrow L(x,y,z)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}(z-0)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}z$

59. $V=\pi r^2h \Rightarrow dV=2\pi rh\,dr+\pi r^2\,dh \Rightarrow dV|_{(1.5,5280)}=2\pi(1.5)(5280)\,dr+\pi(1.5)^2\,dh=15{,}840\pi\,dr+2.25\pi\,dh$.
You should be more careful with the diameter since it has a greater effect on dV.

60. $df=(2x-y)\,dx+(-x+2y)\,dy \Rightarrow df|_{(1,2)}=3\,dy \Rightarrow$ f is more sensitive to changes in y; in fact, near the point $(1,2)$ a change in x does not change f.

61. $dI=\frac{1}{R}dV-\frac{V}{R^2}\,dR \Rightarrow dI\Big|_{(24,100)}=\frac{1}{100}\,dV-\frac{24}{100^2}\,dR \Rightarrow dI\Big|_{dV=-1,\,dR=-20}=-0.01+(480)(.0001)=0.038$,
or increases by 0.038 amps; % change in $V=(100)\left(-\frac{1}{24}\right)\approx-4.17\%$; % change in $R=\left(-\frac{20}{100}\right)(100)=-20\%$;
$I=\frac{24}{100}=0.24 \Rightarrow$ estimated % change in $I=\frac{dI}{I}\times100=\frac{0.038}{0.24}\times100\approx15.83\% \Rightarrow$ more sensitive to voltage change.

62. $A=\pi ab \Rightarrow dA=\pi b\,da+\pi a\,db \Rightarrow dA|_{(10,16)}=16\pi\,da+10\pi\,db$; $da=\pm0.1$ and $db=\pm0.1$
$\Rightarrow dA=\pm26\pi(0.1)=\pm2.6\pi$ and $A=\pi(10)(16)=160\pi \Rightarrow \left|\frac{dA}{A}\times100\right|=\left|\frac{2.6\pi}{160\pi}\times100\right|\approx1.625\%$

63. (a) $y=uv \Rightarrow dy=v\,du+u\,dv$; percentage change in $u\le2\% \Rightarrow |du|\le0.02$, and percentage change in $v\le3\%$
$\Rightarrow |dv|\le0.03$; $\frac{dy}{y}=\frac{v\,du+u\,dv}{uv}=\frac{du}{u}+\frac{dv}{v} \Rightarrow \left|\frac{dy}{y}\times100\right|=\left|\frac{du}{u}\times100+\frac{dv}{v}\times100\right|\le\left|\frac{du}{u}\times100\right|+\left|\frac{dv}{v}\times100\right|$
$\le2\%+3\%=5\%$
(b) $z=u+v \Rightarrow \frac{dz}{z}=\frac{du+dv}{u+v}=\frac{du}{u+v}+\frac{dv}{u+v}\le\frac{du}{u}+\frac{dv}{v}$ (since $u>0$, $v>0$)
$\Rightarrow \left|\frac{dz}{z}\times100\right|\le\left|\frac{du}{u}\times100+\frac{dv}{v}\times100\right|=\left|\frac{dy}{y}\times100\right|$

64. $C=\frac{7}{71.84w^{0.425}h^{0.725}} \Rightarrow C_w=\frac{(-0.425)(7)}{71.84w^{1.425}h^{0.725}}$ and $C_h=\frac{(-0.725)(7)}{71.84w^{0.425}h^{1.725}}$
$\Rightarrow dC=\frac{-2.975}{71.84w^{1.425}h^{0.725}}\,dw+\frac{-5.075}{71.84w^{0.425}h^{1.725}}\,dh$; thus when $w=70$ and $h=180$ we have
$dC|_{(70,180)}\approx-(0.00000225)\,dw-(0.0000149)\,dh \Rightarrow$ 1 cm error in height has more effect

65. $f_x(x,y) = 2x - y + 2 = 0$ and $f_y(x,y) = -x + 2y + 2 = 0 \Rightarrow x = -2$ and $y = -2 \Rightarrow (-2,-2)$ is the critical point; $f_{xx}(-2,-2) = 2$, $f_{yy}(-2,-2) = 2$, $f_{xy}(-2,-2) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2,-2) = -8$

66. $f_x(x,y) = 10x + 4y + 4 = 0$ and $f_y(x,y) = 4x - 4y - 4 = 0 \Rightarrow x = 0$ and $y = -1 \Rightarrow (0,-1)$ is the critical point; $f_{xx}(0,-1) = 10$, $f_{yy}(0,-1) = -4$, $f_{xy}(0,-1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -56 < 0 \Rightarrow$ saddle point with $f(0,-1) = 2$

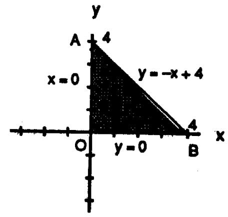
67. $f_x(x,y) = 6x^2 + 3y = 0$ and $f_y(x,y) = 3x + 6y^2 = 0 \Rightarrow y = -2x^2$ and $3x + 6(4x^4) = 0 \Rightarrow x(1 + 8x^3) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = -\frac{1}{2}$ and $y = -\frac{1}{2} \Rightarrow$ the critical points are $(0,0)$ and $\left(-\frac{1}{2},-\frac{1}{2}\right)$. For $(0,0)$: $f_{xx}(0,0) = 12x\big|_{(0,0)} = 0$, $f_{yy}(0,0) = 12y\big|_{(0,0)} = 0$, $f_{xy}(0,0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0,0) = 0$. For $\left(-\frac{1}{2},-\frac{1}{2}\right)$: $f_{xx} = -6$, $f_{yy} = -6$, $f_{xy} = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum value of $f\left(-\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{4}$

68. $f_x(x,y) = 3x^2 - 3y = 0$ and $f_y(x,y) = 3y^2 - 3x = 0 \Rightarrow y = x^2$ and $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow$ the critical points are $(0,0)$ and $(1,1)$. For $(0,0)$: $f_{xx}(0,0) = 6x\big|_{(0,0)} = 0$, $f_{yy}(0,0) = 6y\big|_{(0,0)} = 0$, $f_{xy}(0,0) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0,0) = 15$. For $(1,1)$: $f_{xx}(1,1) = 6$, $f_{yy}(1,1) = 6$, $f_{xy}(1,1) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(1,1) = 14$

69. $f_x(x,y) = 3x^2 + 6x = 0$ and $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow x(x+2) = 0$ and $y(y-2) = 0 \Rightarrow x = 0$ or $x = -2$ and $y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$. For $(0,0)$: $f_{xx}(0,0) = 6x + 6\big|_{(0,0)}$ $= 6$, $f_{yy}(0,0) = 6y - 6\big|_{(0,0)} = -6$, $f_{xy}(0,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(0,0) = 0$. For $(0,2)$: $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6$, $f_{xy}(0,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(0,2) = -4$. For $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum value of $f(-2,0) = 4$. For $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(-2,2) = 0$.

70. $f_x(x,y) = 4x^3 - 16x = 0 \Rightarrow 4x(x^2 - 4) = 0 \Rightarrow x = 0, 2, -2$; $f_y(x,y) = 6y - 6 = 0 \Rightarrow y = 1$. Therefore the critical points are $(0,1)$, $(2,1)$, and $(-2,1)$. For $(0,1)$: $f_{xx}(0,1) = 12x^2 - 16\big|_{(0,1)} = -16$, $f_{yy}(0,1) = 6$, $f_{xy}(0,1) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -96 < 0 \Rightarrow$ saddle point with $f(0,1) = -3$. For $(2,1)$: $f_{xx}(2,1) = 32$, $f_{yy}(2,1) = 6$, $f_{xy}(2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(2,1) = -19$. For $(-2,1)$: $f_{xx}(-2,1) = 32$, $f_{yy}(-2,1) = 6$, $f_{xy}(-2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2,1) = -19$.

71. (i) On OA, $f(x,y) = f(0,y) = y^2 + 3y$ for $0 \le y \le 4$ $\Rightarrow f'(0,y) = 2y + 3 = 0 \Rightarrow y = -\frac{3}{2}$. But $\left(0, -\frac{3}{2}\right)$ is not in the region.
Endpoints: $f(0,0) = 0$ and $f(0,4) = 28$.

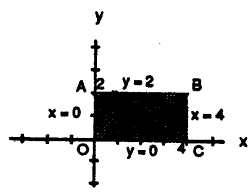


(ii) On AB, $f(x,y) = f(x,-x+4) = x^2 - 10x + 28$ for $0 \le x \le 4 \Rightarrow f'(x,-x+4) = 2x - 10 = 0$ $\Rightarrow x = 5,\ y = -1$. But $(5,-1)$ is not in the region.
Endpoints: $f(4,0) = 4$ and $f(0,4) = 28$.

(iii) On OB, $f(x,y) = f(x,0) = x^2 - 3x$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 3 \Rightarrow x = \frac{3}{2}$ and $y = 0 \Rightarrow \left(\frac{3}{2}, 0\right)$ is a critical point with $f\left(\frac{3}{2}, 0\right) = -\frac{9}{4}$. Endpoints: $f(0,0) = 0$ and $f(4,0) = 4$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x + y - 3 = 0$ and $f_y(x,y) = x + 2y + 3 = 0 \Rightarrow x = 3$ and $y = -3$. But $(3,-3)$ is not in the region. Therefore the absolute maximum is 28 at $(0,4)$ and the absolute minimum is $-\frac{9}{4}$ at $\left(\frac{3}{2}, 0\right)$.

72. (i) On OA, $f(x,y) = f(0,y) = -y^2 + 4y + 1$ for $0 \le y \le 2$ $\Rightarrow f'(0,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 0$. But $(0,2)$ is not in the interior of OA.
Endpoints: $f(0,0) = 1$ and $f(0,2) = 5$.



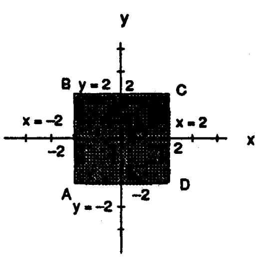
(ii) On AB, $f(x,y) = f(x,2) = x^2 - 2x + 5$ for $0 \le x \le 4$ $\Rightarrow f'(x,2) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 4$.
Endpoints: $f(4,2) = 13$ and $f(0,2) = 5$.

(iii) On BC, $f(x,y) = f(4,y) = -y^2 + 4y + 9$ for $0 \le y \le 2 \Rightarrow f'(4,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 4$. But $(4,2)$ is not in the interior of BC. Endpoints: $f(4,0) = 9$ and $f(4,2) = 13$.

(iv) On OC, $f(x,y) = f(x,0) = x^2 - 2x + 1$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 0$. Endpoints: $f(0,0) = 1$ and $f(4,0) = 9$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2$. But $(1,2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4,2)$ and the absolute minimum is 0 at $(1,0)$.

73. (i) On AB, $f(x,y) = f(-2,y) = y^2 - y - 4$ for $-2 \le y \le 2$ $\Rightarrow f'(-2,y) = 2y - 1 \Rightarrow y = \frac{1}{2}$ and $x = -2 \Rightarrow \left(-2, \frac{1}{2}\right)$ is an interior critical point in AB with $f\left(-2, \frac{1}{2}\right) = -\frac{17}{4}$.
Endpoints: $f(-2,-2) = 2$ and $f(2,2) = -2$.



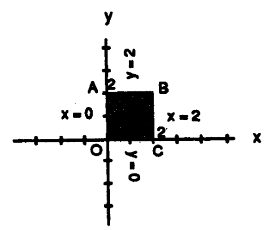
(ii) On BC, $f(x,y) = f(x,2) = -2$ for $-2 \le x \le 2 \Rightarrow f'(x,2) = 0 \Rightarrow$ no critical points in the interior of BC. Endpoints: $f(-2,2) = -2$ and $f(2,2) = -2$.

(iii) On CD, $f(x,y) = f(2,y) = y^2 - 5y + 4$ for $-2 \le y \le 2$ $\Rightarrow f'(2,y) = 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ and $x = 2$. But $\left(2, \frac{5}{2}\right)$ is not in the region. Endpoints: $f(2,-2) = 18$ and $f(2,2) = -2$.

(iv) On AD, $f(x,y) = f(x,-2) = 4x + 10$ for $-2 \le x \le 2 \Rightarrow f'(x,-2) = 4 \Rightarrow$ no critical points in the interior of AD. Endpoints: $f(-2,-2) = 2$ and $f(2,-2) = 18$.

(v) For the interior of the square, $f_x(x,y) = -y + 2 = 0$ and $f_y(x,y) = 2y - x - 3 = 0 \Rightarrow y = 2$ and $x = 1$ $\Rightarrow (1,2)$ is an interior critical point of the square with $f(1,2) = -2$. Therefore the absolute maximum is 18 at $(2,-2)$ and the absolute minimum is $-\frac{17}{4}$ at $\left(-2, \frac{1}{2}\right)$.

74. (i) On OA, $f(x,y) = f(0,y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(0,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 0 \Rightarrow (0,1)$ is an interior critical point of OA with $f(0,1) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.



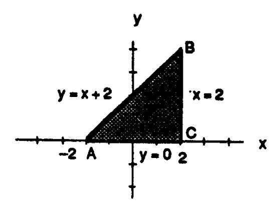
(ii) On AB, $f(x,y) = f(x,2) = 2x - x^2$ for $0 \le x \le 2$ $\Rightarrow f'(x,2) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 1$. Endpoints: $f(0,2) = 0$ and $f(2,2) = 0$.

(iii) On BC, $f(x,y) = f(2,y) = 2y - y^2$ for $0 \le y \le 2 \Rightarrow f'(2,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 2$ $\Rightarrow (2,1)$ is an interior critical point of BC with $f(2,1) = 1$. Endpoints: $f(2,0) = 0$ and $f(2,2) = 0$.

(iv) On OC, $f(x,y) = f(x,0) = 2x - x^2$ for $0 \le x \le 2 \Rightarrow f'(x,0) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2 - 2x = 0$ and $f_y(x,y) = 2 - 2y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow (1,1)$ is an interior critical point of the square with $f(1,1) = 2$. Therefore the absolute maximum is 2 at $(1,1)$ and the absolute minimum is 0 at the four corners $(0,0)$, $(0,2)$, $(2,2)$, and $(2,0)$.

75. (i) On AB, $f(x,y) = f(x,x+2) = -2x + 4$ for $-2 \le x \le 2$ $\Rightarrow f'(x,x+2) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints: $f(-2,0) = 8$ and $f(2,4) = 0$.



(ii) On BC, $f(x,y) = f(2,y) = -y^2 + 4y$ for $0 \le y \le 4$ $\Rightarrow f'(2,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2 \Rightarrow (2,2)$ is an interior critical point of BC with $f(2,2) = 4$. Endpoints: $f(2,0) = 0$ and $f(2,4) = 0$.

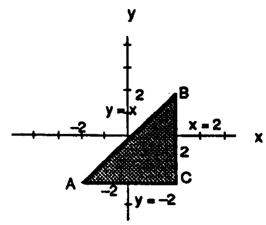
(iii) On AC, $f(x,y) = f(x,0) = x^2 - 2x$ for $-2 \le x \le 2$ $\Rightarrow f'(x,0) = 2x - 2 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of AC with $f(1,0) = -1$.

Endpoints: $f(-2,0) = 8$ and $f(2,0) = 0$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is an interior critical point of the region with $f(1,2) = 3$. Therefore the absolute maximum is 8 at $(-2,0)$ and the absolute minimum is -1 at $(1,0)$.

76. (i) On AB, $f(x,y) = f(x,x) = 4x^2 - 2x^4 + 16$ for $-2 \le x \le 2$ $\Rightarrow f'(x,x) = 8x - 8x^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$, or $x = -1$ and $y = -1 \Rightarrow (0,0)$, $(1,1)$, $(-1,-1)$ are all interior points of AB with $f(0,0) = 16$, $f(1,1) = 18$, and $f(-1,-1) = 18$.

Endpoints: $f(-2,-2) = 0$ and $f(2,2) = 0$.



(ii) On BC, $f(x,y) = f(2,y) = 8y - y^4$ for $-2 \le y \le 2$ $\Rightarrow f'(2,y) = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$ and $x = 2 \Rightarrow (2, \sqrt[3]{2})$ is an interior critical point of BC with $f(2,\sqrt[3]{2}) = 6\sqrt[3]{2}$.

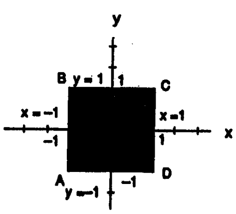
Endpoints: $f(2,-2) = -32$ and $f(2,2) = 0$.

(iii) On AC, $f(x,y) = f(x,-2) = -8x - x^4$ for $-2 \le x \le 2 \Rightarrow f'(x,-2) = -8 - 4x^3 = 0 \Rightarrow x = \sqrt[3]{-2}$ and $y = -2$ $\Rightarrow (\sqrt[3]{-2}, -2)$ is an interior critical point of AC with $f(\sqrt[3]{-2}, -2) = 6\sqrt[3]{2}$. Endpoints: $f(-2,-2) = 0$ and $f(2,-2) = -32$.

(iv) For the interior of the triangular region, $f_x(x,y) = 4y - 4x^3 = 0$ and $f_y(x,y) = 4x - 4y^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$. But neither of the points $(0,0)$ and $(1,1)$ are interior to the region. Therefore the absolute maximum is 18 at $(1,1)$ and $(-1,-1)$, and the absolute minimum is -32 at $(2,-2)$.

77. (i) On AB, $f(x,y) = f(-1,y) = y^3 - 3y^2 + 2$ for $-1 \le y \le 1$ $\Rightarrow f'(-1,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = -1$, or $y = 2$ and $x = -1 \Rightarrow (-1,0)$ is an interior critical point of AB with $f(-1,0) = 2$; $(-1,2)$ is outside the boundary.

Endpoints: $f(-1,-1) = -2$ and $f(-1,1) = 0$.



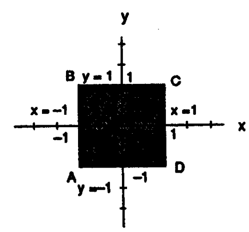
(ii) On BC, $f(x,y) = f(x,1) = x^3 + 3x^2 - 2$ for $-1 \le x \le 1 \Rightarrow f'(x,1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = 1$, or $x = 2$ and $y = 1 \Rightarrow (0,1)$ is an interior critical point of BC with $f(0,1) = -2$; $(2,1)$ is outside the boundary. Endpoints: $f(-1,1) = 0$ and $f(1,1) = 2$.

(iii) On CD, $f(x,y) = f(1,y) = y^3 - 3y^2 + 4$ for $-1 \le y \le 1 \Rightarrow f'(1,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = 1$, or $y = 2$ and $x = 1 \Rightarrow (1,0)$ is an interior critical point of CD with $f(1,0) = 4$; $(1,2)$ is outside the boundary. Endpoints: $f(1,1) = 2$ and $f(1,-1) = 0$.

(iv) On AD, $f(x,y) = f(x,-1) = x^3 + 3x^2 - 4$ for $-1 \le x \le 1 \Rightarrow f'(x,-1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = -1$, or $x = -2$ and $y = -1 \Rightarrow (0,-1)$ is an interior point of AD with $f(0,-1) = -4$; $(-2,-1)$ is outside the boundary. Endpoints: $f(-1,-1) = -2$ and $f(1,-1) = 0$.

(v) For the interior of the square, $f_x(x,y) = 3x^2 + 6x = 0$ and $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow x = 0$ or $x = -2$, and $y = 0$ or $y = 2 \Rightarrow (0,0)$ is an interior critical point of the square region with $f(0,0) = 0$; the points $(0,2)$, $(-2,0)$, and $(-2,2)$ are outside the region. Therefore the absolute maximum is 4 at $(1,0)$ and the absolute minimum is -4 at $(0,-1)$.

78. (i) On AB, $f(x,y) = f(-1,y) = y^3 - 3y$ for $-1 \le y \le 1$ $\Rightarrow f'(-1,y) = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$ and $x = -1$ yielding the corner points $(-1,-1)$ and $(-1,1)$ with $f(-1,-1) = 2$ and $f(-1,1) = -2$.



(ii) On BC, $f(x,y) = f(x,1) = x^3 + 3x + 2$ for $-1 \le x \le 1$ $\Rightarrow f'(x,1) = 3x^2 + 3 = 0 \Rightarrow$ no solution.
Endpoints: $f(-1,1) = -2$ and $f(1,1) = 6$.

(iii) On CD, $f(x,y) = f(1,y) = y^3 + 3y + 2$ for $-1 \le y \le 1$ $\Rightarrow f'(1,y) = 3y^2 + 3 = 0 \Rightarrow$ no solution.
Endpoints: $f(1,1) = 6$ and $f(1,-1) = -2$.

(iv) On AD, $f(x,y) = f(x,-1) = x^3 - 3x$ for $-1 \le x \le 1 \Rightarrow f'(x,-1) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ and $y = -1$ yielding the corner points $(-1,-1)$ and $(1,-1)$ with $f(-1,-1) = 2$ and $f(1,-1) = -2$

(v) For the interior of the square, $f_x(x,y) = 3x^2 + 3y = 0$ and $f_y(x,y) = 3y^2 + 3x = 0 \Rightarrow y = -x^2$ and $x^4 + x = 0 \Rightarrow x = 0$ or $x = -1 \Rightarrow y = 0$ or $y = -1 \Rightarrow (0,0)$ is an interior critical point of the square region with $f(0,0) = 1$; $(-1,-1)$ is on the boundary. Therefore the absolute maximum is 6 at $(1,1)$ and the absolute minimum is -2 at $(1,-1)$ and $(-1,1)$.

79. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3x^2\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 3x^2 = 2x\lambda$ and $2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$.

CASE 1: $\lambda = 1 \Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$; $x = 0 \Rightarrow y = \pm 1$ yielding the points $(0,1)$ and $(0,-1)$; $x = \frac{2}{3}$ $\Rightarrow y = \pm\frac{\sqrt{5}}{3}$ yielding the points $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$.

CASE 2: $y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ yielding the points $(1,0)$ and $(-1,0)$.

Evaluations give $f(0, \pm 1) = 1$, $f\left(\frac{2}{3}, \pm\frac{\sqrt{5}}{3}\right) = \frac{23}{27}$, $f(1,0) = 1$, and $f(-1,0) = -1$. Therefore the absolute maximum is 1 at $(0, \pm 1)$ and $(1,0)$, and the absolute minimum is -1 at $(-1,0)$.

80. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x = 0$ or $4\lambda^2 = 1$.

CASE 1: $x = 0 \Rightarrow y = 0$ but $(0,0)$ does not lie on the circle, so no solution.

CASE 2: $4\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = y = \pm\frac{1}{\sqrt{2}}$ yielding the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\lambda = -\frac{1}{2}$, $y = -x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ and

$y = -x$ yielding the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Evaluations give the absolute maximum value $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and the absolute minimum value $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$.

81. (i) $f(x,y) = x^2 + 3y^2 + 2y$ on $x^2 + y^2 = 1 \Rightarrow \nabla f = 2x\mathbf{i} + (6y+2)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2x\mathbf{i} + (6y+2)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x = 2x\lambda$ and $6y + 2 = 2y\lambda \Rightarrow \lambda = 1$ or $x = 0$.

CASE 1: $\lambda = 1 \Rightarrow 6y + 2 = 2y \Rightarrow y = -\frac{1}{2}$ and $x = \pm\frac{\sqrt{3}}{2}$ yielding the points $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

CASE 2: $x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ yielding the points $(0, \pm 1)$.

Evaluations give $f\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{1}{2}$, $f(0,1) = 5$, and $f(0,-1) = 1$. Therefore $\frac{1}{2}$ and 5 are the extreme values on the boundary of the disk.

(ii) For the interior of the disk, $f_x(x,y) = 2x = 0$ and $f_y(x,y) = 6y + 2 = 0 \Rightarrow x = 0$ and $y = -\frac{1}{3}$ $\Rightarrow \left(0, -\frac{1}{3}\right)$ is an interior critical point with $f\left(0, -\frac{1}{3}\right) = -\frac{1}{3}$. Therefore the absolute maximum of f on the disk is 5 at $(0,1)$ and the absolute minimum of f on the disk is $-\frac{1}{3}$ at $\left(0, -\frac{1}{3}\right)$.

82. (i) $f(x,y) = x^2 + y^2 - 3x - xy$ on $x^2 + y^2 = 9 \Rightarrow \nabla f = (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x - 3 - y = 2x\lambda$ and $2y - x = 2y\lambda$ $\Rightarrow 2x(1-\lambda) - y = 3$ and $-x + 2y(1-\lambda) = 0 \Rightarrow 1 - \lambda = \frac{x}{2y}$ and $(2x)\left(\frac{x}{2y}\right) - y = 3 \Rightarrow x^2 - y^2 = 3y$ $\Rightarrow x^2 = y^2 + 3y$. Thus, $9 = x^2 + y^2 = y^2 + 3y + y^2 \Rightarrow 2y^2 + 3y - 9 = 0 \Rightarrow (2y - 3)(y + 3) = 0$ $\Rightarrow y = -3, \frac{3}{2}$. For $y = -3$, $x^2 + y^2 = 9 \Rightarrow x = 0$ yielding the point $(0, -3)$. For $y = \frac{3}{2}$, $x^2 + y^2 = 9$ $\Rightarrow x^2 + \frac{9}{4} = 9 \Rightarrow x^2 = \frac{27}{4} \Rightarrow x = \pm\frac{3\sqrt{3}}{2}$. Evaluations give $f(0,-3) = 9$, $f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4}$ ≈ 20.691, and $f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4} \approx -2.691$.

(ii) For the interior of the disk, $f_x(x,y) = 2x - 3 - y = 0$ and $f_y(x,y) = 2y - x = 0 \Rightarrow x = 2$ and $y = 1$ $\Rightarrow (2,1)$ is an interior critical point of the disk with $f(2,1) = -3$. Therefore, the absolute maximum of f on the disk is $9 + \frac{27\sqrt{3}}{4}$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and the absolute minimum of f on the disk is -3 at $(2,1)$.

83. $\nabla f = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-1 = 2y\lambda$, $1 = 2z\lambda \Rightarrow x = -y = z = \frac{1}{2\lambda}$. Thus $x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ yielding the points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Evaluations give the absolute maximum value of $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and the absolute minimum value of $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$.

84. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and $g(x,y,z) = z^2 - xy - 4$. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = -\lambda y$, $2y = -\lambda x$, and $2z = 2\lambda z$ $\Rightarrow z = 0$ or $\lambda = 1$.

CASE 1: $z = 0 \Rightarrow xy = -4 \Rightarrow x = -\frac{4}{y}$ and $y = -\frac{4}{x} \Rightarrow 2\left(-\frac{4}{y}\right) = -\lambda y$ and $2\left(-\frac{4}{x}\right) = -\lambda x \Rightarrow \frac{8}{\lambda} = y^2$ and $\frac{8}{\lambda} = x^2$ $\Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. But $y = x \Rightarrow x^2 = -4$ leads to no solution, so $y = -x \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ yielding the points $(-2,2,0)$ and $(2,-2,0)$.

CASE 2: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow 2y = -\left(-\frac{y}{2}\right) \Rightarrow 4y = y \Rightarrow y = 0 \Rightarrow x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0,0,-2)$ and $(0,0,2)$.

Evaluations give $f(-2,2,0) = f(2,-2,0) = 8$ and $f(0,0,-2) = f(0,0,2) = 4$. Thus the points $(0,0,-2)$ and $(0,0,2)$ on the surface are closest to the origin.

85. The cost is $f(x,y,z) = 2axy + 2bxz + 2cyz$ subject to the constraint $xyz = V$. Then $\nabla f = \lambda \nabla g$ $\Rightarrow 2ay + 2bz = \lambda yz$, $2ax + 2cz = \lambda xz$, and $2bx + 2cy = \lambda xy \Rightarrow 2axy + 2bxz = \lambda xyz$, $2axy + 2cyz = \lambda xyz$, and $2bxz + 2cyz = \lambda xyz \Rightarrow 2axy + 2bxz = 2axy + 2cyz \Rightarrow y = \left(\frac{b}{c}\right)x$. Also $2axy + 2bxz = 2bxz + 2cyz \Rightarrow z = \left(\frac{a}{c}\right)x$. Then $x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) = V \Rightarrow x^3 = \frac{c^2V}{ab} \Rightarrow \text{width} = x = \left(\frac{c^2V}{ab}\right)^{1/3}$, Depth $= y = \left(\frac{b}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{b^2V}{ac}\right)^{1/3}$, and Height $= z = \left(\frac{a}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{a^2V}{bc}\right)^{1/3}$.

86. The volume of the pyramid in the first octant formed by the plane is $V(a,b,c) = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$. The point $(2,1,2)$ on the plane $\Rightarrow \frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$. We want to maximize V subject to the constraint $2bc + ac + 2ab = abc$. Thus, $\nabla V = \frac{bc}{6}\mathbf{i} + \frac{ac}{6}\mathbf{j} + \frac{ab}{6}\mathbf{k}$ and $\nabla g = (c + 2b - bc)\mathbf{i} + (2c + 2a - ac)\mathbf{j} + (2b + a - ab)\mathbf{k}$ so that $\nabla V = \lambda \nabla g$ $\Rightarrow \frac{bc}{6} = \lambda(c + 2b - bc)$, $\frac{ac}{6} = \lambda(2c + 2a - ac)$, and $\frac{ab}{6} = \lambda(2b + a - ab) \Rightarrow \frac{abc}{6} = \lambda(ac + 2ab - abc)$, $\frac{abc}{6} = \lambda(2bc + 2ab - abc)$, and $\frac{abc}{6} = \lambda(2bc + ac - abc) \Rightarrow \lambda ac = 2\lambda bc$ and $2\lambda ab = 2\lambda bc$. Now $\lambda \neq 0$ since $a \neq 0$, $b \neq 0$, and $c \neq 0 \Rightarrow ac = 2bc$ and $ab = bc \Rightarrow a = 2b = c$. Substituting into the constraint equation gives $\frac{2}{a} + \frac{2}{a} + \frac{2}{a} = 1 \Rightarrow a = 6 \Rightarrow b = 3$ and $c = 6$. Therefore the desired plane is $\frac{x}{6} + \frac{y}{3} + \frac{z}{6} = 1$ or $x + 2y + z = 6$.

87. $\nabla f = (y + z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h$ $\Rightarrow (y + z)\mathbf{i} + x\mathbf{j} + x\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(z\mathbf{i} + x\mathbf{k}) \Rightarrow y + z = 2\lambda x + \mu z$, $x = 2\lambda y$, $x = \mu x \Rightarrow x = 0$ or $\mu = 1$.

CASE 1: $x = 0$ which is impossible since $xz = 1$.

CASE 2: $\mu = 1 \Rightarrow y + z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz = 1$ we obtain the points $(1,0,1)$ and $(-1,0,-1)$. If $4\lambda^2 = 1$, then $\lambda = \pm\frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y = -x$ so $x^2 + y^2 = 1 \Rightarrow x^2 = \frac{1}{2}$ $\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.

Evaluations give $f(1,0,1) = 1$, $f(-1,0,-1) = 1$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{3}{2}$, and $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$, and the absolute minimum is $\frac{1}{2}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

88. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla \mathbf{g} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\nabla h = 4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla \mathbf{g} + \mu \nabla h \Rightarrow 2x = \lambda + 4x\mu$, $2y = \lambda + 4y\mu$, and $2z = \lambda - 2z\mu \Rightarrow \lambda = 2x(1 - 2\mu) = 2y(1 - 2\mu) = 2z(1 + 2\mu) \Rightarrow x = y$ or $\mu = \frac{1}{2}$.
CASE 1: $x = y \Rightarrow z^2 = 4x^2 \Rightarrow z = \pm 2x$ so that $x + y + z = 1 \Rightarrow x + x + 2x = 1$ or $x + x - 2x = 1$ (impossible) $\Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4}$ and $z = \frac{1}{2}$ yielding the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$.
CASE 2: $\mu = \frac{1}{2} \Rightarrow \lambda = 0 \Rightarrow 0 = 2z(1+1) \Rightarrow z = 0$ so that $2x^2 + 2y^2 = 0 \Rightarrow x = y = 0$. But the origin $(0,0,0)$ fails to satisfy the first constraint $x + y + z = 1$.
Therefore, the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

89. (a) y, z are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$ $= \left(2xe^{yz}\right)\frac{\partial x}{\partial y} + \left(zx^2e^{yz}\right)(1) + \left(yx^2e^{yz}\right)(0)$; $z = x^2 - y^2 \Rightarrow 0 = 2x\frac{\partial x}{\partial y} - 2y \Rightarrow \frac{\partial x}{\partial y} = \frac{y}{x}$; therefore, $\left(\frac{\partial w}{\partial y}\right)_z = \left(2xe^{yz}\right)\left(\frac{y}{x}\right) + zx^2e^{yz} = \left(2y + zx^2\right)e^{yz}$

(b) z, x are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= \left(2xe^{yz}\right)(0) + \left(zx^2e^{yz}\right)\frac{\partial y}{\partial z} + \left(yx^2e^{yz}\right)(1)$; $z = x^2 - y^2 \Rightarrow 1 = 0 - 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = -\frac{1}{2y}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_x = \left(zx^2e^{yz}\right)\left(-\frac{1}{2y}\right) + yx^2e^{yz} = x^2e^{yz}\left(y - \frac{z}{2y}\right)$

(c) z, y are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= \left(2xe^{yz}\right)\frac{\partial x}{\partial z} + \left(zx^2e^{yz}\right)(0) + \left(yx^2e^{yz}\right)(1)$; $z = x^2 - y^2 \Rightarrow 1 = 2x\frac{\partial x}{\partial z} - 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_y = \left(2xe^{yz}\right)\left(\frac{1}{2x}\right) + yx^2e^{yz} = \left(1 + x^2y\right)e^{yz}$

90. (a) T, P are independent with $U = f(P,V,T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial T} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T}$
$= \left(\frac{\partial U}{\partial P}\right)(0) + \left(\frac{\partial U}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right) + \left(\frac{\partial U}{\partial T}\right)(1)$; $PV = nRT \Rightarrow P\frac{\partial V}{\partial T} = nR \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$; therefore,
$\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial V}\right)\left(\frac{nR}{P}\right) + \frac{\partial U}{\partial T}$

(b) V, T are independent with $U = f(P,V,T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial V} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial V} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial V} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial V}$
$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{\partial P}{\partial V}\right) + \left(\frac{\partial U}{\partial V}\right)(1) + \left(\frac{\partial U}{\partial T}\right)(0)$; $PV = nRT \Rightarrow V\frac{\partial P}{\partial V} + P = (nR)\left(\frac{\partial T}{\partial V}\right) = 0 \Rightarrow \frac{\partial P}{\partial V} = -\frac{P}{V}$; therefore,
$\left(\frac{\partial U}{\partial V}\right)_T = \left(\frac{\partial U}{\partial P}\right)\left(-\frac{P}{V}\right) + \frac{\partial U}{\partial V}$

91. Note that $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Thus,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{-y}{x^2+y^2}\right) = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$;
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial y} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{x}{x^2+y^2}\right) = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$

92. $z_x = f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x} = af_u + af_v$, and $z_y = f_u\frac{\partial u}{\partial y} + f_v\frac{\partial v}{\partial y} = bf_u - bf_v$

93. $\frac{\partial u}{\partial y} = b$ and $\frac{\partial u}{\partial x} = a \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = a\frac{dw}{du}$ and $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = b\frac{dw}{du} \Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{dw}{du}$ and $\frac{1}{b}\frac{\partial w}{\partial y} = \frac{dw}{du}$
$\Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{1}{b}\frac{\partial w}{\partial y} \Rightarrow b\frac{\partial w}{\partial x} = a\frac{\partial w}{\partial y}$

94. $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+2z} = \frac{2(r+s)}{(r+s)^2+(r-s)^2+4rs} = \frac{2(r+s)}{2(r^2+2rs+s^2)} = \frac{1}{r+s}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+2z} = \frac{2(r-s)}{2(r+s)^2} = \frac{r-s}{(r+s)^2}$,
and $\frac{\partial w}{\partial z} = \frac{2}{x^2+y^2+2z} = \frac{1}{(r+s)^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{r+s} + \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2s) = \frac{2r+2s}{(r+s)^2}$
$= \frac{2}{r+s}$ and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{r+s} - \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2r) = \frac{2}{r+s}$

95. $e^u\cos v - x = 0 \Rightarrow (e^u\cos v)\frac{\partial u}{\partial x} - (e^u\sin v)\frac{\partial v}{\partial x} = 1$; $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial x} + (e^u\cos v)\frac{\partial v}{\partial x} = 0$.
Solving this system yields $\frac{\partial u}{\partial x} = e^{-u}\cos v$ and $\frac{\partial v}{\partial x} = -e^{-u}\sin v$. Similarly, $e^u\cos v - x = 0$
$\Rightarrow (e^u\cos v)\frac{\partial u}{\partial y} - (e^u\sin v)\frac{\partial v}{\partial y} = 0$ and $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial y} + (e^u\cos v)\frac{\partial v}{\partial y} = 1$. Solving this second system yields $\frac{\partial u}{\partial y} = e^{-u}\sin v$ and $\frac{\partial v}{\partial y} = e^{-u}\cos v$. Therefore $\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right)\cdot\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right)$
$= [(e^{-u}\cos v)\mathbf{i} + (e^{-u}\sin v)\mathbf{j}]\cdot[(-e^{-u}\sin v)\mathbf{i} + (e^{-u}\cos v)\mathbf{j}] = 0 \Rightarrow$ the vectors are orthogonal $\Rightarrow$ the angle between the vectors is the constant $\frac{\pi}{2}$.

96. $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (-r \sin \theta)\frac{\partial f}{\partial x} + (r \cos \theta)\frac{\partial f}{\partial y}$

$\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = (-r \sin \theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial \theta}\right) - (r \cos \theta)\frac{\partial f}{\partial x} + (r \cos \theta)\left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) - (r \sin \theta)\frac{\partial f}{\partial y}$

$= (-r \sin \theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r \cos \theta) + (r \cos \theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r \sin \theta)$

$= (-r \sin \theta + r \cos \theta)(-r \sin \theta + r \cos \theta) - (r \cos \theta + r \sin \theta) = (-2)(-2) - (0+2) = 4 - 2 = 2$ at $(r, \theta) = \left(2, \frac{\pi}{2}\right)$.

97. $(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y + z = \pm 4$. Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t \pm 4, t)$, t a real number.

98. Let $f(x,y,z) = xy + yz + zx - x - z^2 = 0$. If the tangent plane is to be parallel to the xy-plane, then ∇f is perpendicular to the xy-plane $\Rightarrow \nabla f \cdot \mathbf{i} = 0$ and $\nabla f \cdot \mathbf{j} = 0$. Now $\nabla f = (y+z-1)\mathbf{i} + (x+z)\mathbf{j} + (y+x-2z)\mathbf{k}$ so that $\nabla f \cdot \mathbf{i} = y + z - 1 = 0 \Rightarrow y + z = 1 \Rightarrow y = 1 - z$, and $\nabla f \cdot \mathbf{j} = x + z = 0 \Rightarrow x = -z$. Then $-z(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \Rightarrow z - 2z^2 = 0 \Rightarrow z = \frac{1}{2}$ or $z = 0$. Now $z = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$ and $y = \frac{1}{2}$ $\Rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is one desired point; $z = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow (0,1,0)$ is a second desired point.

99. $\nabla f = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \Rightarrow \frac{\partial f}{\partial x} = \lambda x \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + g(y,z)$ for some function $g \Rightarrow \lambda y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$ $\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + h(z)$ for some function $h \Rightarrow yz = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = h'(z) \Rightarrow h(z) = \frac{1}{2}\lambda z^2 + C$ for some arbitrary constant $C \Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + \left(\frac{1}{2}\lambda z^2 + C\right) \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda y^2 + \frac{1}{2}\lambda z^2 + C \Rightarrow f(0,0,a) = \frac{1}{2}\lambda a^2 + C$ and $f(0,0,-a) = \frac{1}{2}\lambda(-a)^2 + C \Rightarrow f(0,0,a) = f(0,0,-a)$ for any constant a, as claimed.

100. $$\left(\frac{df}{ds}\right)_{\mathbf{u},(0,0,0)} = \lim_{s \to 0} \frac{f(0+su_1, 0+su_2, 0+su_3) - f(0,0,0)}{s},\ s > 0$$

$$= \lim_{s \to 0} \frac{\sqrt{s^2u_1^2 + s^2u_2^2 + s^2u_3^2} - 0}{s},\ s > 0$$

$$= \lim_{s \to 0} \frac{s\sqrt{u_1^2 + u_2^2 + u_3^2}}{s} = \lim_{s \to 0} |\mathbf{u}| = 1;$$

however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ fails to exist at the origin $(0,0,0)$

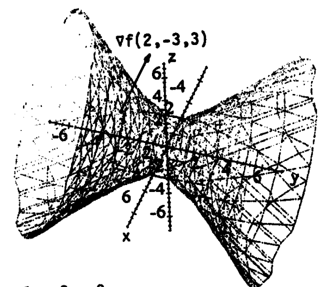
101. Let $f(x,y,z) = xy + z - 2 \Rightarrow \nabla f = y\mathbf{i} + z\mathbf{j} + \mathbf{k}$. At $(1,1,1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the normal line is $x = 1 + t,\ y = 1 + t,\ z = 1 + t$, so at $t = -1 \Rightarrow x = 0,\ y = 0,\ z = 0$ and the normal line passes through the origin.

102. (b) $f(x,y,z) = x^2 - y^2 + z^2 = 4$

$\Rightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$ at $(2,-3,3)$ the gradient is $\nabla f = 4\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ which is normal to the surface

(c) Tangent plane: $4x + 6y + 6z = 8$ or $2x + 3y + 3z = 4$

Normal line: $x = 2 + 4t,\ y = -3 + 6t,\ z = 3 + 6t$



CHAPTER 11 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. By definition, $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$ so we need to calculate the first partial derivatives in the numerator. For $(x,y) \neq (0,0)$ we calculate $f_x(x,y)$ by applying the differentiation rules to the formula for $f(x,y)$: $f_x(x,y) = \frac{x^2y - y^3}{x^2+y^2} + (xy)\frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{x^2y-y^3}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} \Rightarrow f_x(0,h) = -\frac{h^3}{h^2} = -h.$ For $(x,y) = (0,0)$ we apply the definition: $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$. Then by definition $f_{xy}(0,0) = \lim_{h\to 0} \frac{-h-0}{h} = -1$. Similarly, $f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$, so for $(x,y) \neq (0,0)$ we have $f_y(x,y) = \frac{x^3 - xy^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} \Rightarrow f_y(h,0) = \frac{h^3}{h^2} = h$; for $(x,y) = (0,0)$ we obtain $f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$. Then by definition $f_{yx}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$. Note that $f_{xy}(0,0) \neq f_{yx}(0,0)$ in this case.

2. $\frac{\partial w}{\partial x} = 1 + e^x \cos y \Rightarrow w = x + e^x \cos y + g(y)$; $\frac{\partial w}{\partial y} = -e^x \sin y + g'(y) = 2y - e^x \sin y \Rightarrow g'(y) = 2y$ $\Rightarrow g(y) = y^2 + C$; $w = \ln 2$ when $x = \ln 2$ and $y = 0 \Rightarrow \ln 2 = \ln 2 + e^{\ln 2} \cos 0 + 0^2 + C \Rightarrow 0 = 2 + C$ $\Rightarrow C = -2$. Thus, $w = x + e^x \cos y + g(y) = x + e^x \cos y + y^2 - 2$.

3. Substitution of $u + u(x)$ and $v = v(x)$ in $g(u,v)$ gives $g(u(x),v(x))$ which is a function of the independent variable x. Then, $g(u,v) = \int_u^v f(t)\,dt \Rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx} = \left(\frac{\partial}{\partial u}\int_u^v f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$ $= \left(-\frac{\partial}{\partial u}\int_v^u f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx} = -f(u(x))\frac{du}{dx} + f(v(x))\frac{dv}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$

4. Applying the chain rules, $f_x = \frac{df}{dr}\frac{\partial r}{\partial x} \Rightarrow f_{xx} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$. Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial y^2}$ and $f_{zz} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial z^2}$. Moreover, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$ $\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; and $\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$. Next, $f_{xx} + f_{yy} + f_{zz} = 0$

$$\Rightarrow \left(\frac{d^2f}{dr^2}\right)\left(\frac{x^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) + \left(\frac{d^2f}{dr^2}\right)\left(\frac{y^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)$$

$$+ \left(\frac{d^2f}{dr^2}\right)\left(\frac{z^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = 0 \Rightarrow \frac{d^2f}{dr^2} + \left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)\frac{df}{dr} = 0 \Rightarrow \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$$

$\Rightarrow \frac{d}{dr}(f') = \left(-\frac{2}{r}\right)f'$, where $f' = \frac{df}{dr} \Rightarrow \frac{df'}{f'} = -\frac{2\,dr}{r} \Rightarrow \ln f' = -2\ln r + \ln C \Rightarrow f' = Cr^{-2}$, or $\frac{df}{dr} = Cr^{-2} \Rightarrow f(r) = -\frac{C}{r} + b = \frac{a}{r} + b$ for some constants a and b (setting $a = -C$)

5. (a) Let $u = tx$, $v = ty$, and $w = f(u,v) = f(u(t,x), v(t,y)) = f(tx,ty) = t^n f(x,y)$, where t, x, and y are independent variables. Then $nt^{n-1}f(x,y) = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Now, $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial u}\right)(t) + \left(\frac{\partial w}{\partial v}\right)(0) = t\frac{\partial w}{\partial u} \Rightarrow \frac{\partial w}{\partial u} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial x}\right)$. Likewise, $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \left(\frac{\partial w}{\partial u}\right)(0) + \left(\frac{\partial w}{\partial v}\right)(t) \Rightarrow \frac{\partial w}{\partial v} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. Therefore, $nt^{n-1}f(x,y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} = \left(\frac{x}{t}\right)\left(\frac{\partial w}{\partial x}\right) + \left(\frac{y}{t}\right)\left(\frac{\partial w}{\partial y}\right) = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}$. When $t = 1$, $u = x$, $v = y$, and $w = f(x,y) \Rightarrow \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}$ and $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial x} \Rightarrow nf(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, as claimed.

(b) From part (a), $nt^{n-1}f(x,y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Differentiating with respect to t again we obtain $n(n-1)t^{n-2}f(x,y) = x\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial t} + x\frac{\partial^2 w}{\partial v\partial w}\frac{\partial v}{\partial t} + y\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial t} + y\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial t} = x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial u\partial v} + y^2\frac{\partial^2 w}{\partial v^2}$.
Also from part (a), $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial x} = t^2\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)$ $= \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial v}\right) = t\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y} = t^2\frac{\partial^2 w}{\partial v^2}$, and $\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial y}$ $= t^2\frac{\partial^2 w}{\partial v\partial u} \Rightarrow \left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial u^2}$, $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial v^2}$, and $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial v\partial u}$
$\Rightarrow n(n-1)t^{n-2}f(x,y) = \left(\frac{x^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{2xy}{t^2}\right)\left(\frac{\partial^2 w}{\partial y\partial x}\right) + \left(\frac{y^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)$ for $t \neq 0$. When $t = 1$, $w = f(x,y)$ and we have $n(n-1)f(x,y) = x^2\left(\frac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 f}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)$ as claimed.

6. (a) $\lim\limits_{r\to 0}\frac{\sin 6r}{6r} = \lim\limits_{t\to 0}\frac{\sin t}{t} = 1$, where $t = 6r$

(b) $f_r(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{\sin 6h}{6h}\right)-1}{h} = \lim_{h\to 0} \frac{\sin 6h - 6h}{6h^2} = \lim_{h\to 0} \frac{6\cos 6h - 6}{12h}$

$= \lim_{h\to 0} \frac{-36 \sin 6h}{12} = 0$ (applying l'Hôpital's rule twice)

(c) $f_\theta(r,\theta) = \lim_{h\to 0} \frac{f(r,\theta+h)-f(r,\theta)}{h} = \lim_{h\to 0} \frac{\left(\frac{\sin 6r}{6r}\right)-\left(\frac{\sin 6r}{6r}\right)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

7. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\nabla r = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$

$= \frac{\mathbf{r}}{r}$

(b) $r^n = \left(\sqrt{x^2+y^2+z^2}\right)^n$

$\Rightarrow \nabla(r^n) = nx\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{i} + ny\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{j} + nz\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{k}$

$= nr^{n-2}\mathbf{r}$

(c) Let $n = 2$ in part (b). Then $\frac{1}{2}\nabla\left(r^2\right) = \mathbf{r} \Rightarrow \nabla\left(\frac{1}{2}r^2\right) = \mathbf{r} \Rightarrow \frac{r^2}{2} = \frac{1}{2}\left(x^2+y^2+z^2\right)$ is the function.

(d) $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \Rightarrow \mathbf{r}\cdot d\mathbf{r} = x\,dx + y\,dy + z\,dz$, and $dr = r_x\,dx + r_y\,dy + r_z\,dz = \frac{x}{r}\,dx + \frac{y}{r}\,dy + \frac{z}{r}\,dz$

$\Rightarrow r\,dr = x\,dx + y\,dy + z\,dz = \mathbf{r}\cdot d\mathbf{r}$

(e) $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{A}\cdot\mathbf{r} = ax + by + cz \Rightarrow \nabla(\mathbf{A}\cdot\mathbf{r}) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{A}$

8. $f(g(t),h(t)) = c \Rightarrow 0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$, where $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is the tangent vector

$\Rightarrow \nabla f$ is orthogonal to the tangent vector

9. $f(x,y,z) = xz^2 - yz + \cos xy - 1 \Rightarrow \nabla f = (z^2 - y\sin xy)\mathbf{i} + (-z - x\sin xy)\mathbf{j} + (2xz - y)\mathbf{k} \Rightarrow \nabla f(0,0,1) = \mathbf{i} - \mathbf{j}$

$\Rightarrow$ the tangent plane is $x - y = 0$; $\mathbf{r} = (\ln t)\mathbf{i} + (t\ln t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{r}' = \left(\frac{1}{t}\right)\mathbf{i} + (\ln t + 1)\mathbf{j} + \mathbf{k}$; $x = y = 0$, $z = 1$

$\Rightarrow t = 1 \Rightarrow \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Since $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}) = r'(1)\cdot\nabla f = 0$, $\mathbf{r}$ is parallel to the plane, and

$\mathbf{r}(1) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

10. Let $f(x,y,z) = x^3 + y^3 + z^3 - xyz \Rightarrow \nabla f = \left(3x^2 - yz\right)\mathbf{i} + \left(3y^2 - xz\right)\mathbf{j} + \left(3z^2 - xy\right)\mathbf{k} \Rightarrow \nabla f(0,-1,1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

$\Rightarrow$ the tangent plane is $x + 3y + 3z = 0$; $\mathbf{r} = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + (\cos(t-2))\mathbf{k}$

$\Rightarrow \mathbf{r}' = \left(\frac{3t^2}{4}\right)\mathbf{i} - \left(\frac{4}{t^2}\right)\mathbf{j} - (\sin(t-2))\mathbf{k}$; $x = 0$, $y = -1$, $z = 1 \Rightarrow t = 2 \Rightarrow \mathbf{r}'(2) = 3\mathbf{i} - \mathbf{j}$. Since

$\mathbf{r}'(2)\cdot\nabla f = 0 \Rightarrow \mathbf{r}$ is parallel to the plane, and $\mathbf{r}(2) = -\mathbf{i} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

11. $\frac{\partial z}{\partial x} = 3x^2 - 9y = 0$ and $\frac{\partial z}{\partial y} = 3y^2 - 9x = 0 \Rightarrow y = \frac{1}{3}x^2$ and $3\left(\frac{1}{3}x^2\right)^2 - 9x = 0 \Rightarrow \frac{1}{3}x^4 - 9x = 0$

$\Rightarrow x\left(x^3 - 27\right) = 0 \Rightarrow x = 0$ or $x = 3$. Now $x = 0 \Rightarrow y = 0$ or $(0,0)$ and $x = 3 \Rightarrow y = 3$ or $(3,3)$. Next

$\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2} = 6y$, and $\frac{\partial^2 z}{\partial x\partial y} = -9$. For $(0,0)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\partial y}\right)^2 = -81 \Rightarrow$ no extremum (a saddle point),

and for $(3,3)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\partial y}\right)^2 = 243 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 18 > 0 \Rightarrow$ a local minimum.

12. $f(x,y) = 6xye^{-(2x+3y)} \Rightarrow f_x(x,y) = 6y(1-2x)e^{-(2x+3y)} = 0$ and $f_y(x,y) = 6x(1-3y)e^{-(2x+3y)} = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{3}$. The value $f(0,0) = 0$ is on the boundary, and $f\left(\frac{1}{2},\frac{1}{3}\right) = \frac{1}{e^2}$. On the positive y-axis, $f(0,y) = 0$, and on the positive x-axis, $f(x,0) = 0$. As $x \to \infty$ or $y \to \infty$ we see that $f(x,y) \to 0$. Thus the absolute maximum of f in the closed first quadrant is $\frac{1}{e^2}$ at the point $\left(\frac{1}{2},\frac{1}{3}\right)$.

13. Let $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla f = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k} \Rightarrow$ an equation of the plane tangent at the point $P_0(x_0,y_0,y_0)$ is $\left(\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + \left(\frac{2z_0}{c^2}\right)z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2} = 2$ or $\left(\frac{x_0}{a^2}\right)x + \left(\frac{y_0}{b^2}\right)y + \left(\frac{z_0}{c^2}\right)z = 1$.

The intercepts of the plane are $\left(\frac{a^2}{x_0},0,0\right)$, $\left(0,\frac{b^2}{y_0},0\right)$ and $\left(0,0,\frac{c^2}{z_0}\right)$. The volume of the tetrahedron formed by the plane and the coordinate planes is $V = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) \Rightarrow$ we need to maximize $V(x,y,z) = \frac{(abc)^2}{2}(xyz)^{-1}$ subject to the constraint $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Thus,

$\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{x^2yz}\right) = \frac{2x}{a^2}\lambda$, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xy^2z}\right) = \frac{2y}{b^2}\lambda$, and $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xyz^2}\right) = \frac{2z}{c^2}\lambda$. Multiply the first equation by a^2yz, the second by b^2xz, and the third by c^2xy. Then equate the first and second $\Rightarrow a^2y^2 = b^2x^2$ $\Rightarrow y = \frac{b}{a}x$, $x > 0$; equate the first and third $\Rightarrow a^2z^2 = c^2x^2 \Rightarrow z = \frac{c}{a}x$, $x > 0$; substitute into $f(x,y,z) = 0$ $\Rightarrow x = \sqrt{\frac{a}{3}} \Rightarrow y = \sqrt{\frac{b}{3}} \Rightarrow z = \sqrt{\frac{c}{3}} \Rightarrow V = \frac{\sqrt{3}}{2}abc$.

14. $2(x-u) = -\lambda$, $2(y-v) = \lambda$, $-2(x-u) = \mu$, and $-2(y-v) = 2\mu v \Rightarrow x - u = v - y$, $x - u = \frac{\mu}{2}$, and $y - v = -\mu v$ $\Rightarrow x - \mu = \mu v = \frac{\mu}{2} \Rightarrow v = \frac{1}{2}$ or $\mu = 0$.

CASE 1: $\mu = 0 \Rightarrow x = u$, $y = v$, and $\lambda = 0$; then $y = x + 1 \Rightarrow v = u + 1$ and $v^2 = u \Rightarrow v = v^2 + 1$ $\Rightarrow v^2 - v + 1 = 0 \Rightarrow v = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow$ no real solution.

CASE 2: $v = \frac{1}{2}$ and $u = v^2 \Rightarrow u = \frac{1}{4}$; $x - \frac{1}{4} = \frac{1}{2} - y$ and $y = x + 1 \Rightarrow x - \frac{1}{4} = -x - \frac{1}{2} \Rightarrow 2x = -\frac{1}{4}$ $\Rightarrow x = -\frac{1}{8} \Rightarrow y = \frac{7}{8}$. Then $f\left(-\frac{1}{8},\frac{7}{8},\frac{1}{4},\frac{1}{2}\right) = \left(-\frac{1}{8}-\frac{1}{4}\right)^2 + \left(\frac{7}{8}-\frac{1}{2}\right)^2 = 2\left(\frac{3}{8}\right)^2 \Rightarrow$ the minimum distance is $\frac{3}{8}\sqrt{2}$. (Notice that f has no maximum value.)

15. Let (x_0,y_0) be any point in R. We must show $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ or, equivalently that $\lim_{(h,k)\to(0,0)} |f(x_0+h,y_0+k) - f(x_0,y_0)| = 0$. Consider $f(x_0+h,y_0+k) - f(x_0,y_0)$ $= [f(x_0+h,y_0+k) - f(x_0,y_0+k)] + [f(x_0,y_0+k) - f(x_0,y_0)]$. Let $F(x) = f(x,y_0+k)$ and apply the Mean Value Theorem: there exists ξ with $x_0 < \xi < x_0 + h$ such that $F'(\xi)h = F(x_0+h) - F(x_0) \Rightarrow hf_x(\xi,y_0+k)$ $= f(x_0+h,y_0+k) - f(x_0,y_0+k)$. Similarly, $k\,f_y(x_0,\eta) = f(x_0,y_0+k) - f(x_0,y_0)$ for some η with

$y_0 < \eta < y_0 + k$. Then $|f(x_0+h, y_0+k) - f(x_0, y_0)| \le |hf_x(\xi, y_0+k)| + |kf_y(x_0, \eta)|$. If M, N are positive real numbers such that $|f_x| \le M$ and $|f_y| \le N$ for all (x, y) in the xy-plane, then $|f(x_0+h, y_0+k) - f(x_0, y_0)|$ $\le M|h| + N|k|$. As $(h,k) \to 0$, $|f(x_0+h, y_0+k) - f(x_0, y_0)| \to 0 \Rightarrow \lim_{(h,k)\to(0,0)} |f(x_0+h, y_0+k) - f(x_0, y_0)|$ $= 0 \Rightarrow$ f is continuous at (x_0, y_0).

16. At extreme values, ∇f and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal because $\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$ by the First Derivative Theorem for Local Extreme Values.

17. $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x,y) = h(y)$ is a function of y only. Also, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} = 0 \Rightarrow g(x,y) = k(x)$ is a function of x only. Moreover, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow h'(y) = k'(x)$ for all x and y. This can happen only if $h'(y) = k'(x) = c$ is a constant. Integration gives $h(y) = cy + c_1$ and $k(x) = cx + c_2$, where c_1 and c_2 are constants. Therefore $f(x,y) = cy + c_1$ and $g(x,y) = cx + c_2$. Then $f(1,2) = g(1,2) = 5 \Rightarrow 5 = 2c + c_1 = c + c_2$, and $f(0,0) = 4 \Rightarrow c_1 = 4 \Rightarrow c = \frac{1}{2}$ $\Rightarrow c_2 = \frac{9}{2}$. Thus, $f(x,y) = \frac{1}{2}y + 4$ and $g(x,y) = \frac{1}{2}x + \frac{9}{2}$.

18. Let $g(x,y) = D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$. Then $D_{\mathbf{u}}g(x,y) = g_x(x,y)a + g_y(x,y)b$ $= f_{xx}(x,y)a^2 + f_{yx}(x,y)ab + f_{xy}(x,y)ba + f_{yy}(x,y)b^2 = f_{xx}(x,y)a^2 + 2f_{xy}(x,y)ab + f_{yy}(x,y)b^2$.

19. Since the particle is heat-seeking, at each point (x,y) it moves in the direction of maximal temperature increase, that is in the direction of $\nabla T(x,y) = (e^{-2y}\sin x)\mathbf{i} + (2e^{-2y}\cos x)\mathbf{j}$. Since $\nabla T(x,y)$ is parallel to the particle's velocity vector, it is tangent to the path $y = f(x)$ of the particle $\Rightarrow f'(x) = \frac{2e^{-2y}\cos x}{e^{-2y}\sin x} = 2\cot x$. Integration gives $f(x) = 2\ln|\sin x| + C$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow 0 = 2\ln\left|\sin\frac{\pi}{4}\right| + C \Rightarrow C = -2\ln\frac{\sqrt{2}}{2} = \ln\left(\frac{2}{\sqrt{2}}\right)^2$ $= \ln 2$. Therefore, the path of the particle is the graph of $y = 2\ln|\sin x| + \ln 2$.

20. The line of travel is $x = t$, $y = t$, $z = 30 - 5t$, and the bullet hits the surface $z = 2x^2 + 3y^2$ when $30 - 5t = 2t^2 + 3t^2 \Rightarrow t^2 + t - 6 = 0 \Rightarrow (t+3)(t-2) = 0 \Rightarrow t = 2$ (since $t > 0$). Thus the bullet hits the surface at the point $(2,2,20)$. Now, the vector $4x\mathbf{i} + 6y\mathbf{j} - \mathbf{k}$ is normal to the surface at any (x,y,z), so that $\mathbf{n} = 8\mathbf{i} + 12\mathbf{j} - \mathbf{k}$ is normal to the surface at $(2,2,20)$. If $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$, then the velocity of the particle after the ricochet is $\mathbf{w} = \mathbf{v} - 2\,\mathrm{proj}_{\mathbf{n}}\,\mathbf{v} = \mathbf{v} - \left(\frac{2\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n} = \mathbf{v} - \left(\frac{2\cdot 25}{209}\right)\mathbf{n} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \left(\frac{400}{209}\mathbf{i} + \frac{600}{209}\mathbf{j} - \frac{50}{209}\mathbf{k}\right)$ $= -\frac{191}{209}\mathbf{i} - \frac{391}{209}\mathbf{j} - \frac{995}{209}\mathbf{k}$.

21. (a) $\mathbf{k}$ is a vector normal to $z = 10 - x^2 - y^2$ at the point $(0,0,10)$. So directions tangential to S at $(0,0,10)$ will be unit vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Also, $\nabla T(x,y,z) = (2xy + 4)\mathbf{i} + (x^2 + 2yz + 14)\mathbf{j} + (y^2 + 1)\mathbf{k}$ $\Rightarrow \nabla T(0,0,10) = 4\mathbf{i} + 14\mathbf{j} + \mathbf{k}$. We seek the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ such that $D_{\mathbf{u}}T(0,0,10)$ $= (4\mathbf{i} + 14\mathbf{j} + \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j}) = (4\mathbf{i} + 14\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j})$ is a maximum. The maximum will occur when $a\mathbf{i} + b\mathbf{j}$ has the same direction as $4\mathbf{i} + 14\mathbf{j}$, or $\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$.

(b) A vector normal to S at $(1,1,8)$ is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Now, $\nabla T(1,1,8) = 6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$ and we seek the unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}T(1,1,8) = \nabla T \cdot \mathbf{u}$ has its largest value. Now write $\nabla T = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is parallel to ∇T and $\mathbf{w}$ is orthogonal to ∇T. Then $D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u} = (\mathbf{v}+\mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$. Thus $D_{\mathbf{u}}T(1,1,8)$ is a maximum when $\mathbf{u}$ has the same direction as $\mathbf{w}$. Now, $\mathbf{w} = \nabla T - \left(\frac{\nabla T \cdot \mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$

$= (6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}) - \left(\frac{12+62+2}{4+4+1}\right)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \left(6 - \frac{152}{9}\right)\mathbf{i} + \left(31 - \frac{152}{9}\right)\mathbf{j} + \left(2 - \frac{76}{9}\right)\mathbf{k}$

$= -\frac{98}{9}\mathbf{i} + \frac{127}{9}\mathbf{j} - \frac{58}{9}\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = -\frac{1}{\sqrt{29{,}097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$.

22. Suppose the surface (boundary) of the mineral deposit is the graph of $z = f(x,y)$ (where the z-axis points up into the air). Then $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is an outer normal to the mineral deposit at (x,y) and $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ points in the direction of steepest ascent of the mineral deposit. This is in the direction of the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ at $(0,0)$ (the location of the 1st borehole) that the geologists should drill their fourth borehole. To approximate this vector we use the fact that $(0,0,-1000)$, $(0,100,-950)$, and $(100,0,-1025)$ lie on the graph of $z = f(x,y)$. The plane containing these three points is a good approximation to the tangent plane to $z = f(x,y)$ at the point $(0,0,0)$. A normal to this plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 100 & 50 \\ 100 & 0 & -25 \end{vmatrix} = -2500\mathbf{i} + 5000\mathbf{j} - 10{,}000\mathbf{k}$, or $-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. So at $(0,0)$ the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is approximately $-\mathbf{i} + 2\mathbf{j}$. Thus the geologists should drill their fourth borehole in the direction of $\frac{1}{\sqrt{5}}(-\mathbf{i} + 2\mathbf{j})$ from the first borehole.

23. $w = e^{rt} \sin \pi x \Rightarrow w_t = re^{rt} \sin \pi x$ and $w_x = \pi e^{rt} \cos \pi x \Rightarrow w_{xx} = -\pi^2 e^{rt} \sin \pi x$; $w_{xx} = \frac{1}{c^2} w_t$, where c^2 is the positive constant determined by the material of the rod $\Rightarrow -\pi^2 e^{rt} \sin \pi x = \frac{1}{c^2}\left(re^{rt} \sin \pi x\right)$

$\Rightarrow \left(r + c^2\pi^2\right)e^{rt} \sin \pi x = 0 \Rightarrow r = -c^2\pi^2 \Rightarrow w = e^{-c^2\pi^2 t} \sin \pi x$

24. $w = e^{rt} \sin kx \Rightarrow w_t = re^{rt} \sin kx$ and $w_x = ke^{rt} \cos kx \Rightarrow w_{xx} = -k^2 e^{rt} \sin kx$; $w_{xx} = \frac{1}{c^2} w_t$

$\Rightarrow -k^2 e^{rt} \sin kx = \frac{1}{c^2}\left(re^{rt} \sin kx\right) \Rightarrow \left(r + c^2k^2\right)e^{rt} \sin kx = 0 \Rightarrow r = -c^2k^2 \Rightarrow w = e^{-c^2k^2t} \sin kx$.

Now, $w(L,t) = 0 \Rightarrow e^{-c^2k^2t} \sin kL = 0 \Rightarrow kL = n\pi$ for n an integer $\Rightarrow k = \frac{n\pi}{L} \Rightarrow w = e^{-c^2n^2\pi^2t/L^2} \sin\left(\frac{n\pi}{L}x\right)$.

As $t \to \infty$, $w \to 0$ since $\left|\sin\left(\frac{n\pi}{L}x\right)\right| \le 1$ and $e^{-c^2n^2\pi^2t/L^2} \to 0$.