$= \frac{1}{2} \int_{-2}^{2} (2-x)\left[\frac{4-x^2}{4} - \frac{4-x^2}{4}\right] dx = 0 \Rightarrow \bar{x} = -\frac{1}{2}$ and $\bar{y} = 0$

(b) $M_{xy} = \int_{-2}^{2} \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} \int_{0}^{2-x} z\, dz\, dy\, dx = \frac{1}{2} \int_{-2}^{2} \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} (2-x)^2\, dy\, dx = \frac{1}{2} \int_{-2}^{2} (2-x)^2\left(\sqrt{4-x^2}\right) dx$

$= 5\pi \Rightarrow \bar{z} = \frac{5}{4}$

7. (a) $M = 4 \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\, dy\, dx = 4 \int_{0}^{\pi/2} \int_{0}^{2} \int_{r^2}^{4} r\, dz\, dr\, d\theta = 4 \int_{0}^{\pi/2} \int_{0}^{2} \left(4r - r^3\right) dr\, d\theta = 4 \int_{0}^{\pi/2} 4\, d\theta = 8\pi;$

$M_{xy} = \int_{0}^{2\pi} \int_{0}^{2} \int_{r^2}^{4} zr\, dz\, dr\, d\theta = \int_{0}^{2\pi} \int_{0}^{2} \frac{r}{2}\left(16 - r^4\right) dr\, d\theta = \frac{32}{3} \int_{0}^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{8}{3}$, and $\bar{x} = \bar{y} = 0$, by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_{0}^{2\pi} \int_{0}^{\sqrt{c}} \int_{r^2}^{c} r\, dz\, dr\, d\theta = \int_{0}^{2\pi} \int_{0}^{\sqrt{c}} \left(cr - r^3\right) dr\, d\theta = \int_{0}^{2\pi} \frac{c^2}{4}\, d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2}$, since $c > 0$

8. $M = 8$; $M_{xy} = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} z\, dz\, dy\, dx = \int_{-1}^{1} \int_{3}^{5} \left[\frac{z^2}{2}\right]_{-1}^{1} dy\, dx = 0$; $M_{yz} = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} x\, dz\, dy\, dx$

$= 2 \int_{-1}^{1} \int_{3}^{5} x\, dy\, dx = 4 \int_{-1}^{1} x^2\, dx = 0$; $M_{xz} = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} y\, dz\, dy\, dx = 2 \int_{-1}^{1} \int_{3}^{5} y\, dy\, dx = 16 \int_{-1}^{1} dx = 32$

$\Rightarrow \bar{x} = 0$, $\bar{y} = 4$, $\bar{z} = 0$; $I_x = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} \left(y^2 + z^2\right) dz\, dy\, dx = \int_{-1}^{1} \int_{3}^{5} \left(2y^2 + \frac{2}{3}\right) dy\, dx = \frac{2}{3} \int_{-1}^{1} 100\, dx = \frac{400}{3};$

$I_y = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} \left(x^2 + z^2\right) dz\, dy\, dx = \int_{-1}^{1} \int_{3}^{5} \left(2x^2 + \frac{2}{3}\right) dy\, dx = \frac{4}{3} \int_{-1}^{1} \left(3x^2 + 1\right) dx = \frac{16}{3};$

$I_z = \int_{-1}^{1} \int_{3}^{5} \int_{-1}^{1} \left(x^2 + y^2\right) dz\, dy\, dx = 2 \int_{-1}^{1} \int_{3}^{5} \left(x^2 + y^2\right) dy\, dx = 2 \int_{-1}^{1} \left(2x^2 + \frac{98}{3}\right) dx = \frac{400}{3} \Rightarrow R_x = R_z = \sqrt{\frac{50}{3}}$

and $R_y = \sqrt{\frac{2}{3}}$

9. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2} \int_{-2}^{4} \int_{-1}^{(2-y)/2} \left[(y-6)^2 + z^2\right] dz\, dy\, dx$

$= \int_{-2}^{2} \int_{-2}^{4} \left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3}\right] dy\, dx$; let $t = 2 - y \Rightarrow I_L = 4 \int_{-2}^{4} \left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3}\right) dt = 1386;$

$M = \frac{1}{2}(3)(6)(4) = 36 \Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{77}{2}}$

10. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2} \left[(x-4)^2 + y^2\right] dz\,dy\,dx$

$= \frac{1}{2}\int_{-2}^{2}\int_{-2}^{4} \left(x^2 - 8x + 16 + y^2\right)(4-y)\,dy\,dx = \int_{-2}^{2} \left(9x^2 - 72x + 162\right) dx = 696;\ M = \frac{1}{2}(3)(6)(4) = 36$

$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{58}{3}}$

11. $M = 8;\ I_L = \int_0^4\int_0^2\int_0^1 \left[z^2 + (y-2)^2\right] dz\,dy\,dx = \int_0^4\int_0^2 \left(y^2 - 4y + \frac{13}{3}\right) dy\,dx = \frac{10}{3}\int_0^4 dx = \frac{40}{3}$

$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{5}{3}}$

12. $M = 8;\ I_L = \int_0^4\int_0^2\int_0^1 \left[(x-4)^2 + y^2\right] dz\,dy\,dx = \int_0^4\int_0^2 \left[(x-4)^2 + y^2\right] dy\,dx = \int_0^4 \left[2(x-4)^2 + \frac{8}{3}\right] dx = \frac{160}{3}$

$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{20}{3}}$

13. (a) $M = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx = \int_0^2\int_0^{2-x} \left(4x - 2x^2 - 2xy\right) dy\,dx = \int_0^2 \left(x^3 - 4x^2 + 4x\right) dx = \frac{4}{3}$

(b) $M_{xy} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2xz\,dz\,dy\,dx = \int_0^2\int_0^{2-x} x(2-x-y)^2\,dy\,dx = \int_0^2 \frac{x(2-x)^3}{3}\,dx = \frac{8}{15};\ M_{xz} = \frac{8}{15}$ by

symmetry; $M_{yz} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x^2\,dz\,dy\,dx = \int_0^2\int_0^{2-x} 2x^2(2-x-y)\,dy\,dx = \int_0^2 \left(2x - x^2\right)^2 dx = \frac{16}{15}$

$\Rightarrow \overline{x} = \frac{4}{5}$, and $\overline{y} = \overline{z} = \frac{2}{5}$

14. (a) $M = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy\left(4 - x^2\right) dy\,dx = \frac{k}{2}\int_0^2 \left(4x^2 - x^4\right) dx = \frac{32k}{15}$

(b) $M_{yz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kx^2y\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} x^2y\left(4 - x^2\right) dy\,dx = \frac{k}{2}\int_0^2 \left(4x^3 - x^5\right) dx = \frac{8k}{3}$

$\Rightarrow \overline{x} = \frac{5}{4};\ M_{xz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy^2\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy^2\left(4 - x^2\right) dy\,dx = \frac{k}{3}\int_0^2 \left(4x^{5/2} - x^{9/2}\right) dx$

$= \frac{256\sqrt{2}k}{231} \Rightarrow \bar{y} = \frac{40\sqrt{2}}{77}$; $M_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz \, dz\, dy\, dx = \frac{k}{2} \int_0^2 \int_0^{\sqrt{x}} xy\left(4-x^2\right)^2 dy\, dx$

$= \frac{k}{4} \int_0^2 \left(16x^2 - 8x^4 + x^6\right) dx = \frac{256k}{105} \Rightarrow \bar{z} = \frac{8}{7}$

15. (a) $M = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1)\, dz\, dy\, dx = \int_0^1 \int_0^1 \left(x+y+\frac{3}{2}\right) dy\, dx = \int_0^1 (x+2)\, dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1)\, dz\, dy\, dx = \frac{1}{2} \int_0^1 \int_0^1 \left(x+y+\frac{5}{3}\right) dy\, dx = \frac{1}{2} \int_0^1 \left(x+\frac{13}{6}\right) dx = \frac{4}{3}$

$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$

(c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2)(x+y+z+1)\, dz\, dy\, dx = \int_0^1 \int_0^1 (x^2+y^2)\left(x+y+\frac{3}{2}\right) dy\, dx$

$= \int_0^1 \left(x^3 + 2x^2 + \frac{1}{3}x + \frac{3}{4}\right) dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}$, by symmetry

(d) $R_x = R_y = R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{11}{15}}$

16. The plane $y + 2z = 2$ is the top of the wedge.

(a) $M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\, dz\, dy\, dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right) dy\, dx = 18$

(b) $M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1)\, dz\, dy\, dx = \int_{-1}^1 \int_{-2}^4 x(x+1)\left(2-\frac{y}{2}\right) dy\, dx = 6;$

$M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1)\, dz\, dy\, dx = \int_{-1}^1 \int_{-2}^4 y(x+1)\left(2-\frac{y}{2}\right) dy\, dx = 0;$

$M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1)\, dz\, dy\, dx = \frac{1}{2} \int_{-1}^1 \int_{-2}^4 (x+1)\left(\frac{y^2}{4} - y\right) dy\, dx = 0 \Rightarrow \bar{x} = \frac{4}{3}$, and $\bar{y} = \bar{z} = 0$

(c) $I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(y^2+z^2)\, dz\, dy\, dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2y^2 + \frac{1}{3} - \frac{y^3}{2} + \frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\, dx = 45;$

$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2+z^2)\, dz\, dy\, dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2x^2 + \frac{1}{3} - \frac{x^2 y}{2} + \frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\, dx = 15;$

$$I_z = \int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2} (x+1)(x^2+y^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_{-2}^{4} (x+1)\left(2-\frac{y}{2}\right)(x^2+y^2)\,dy\,dx = 42$$

(d) $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{5}{2}}$, $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{5}{6}}$, and $R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{3}}$

17. $M = \int_0^1 \int_{z-1}^{1-z} \int_0^{\sqrt{z}} (2y+5)\,dy\,dx\,dz = \int_0^1 \int_{z-1}^{1-z} (z+5\sqrt{z})\,dx\,dz = \int_0^1 2(z+5\sqrt{z})(1-z)\,dz$

$= 2\int_0^1 \left(5z^{1/2}+z-5z^{3/2}-z^2\right) dz = 2\left[\frac{10}{3}z^{3/2}+\frac{1}{2}z^2-2z^{5/2}-\frac{1}{3}z^3\right]_0^1 = 2\left(\frac{9}{3}-\frac{3}{2}\right) = 3$

18. $M = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\left[16-4(x^2+y^2)\right] dy\,dx$

$= 4\int_0^{2\pi}\int_0^2 r(4-r^2)\,r\,dr\,d\theta = 4\int_0^{2\pi}\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 d\theta = 4\int_0^{2\pi}\frac{64}{15}\,d\theta = \frac{512\pi}{15}$

19. (a) Let ΔV_i be the volume of the ith piece, and let (x_i, y_i, z_i) be a point in the ith piece. Then the work done by gravity in moving the ith piece to the xy-plane is approximately $W_i = m_i g z_i = (x_i + y_i + z_i + 1)g\,\Delta V_i\, z_i$ $\Rightarrow$ the total work done is the triple integral $W = \int_0^1\int_0^1\int_0^1 (x+y+z+1)gz\,dz\,dy\,dx$

$= g\int_0^1\int_0^1 \left[\frac{1}{2}xz^2+\frac{1}{2}yz^2+\frac{1}{3}z^3+\frac{1}{2}z^2\right]_0^1 dy\,dx = g\int_0^1\int_0^1 \left(\frac{1}{2}x+\frac{1}{2}y+\frac{5}{6}\right) dy\,dx = g\int_0^1 \left[\frac{1}{2}xy+\frac{1}{4}y^2+\frac{5}{6}y\right]_0^1 dx$

$= g\int_0^1 \left(\frac{1}{2}x+\frac{13}{12}\right) dx = g\left[\frac{x^2}{4}+\frac{13}{12}x\right]_0^1 = g\left(\frac{16}{12}\right) = \frac{4}{3}g$

(b) From Exercise 15 the center of mass is $\left(\frac{8}{15},\frac{8}{15},\frac{8}{15}\right)$ and the mass of the liquid is $\frac{5}{2}$ $\Rightarrow$ the work done by gravity in moving the center of mass to the xy-plane is $W = mgd = \left(\frac{5}{2}\right)(g)\left(\frac{8}{15}\right) = \frac{4}{3}g$, which is the same as the work done in part (a).

20. (a) From Exercise 19(a) we see that the work done is $W = g\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxyz\,dz\,dy\,dx$

$= k\cdot g\int_0^2\int_0^{\sqrt{x}} \frac{1}{2}xy\left(4-x^2\right)^2 dy\,dx = \frac{k\cdot g}{4}\int_0^2 x^2\left(4-x^2\right)^2 dx = \frac{k\cdot g}{4}\int_0^2 \left(16x^2-8x^4+x^6\right) dx$

$= \frac{k\cdot g}{4}\left[\frac{16}{3}x^3-\frac{8}{5}x^5+\frac{1}{7}x^7\right]_0^2 = \frac{256k\cdot g}{105}$

(b) From Exercise 14 the center of mass is $\left(\frac{5}{4}, \frac{40\sqrt{2}}{77}, \frac{8}{7}\right)$ and the mass of the liquid is $\frac{32k}{15} \Rightarrow$ the work done by gravity in moving the center of mass to the xy-plane is $W = mgd = \left(\frac{32k}{15}\right)(g)\left(\frac{8}{7}\right) = \frac{256k \cdot g}{105}$

21. (a) $\bar{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint\limits_R x\delta(x,y,z)\, dx\, dy\, dz = 0 \Rightarrow M_{yz} = 0$

(b) $I_L = \iiint\limits_R |\mathbf{v} - h\mathbf{i}|^2\, dm = \iiint\limits_R |(x-h)\mathbf{i} + y\mathbf{j}|^2\, dm = \iiint\limits_R (x^2 - 2xh + h^2 + y^2)\, dm$

$= \iiint\limits_R (x^2+y^2)\, dm - 2h \iiint\limits_R x\, dm + h^2 \iiint\limits_R dm = I_x - 0 + h^2 m = I_{c.m.} + h^2 m$

22. $I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

23. (a) $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \Rightarrow I_z = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc(a^2+b^2)}{4}$

$= \frac{abc(a^2+b^2)}{3} - \frac{abc(a^2+b^2)}{4} = \frac{abc(a^2+b^2)}{12}$; $R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$

(b) $I_L = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4} + \left(\frac{b}{2} - 2b\right)^2}\right)^2 = \frac{abc(a^2+b^2)}{12} + \frac{abc(a^2+9b^2)}{4} = \frac{abc(4a^2+28b^2)}{12}$

$= \frac{abc(a^2+7b^2)}{3}$; $R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$

24. $M = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} dz\, dy\, dx = \int_{-3}^{3}\int_{-2}^{4} \frac{2}{3}(4-y)\, dy\, dx = \int_{-3}^{3} \frac{2}{3}\left[4y - \frac{y^2}{2}\right]_{-2}^{4} dx = 12\int_{-3}^{3} dx = 72$;

$\bar{x} = \bar{y} = \bar{z} = 0$ from Exercise 2 $\Rightarrow I_x = I_{c.m.} + 72\left(\sqrt{0^2+0^2}\right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72\left(\sqrt{16 + \frac{16}{9}}\right)^2$

$= 208 + 72\left(\frac{160}{9}\right) = 1488$

25. $M_{yz_{B_1 \cup B_2}} = \iiint\limits_{B_1} x\, dV_1 + \iiint\limits_{B_2} x\, dV_2 = M_{(yz)_1} + M_{(yz)_2} \Rightarrow \bar{x} = \frac{M_{(yz)_1} + M_{(yz)_2}}{m_1 + m_2}$; similarly,

$\bar{y} = \frac{M_{(xz)_1} + M_{(xz)_2}}{m_1 + m_2}$ and $\bar{z} = \frac{M_{(xy)_1} + M_{(xy)_2}}{m_1 + m_2} \Rightarrow \mathbf{c} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

$= \frac{1}{m_1+m_2}\left[\left(M_{(yz)_1} + M_{(yz)_2}\right)\mathbf{i} + \left(M_{(xz)_1} + M_{(xz)_2}\right)\mathbf{j} + \left(M_{(xy)_1} + M_{(xy)_2}\right)\mathbf{k}\right]$

$= \frac{1}{m_1+m_2}\left[(m_1\bar{x}_1 + m_2\bar{x}_2)\mathbf{i} + (m_1\bar{y}_1 + m_2\bar{y}_2)\mathbf{j} + (m_1\bar{z}_1 + m_2\bar{z}_2)\mathbf{k}\right]$

$= \frac{1}{m_1+m_2}[m_1(\bar{x}_1\mathbf{i}+\bar{y}_1\mathbf{j}+\bar{z}_1\mathbf{k})+m_2(\bar{x}_2\mathbf{i}+\bar{y}_2\mathbf{j}+\bar{z}_2\mathbf{k})] = \frac{m_1\mathbf{c}_1+m_2\mathbf{c}_2}{m_1+m_2}$

26. (a) $\mathbf{c} = \frac{12\left(\mathbf{i}+\frac{3}{2}\mathbf{j}+\mathbf{k}\right)+2\left(\frac{1}{2}\mathbf{i}+4\mathbf{j}+\frac{1}{2}\mathbf{k}\right)}{12+2} = \frac{\frac{13}{2}\mathbf{i}+13\mathbf{j}+\frac{13}{2}\mathbf{k}}{7} \Rightarrow \bar{x}=\frac{13}{14},\ \bar{y}=\frac{13}{7},\ \bar{z}=\frac{13}{14}$

(b) $\mathbf{c} = \frac{12\left(\mathbf{i}+\frac{3}{2}\mathbf{j}+\mathbf{k}\right)+12\left(\mathbf{i}+\frac{11}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}\right)}{12+12} = \frac{2\mathbf{i}+7\mathbf{j}+\frac{1}{2}\mathbf{k}}{2} \Rightarrow \bar{x}=1,\ \bar{y}=\frac{7}{2},\ \bar{z}=\frac{1}{4}$

(c) $\mathbf{c} = \frac{2\left(\frac{1}{2}\mathbf{i}+4\mathbf{j}+\frac{1}{2}\mathbf{k}\right)+12\left(\mathbf{i}+\frac{11}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}\right)}{2+12} = \frac{13\mathbf{i}+74\mathbf{j}-5\mathbf{k}}{14} \Rightarrow \bar{x}=\frac{13}{14},\ \bar{y}=\frac{37}{7},\ \bar{z}=-\frac{5}{14}$

(d) $\mathbf{c} = \frac{12\left(\mathbf{i}+\frac{3}{2}\mathbf{j}+\mathbf{k}\right)+2\left(\frac{1}{2}\mathbf{i}+4\mathbf{j}+\frac{1}{2}\mathbf{k}\right)+12\left(\mathbf{i}+\frac{11}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}\right)}{12+2+12} = \frac{25\mathbf{i}+92\mathbf{j}+7\mathbf{k}}{26} \Rightarrow \bar{x}=\frac{25}{26},\ \bar{y}=\frac{46}{13},\ \bar{z}=\frac{7}{26}$

27. (a) $\mathbf{c} = \frac{\left(\frac{\pi a^2 h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right)+\left(\frac{2\pi a^3}{3}\right)\left(-\frac{3a}{8}\mathbf{k}\right)}{m_1+m_2} = \frac{\left(\frac{a^2\pi}{3}\right)\left(\frac{h^2-3a^2}{4}\mathbf{k}\right)}{m_1+m_2}$, where $m_1=\frac{\pi a^2 h}{3}$ and $m_2=\frac{2\pi a^3}{3}$; if $\frac{h^2-3a^2}{4}=0$, or $h=a\sqrt{3}$, then the centroid is on the common base

(b) See the solution to Exercise 55, Section 12.2, to see that $h=a\sqrt{2}$.

28. $\mathbf{c} = \frac{\left(\frac{s^2h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right)+s^3\left(-\frac{s}{2}\mathbf{k}\right)}{m_1+m_2} = \frac{\left(\frac{s^2}{12}\right)[(h^2-6s^2)\mathbf{k}]}{m_1+m_2}$, where $m_1=\frac{s^2h}{3}$ and $m_2=s^3$; if $h^2-6s^2<0$, or $h<\sqrt{6}s$, then the centroid is in the base of the pyramid.

12.6 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 \left[r(2-r^2)^{1/2}-r^2\right] dr\ d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{3/2}-\frac{r^3}{3}\right]_0^1 d\theta$
$= \int_0^{2\pi}\left(\frac{2^{3/2}}{3}-\frac{2}{3}\right) d\theta = \frac{4\pi\left(\sqrt{2}-1\right)}{3}$

2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^3\left[r(18-r^2)^{1/2}-\frac{r^3}{3}\right] dr\ d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(18-r^2)^{3/2}-\frac{r^4}{12}\right]_0^3 d\theta$
$= \frac{9\pi\left(8\sqrt{2}-7\right)}{2}$

3. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^{\theta/2\pi}\left(3r+24r^3\right) dr\ d\theta = \int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/2\pi} d\theta = \frac{3}{2}\int_0^{2\pi}\left(\frac{\theta^2}{4\pi^2}+\frac{4\theta^4}{16\pi^4}\right) d\theta$

$$= \frac{3}{2}\left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{5\pi^4}\right]_0^{2\pi} = \frac{17\pi}{5}$$

4. $$\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{\theta/\pi} \tfrac{1}{2}[9(4-r^2)-(4-r^2)]r\,dr\,d\theta = 4\int_0^{\pi}\int_0^{\theta/\pi} (4r-r^3)\,dr\,d\theta$$

$$= 4\int_0^{\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi}\left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4}\right)d\theta = \frac{37\pi}{15}$$

5. $$\int_0^{2\pi}\int_0^{1}\int_r^{(2-r^2)^{-1/2}} 3\,dz\,r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^{1}\left[r(2-r^2)^{-1/2} - r^2\right]dr\,d\theta = 3\int_0^{2\pi}\left[-(2-r^2)^{1/2} - \frac{r^3}{3}\right]_0^1 d\theta$$

$$= 3\int_0^{2\pi}\left(\sqrt{2} - \tfrac{4}{3}\right)d\theta = \pi(6\sqrt{2} - 8)$$

6. $$\int_0^{2\pi}\int_0^{1}\int_{-1/2}^{1/2} (r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1}\left(r^3\sin^2\theta + \frac{r}{12}\right)dr\,d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4} + \frac{1}{24}\right)d\theta = \frac{\pi}{3}$$

7. $$\int_0^{2\pi}\int_0^{3}\int_0^{z/3} r^3\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^{3}\frac{z^4}{324}\,dz\,d\theta = \int_0^{2\pi}\frac{3}{20}\,d\theta = \frac{3\pi}{10}$$

8. $$\int_{-1}^{1}\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz = \int_{-1}^{1}\int_0^{2\pi} 2(1+\cos\theta)^2\,d\theta\,dz = \int_{-1}^{1} 6\pi\,d\theta = 12\pi$$

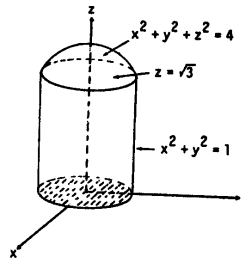
9. $$\int_0^{1}\int_0^{\sqrt{z}}\int_0^{2\pi} (r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz = \int_0^{1}\int_0^{\sqrt{z}}\left[\frac{r^2\theta}{2} + \frac{r^2\sin 2\theta}{4} + z^2\theta\right]_0^{2\pi} r\,dr\,dz = \int_0^{1}\int_0^{\sqrt{z}} (\pi r^3 + 2\pi r z^2)\,dr\,dz$$

$$= \int_0^{1}\left[\frac{\pi r^4}{4} + \pi r^2 z^2\right]_0^{\sqrt{z}} dz = \int_0^{1}\left(\frac{\pi z^2}{4} + \pi z^3\right)dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4}\right]_0^1 = \frac{\pi}{3}$$

10. $$\int_0^{2}\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi} (r\sin\theta + 1)\,r\,d\theta\,dz\,dr = \int_0^{2}\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\,dz\,dr = 2\pi\int_0^{2}\left[r(4-r^2)^{1/2} - r^2 + 2r\right]dr$$

$$= 2\pi\left[-\tfrac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} + r^2\right]_0^2 = 2\pi\left[-\tfrac{8}{3} + 4 + \tfrac{1}{3}(4)^{3/2}\right] = 8\pi$$

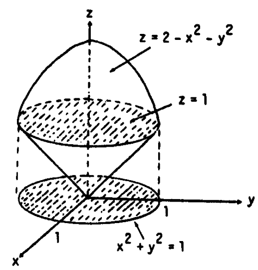
11. (a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\ r\ dr\ d\theta$



(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\ dr\ dz\ d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\ dr\ dz\ d\theta$

(c) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\ d\theta\ dz\ dr$

12. (a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\ r\ dr\ d\theta$



(b) $\int_0^{2\pi}\int_0^1\int_0^z r\ dr\ dz\ d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\ dr\ dz\ d\theta$

(c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\ d\theta\ dz\ dr$

13. $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

14. $\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^3\ dz\ dr\ d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^4\cos\theta\ dr\ d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\ d\theta = \frac{2}{5}$

15. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

16. $\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

17. $\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r,\theta,z)\ dz\ r\ dr\ d\theta$

18. $\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

19. $\int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{2-r\sin\theta} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

20. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}\int_0^{2-r\sin\theta} f(r,\theta,z)\ dz\ r\ dr\ d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\, d\phi\, d\theta=\frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi}+\frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right)d\theta$

$=2\int_0^{\pi}\int_0^{\pi}\sin^2\phi\, d\phi\, d\theta=\int_0^{\pi}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi}d\theta=\int_0^{\pi}\pi\, d\theta=\pi^2$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}(\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\int_0^{2\pi}\int_0^{\pi/4}4\cos\phi\sin\phi\, d\phi\, d\theta=\int_0^{2\pi}\left[2\sin^2\phi\right]_0^{\pi/4}d\theta=\int_0^{2\pi}d\theta=2\pi$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\, d\phi\, d\theta=\frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi}d\theta$

$=\frac{1}{96}\int_0^{2\pi}\left(2^4-0\right)d\theta=\frac{16}{96}\int_0^{2\pi}d\theta=\frac{1}{6}(2\pi)=\frac{\pi}{3}$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1}5\rho^3\sin^3\phi\, d\rho\, d\phi\ d\theta=\frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta=\frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi}+\frac{2}{3}\int_0^{\pi}\sin\phi\, d\phi\right)d\theta$

$=\frac{5}{6}\int_0^{3\pi/2}\left[-\cos\phi\right]_0^{\pi}d\theta=\frac{5}{3}\int_0^{3\pi/2}d\theta=\frac{5\pi}{2}$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}3\rho^2\sin\phi\, d\rho\, d\phi\ d\theta=\int_0^{2\pi}\int_0^{\pi/3}\left(8-\sec^3\phi\right)\sin\phi\, d\phi\, d\theta=\int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\sec^2\phi\right]_0^{\pi/3}d\theta$

$=\int_0^{2\pi}\left[(-4-2)-\left(-8-\frac{1}{2}\right)\right]d\theta=\frac{5}{2}\int_0^{2\pi}d\theta=5\pi$

26. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^3\sin\phi\cos\phi\, d\rho\, d\phi\ d\theta=\frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4}\tan\phi\sec^2\phi\, d\phi\, d\theta=\frac{1}{4}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4}d\theta$

$=\frac{1}{8}\int_0^{2\pi}d\theta=\frac{\pi}{4}$

27. $\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\, d\phi\, d\theta\, d\rho=\int_0^{2}\int_{-\pi}^{0}\rho^3\left[-\frac{\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2}d\theta\, d\rho=\int_0^{2}\int_{-\pi}^{0}\frac{\rho^3}{2}\, d\theta\, d\rho=\int_0^{2}\frac{\rho^3\pi}{2}\, d\rho$

$=\left[\frac{\pi\rho^4}{8}\right]_0^{2}=2\pi$

28. $\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2 \sin\phi \, d\theta\, d\rho\, d\phi = 2\pi \int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi} \rho^2 \sin\phi \, d\rho\, d\phi = \frac{2\pi}{3}\int_{\pi/6}^{\pi/3} \left[\rho^3 \sin\phi\right]_{\csc\phi}^{2\csc\phi} d\phi$

$= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3} \csc^2\phi \, d\phi = \frac{28\pi}{3\sqrt{3}}$

29. $\int_0^1\int_0^{\pi}\int_0^{\pi/4} 12\rho \sin^3\phi \, d\phi\, d\theta\, d\rho = \int_0^1\int_0^{\pi}\left(12\rho\left[\frac{-\sin^2\phi\cos\phi}{3}\right]_0^{\pi/4} + 8\rho\int_0^{\pi/4}\sin\phi\, d\phi\right) d\theta\, d\rho$

$= \int_0^1\int_0^{\pi}\left(-\frac{2\rho}{\sqrt{2}} - 8\rho\left[\cos\phi\right]_0^{\pi/4}\right) d\theta\, d\rho = \int_0^1\int_0^{\pi}\left(8\rho - \frac{10\rho}{\sqrt{2}}\right) d\theta\, d\rho = \pi\int_0^1\left(8\rho - \frac{10\rho}{\sqrt{2}}\right) d\rho = \pi\left[4\rho^2 - \frac{5\rho^2}{\sqrt{2}}\right]_0^1$

$= \frac{(4\sqrt{2}-5)\,\pi}{\sqrt{2}}$

30. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho\, d\theta\, d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} (32 - \csc^5\phi)\sin^3\phi \, d\theta\, d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} (32\sin^3\phi - \csc^2\phi)\, d\theta\, d\phi$

$= \pi\int_{\pi/6}^{\pi/2} (32\sin^3\phi - \csc^2\phi)\, d\phi = \pi\left[-\frac{32\sin^2\phi\cos\phi}{3}\right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3}\int_{\pi/6}^{\pi/2}\sin\phi\, d\phi + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$

$= \pi\left(\frac{32\sqrt{3}}{24}\right) - \frac{64\pi}{3}\left[\cos\phi\right]_{\pi/6}^{\pi/2} + \pi\left(\sqrt{3}\right) = \frac{\sqrt{3}}{3}\pi + \left(\frac{64\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}$

31. (a) $x^2 + y^2 = 1 \Rightarrow \rho^2\sin^2\phi = 1$, and $\rho\sin\phi = 1 \Rightarrow \rho = \csc\phi$; thus

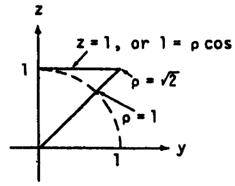
$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

(b) $$\int_0^{2\pi}\int_1^{2}\int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2\sin\phi \, d\phi\, d\rho\, d\theta + \int_0^{2\pi}\int_0^{2}\int_0^{\pi/6} \rho^2\sin\phi \, d\phi\, d\rho\, d\theta$$

32. (a) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

(b) $\int_0^{2\pi}\int_0^1\int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$

$+\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$



33. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \left(8-\cos^3\phi\right)\sin\phi \, d\phi \, d\theta$

$= \frac{1}{3}\int_0^{2\pi} \left[-8\cos\phi + \frac{\cos^4\phi}{4}\right]_0^{\pi/2} d\theta = \frac{1}{3}\int_0^{2\pi} \left(8-\frac{1}{4}\right) d\theta = \left(\frac{31}{12}\right)(2\pi) = \frac{31\pi}{6}$

34. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \left(3\cos\phi + 3\cos^2\phi + \cos^3\phi\right)\sin\phi \, d\phi \, d\theta$

$= \frac{1}{3}\int_0^{2\pi} \left[-\frac{3}{2}\cos^2\phi - \cos^3\phi - \frac{1}{4}\cos^4\phi\right]_0^{\pi/2} d\theta = \frac{1}{3}\int_0^{2\pi} \left(\frac{3}{2}+1+\frac{1}{4}\right) d\theta = \frac{11}{12}\int_0^{2\pi} d\theta = \left(\frac{11}{12}\right)(2\pi) = \frac{11\pi}{6}$

35. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi} \left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} d\theta$

$= \frac{1}{12}(2)^4 \int_0^{2\pi} d\theta = \frac{4}{3}(2\pi) = \frac{8\pi}{3}$

36. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi} \left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi/2} d\theta$

$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}$

37. $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi} \left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2} d\theta$

$= \left(\frac{8}{3}\right)\left(\frac{1}{16}\right)\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

38. $V = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi} \left[-\cos\phi\right]_{\pi/3}^{\pi/2} d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}$

39. (a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$ (b) $8\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(c) $8\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$

40. (a) $\int_0^{\pi/2}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}} dz\, r\, dr\, d\theta$ (b) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(c) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 9\int_0^{\pi/2}\int_0^{\pi/4} \sin\phi\, d\phi\, d\theta = -9\int_0^{\pi/2}\left(\frac{1}{\sqrt{2}}-1\right) d\theta = \frac{9\pi(2-\sqrt{2})}{4}$

41. (a) $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ (b) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(c) $V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$

(d) $V = \int_0^{2\pi}\int_0^{\sqrt{3}} \left[r\left(4-r^2\right)^{1/2} - r\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{\left(4-r^2\right)^{3/2}}{3} - \frac{r^2}{2}\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(-\frac{1}{3}-\frac{3}{2}+\frac{4^{3/2}}{3}\right) d\theta$

$= \frac{5}{6}\int_0^{2\pi} d\theta = \frac{5\pi}{3}$

42. (a) $I_z = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r^2\, dz\, r\, dr\, d\theta$

(b) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \left(\rho^2 \sin^2\phi\right)\left(\rho^2 \sin\phi\right) d\rho\, d\phi\, d\theta$, since $r^2 = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta$ $= \rho^2 \sin^2\phi$

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin^3\phi\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\sin\phi\, d\phi\right) d\theta = \frac{2}{15}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2} d\theta$

$= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$

43. $V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 \left(5r - 4r^3 - r^5\right) dr\, d\theta = 4\int_0^{\pi/2}\left(\frac{5}{2}-1-\frac{1}{6}\right) d\theta$

$$= 4\int_0^{\pi/2} \frac{8}{6}\, d\theta = \frac{8\pi}{3}$$

44. $V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\ r\ dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 \left(r - r^2 + r\sqrt{1-r^2}\right) dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$

$$= 4\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

45. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz\ r\ dr\, d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta} -r^2\sin\theta\ dr\, d\theta = \int_{3\pi/2}^{2\pi}\left(-9\cos^3\theta\right)(\sin\theta)\ d\theta$

$$= \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$$

46. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^{r} dz\ r\ dr\, d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2\ dr\, d\theta = \frac{2}{3}\int_{\pi/2}^{\pi} -27\cos^3\theta\ d\theta$

$$= -18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi}\cos\theta\ d\theta\right) = -12[\sin\theta]_{\pi/2}^{\pi} = 12$$

47. $V = \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\ r\ dr\, d\theta = \int_0^{\pi/2}\int_0^{\sin\theta} r\sqrt{1-r^2}\ dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2} - 1\right] d\theta = -\frac{1}{3}\int_0^{\pi/2}\left(\cos^3\theta - 1\right) d\theta = -\frac{1}{3}\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\cos\theta\ d\theta\right) + \left[\frac{\theta}{3}\right]_0^{\pi/2}$$

$$= -\frac{2}{9}[\sin\theta]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}$$

48. $V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\ r\ dr\, d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}\ dr\, d\theta = \int_0^{\pi/2}\left[-\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$

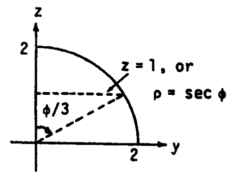
$$= \int_0^{\pi/2}\left[-\left(1-\cos^2\theta\right)^{3/2} + 1\right] d\theta = \int_0^{\pi/2}\left(1-\sin^3\theta\right) d\theta = \left[\theta + \frac{\sin^2\theta\cos\theta}{3}\right]_0^{\pi/2} - \frac{2}{3}\int_0^{\pi/2}\sin\theta\ d\theta$$

$$= \frac{\pi}{2} + \frac{2}{3}[\cos\theta]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}$$

49. $V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3} \frac{a^3}{3}\sin\phi \, d\phi\, d\theta = \frac{a^3}{3}\int_0^{2\pi} \left[-\cos\phi\right]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\right) d\theta = \frac{2\pi a^3}{3}$

50. $V = \int_0^{\pi/6}\int_0^{\pi/2}\int_0^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6}\int_0^{\pi/2} \sin\phi \, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6} d\theta = \frac{a^3\pi}{18}$

51. $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$



$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \left(8\sin\phi - \tan\phi\sec^2\phi\right) d\phi\, d\theta$

$= \frac{1}{3}\int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$

$= \frac{1}{3}\int_0^{2\pi} \left[-4 - \frac{1}{2}(3) + 8\right] d\theta = \frac{1}{3}\int_0^{2\pi} \frac{5}{2}\, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}$

52. $V = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4} \left(8\sec^3\phi - \sec^3\phi\right)\sin\phi \, d\phi\, d\theta$

$= \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4} \sec^3\phi \sin\phi \, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4} \tan\phi\sec^2\phi \, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2} \left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{14}{3}\int_0^{\pi/2} d\theta = \frac{7\pi}{3}$

53. $V = 4\int_0^{\pi/2}\int_0^{1}\int_0^{r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r^3\, dr\, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$

54. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r\, dr\, d\theta = 2\int_0^{\pi/2} d\theta = \pi$

55. $V = 8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{r} dz\, r\, dr\, d\theta = 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\, dr\, d\theta = 8\left(\frac{2\sqrt{2}-1}{3}\right)\int_0^{\pi/2} d\theta = \frac{4\pi\left(2\sqrt{2}-1\right)}{3}$

56. $V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz\ r\ dr\,d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r\sqrt{2-r^2}\ dr\,d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}} d\theta$

$= \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$

57. $V = \int_0^{2\pi} \int_0^2 \int_0^{4-r\sin\theta} dz\ r\ dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(4r - r^2 \sin\theta\right) dr\,d\theta = 8 \int_0^{2\pi} \left(1 - \frac{\sin\theta}{3}\right) d\theta = 16\pi$

58. $V = \int_0^{2\pi} \int_0^2 \int_0^{4-r\cos\theta - r\sin\theta} dz\ r\ dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[4r - r^2(\cos\theta + \sin\theta)\right] dr\,d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos\theta - \sin\theta)\ d\theta = 16\pi$

59. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$

$\Rightarrow V = 4 \int_0^{\pi/2} \int_0^1 \int_{4r^2}^{5-r^2} dz\ r\ dr\,d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(5r - 5r^3\right) dr\,d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}$

60. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow$

$V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz\ r\ dr\,d\theta = 4 \int_0^{\pi/2} \int_1^3 \left(9r - r^3\right) dr\,d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4}\right) d\theta$

$= 64 \int_0^{\pi/2} d\theta = 32\pi$

61. $V = 8 \int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{4-r^2}} dz\ r\ dr\,d\theta = 8 \int_0^{\pi/2} \int_0^1 r\left(4-r^2\right)^{1/2} dr\,d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}\left(4-r^2\right)^{3/2}\right]_0^1 d\theta$

$= -\frac{8}{3} \int_0^{\pi/2} \left(3^{3/2} - 8\right) d\theta = \frac{4\pi\left(8 - 3\sqrt{3}\right)}{3}$

62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$

$\Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \ge 0$. Thus, $x^2 + y^2 = 1$ and the volume is

given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\ r\ dr\,d\theta = 4 \int_0^{\pi/2} \int_0^1 \left[r\left(2-r^2\right)^{1/2} - r^3\right] dr\,d\theta$

$= 4 \int_0^{\pi/2} \left[-\frac{1}{3}\left(2-r^2\right)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

63. average $= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,dr\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 2r^2\,dr\,d\theta = \frac{1}{3\pi}\int_0^{2\pi} d\theta = \frac{2}{3}$

64. average $= \frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\,dz\,dr\,d\theta = \frac{3}{4\pi}\int_0^{2\pi}\int_0^1 2r^2\sqrt{1-r^2}\,dr\,d\theta$

$= \frac{3}{2\pi}\int_0^{2\pi}\left[\frac{1}{8}\sin^{-1}r - \frac{1}{8}r\sqrt{1-r^2}\left(1-r^2\right)\right]_0^1 d\theta = \frac{3}{16\pi}\int_0^{2\pi}\left(\frac{\pi}{2}+0\right)d\theta = \frac{3}{32}\int_0^{2\pi} d\theta = \left(\frac{3}{32}\right)(2\pi) = \frac{3\pi}{16}$

65. average $= \frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{3}{8\pi}\int_0^{2\pi} d\theta = \frac{3}{4}$

66. average $= \frac{1}{\left(\frac{2\pi}{3}\right)}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{8\pi}\int_0^{2\pi}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta = \frac{3}{8\pi}\int_0^{2\pi}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} d\theta$

$= \frac{3}{16\pi}\int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}$

67. $M = 4\int_0^{\pi/2}\int_0^1\int_0^r dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 r^2\,dr\,d\theta = \frac{4}{3}\int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^r z\,dz\,r\,dr\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\bar{x} = \bar{y} = 0$, by symmetry

68. $M = \int_0^{\pi/2}\int_0^2\int_0^r dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta = \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2}\int_0^2\int_0^r x\,dz\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^2 r^3\cos\theta\,dr\,d\theta = 4\int_0^{\pi/2}\cos\theta\,d\theta = 4$; $M_{xz} = \int_0^{\pi/2}\int_0^2\int_0^r y\,dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 r^3\sin\theta\,dr\,d\theta$

$= 4\int_0^{\pi/2}\sin\theta\,d\theta = 4$; $M_{xy} = \int_0^{\pi/2}\int_0^2\int_0^r z\,dz\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^2 r^3\,dr\,d\theta = 2\int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,

$\bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

69. $M = \frac{8\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 z\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = 4\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta$

$= 4 \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2}\right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8}\right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \Rightarrow \bar{z} = \frac{M_{xy}}{M} = (\pi)\left(\frac{3}{8\pi}\right) = \frac{3}{8}$, and $\bar{x} = \bar{y} = 0$, by symmetry

70. $M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2 - \sqrt{2}}{2} \, d\theta = \frac{\pi a^3 (2 - \sqrt{2})}{3}$;

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi \, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$

$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right)\left[\frac{3}{\pi a^3 (2 - \sqrt{2})}\right] = \left(\frac{3a}{8}\right)\left(\frac{2 + \sqrt{2}}{2}\right) = \frac{3(2 + \sqrt{2})a}{16}$, and $\bar{x} = \bar{y} = 0$, by symmetry

71. $M = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r^{3/2} \, dr \, d\theta = \frac{64}{5} \int_0^{2\pi} d\theta = \frac{128\pi}{5}$; $M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} z \, dz \, r \, dr \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \int_0^4 r^2 \, dr \, d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry

72. $M = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \int_0^1 2r\sqrt{1 - r^2} \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left[-\frac{2}{3}(1 - r^2)^{3/2}\right]_0^1 d\theta$

$= \frac{2}{3} \int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}$; $M_{yz} = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \cos \theta \, dz \, dr \, d\theta = 2 \int_{-\pi/3}^{\pi/3} \int_0^1 r^2 \sqrt{1 - r^2} \cos \theta \, dr \, d\theta$

$= 2 \int_{-\pi/3}^{\pi/3} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r\sqrt{1 - r^2}(1 - 2r^2)\right]_0^1 \cos \theta \, d\theta = \frac{\pi}{8} \int_{-\pi/3}^{\pi/3} \cos \theta \, d\theta = \frac{\pi}{8} [\sin \theta]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right)\left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8}$

$\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}$, and $\bar{y} = \bar{z} = 0$, by symmetry

73. $I_z = \int_0^{2\pi} \int_1^2 \int_0^4 (x^2 + y^2) \, dz \, r \, dr \, d\theta = 4 \int_0^{2\pi} \int_r^2 r^3 \, dr \, d\theta = \int_0^{2\pi} 15 \, d\theta = 30\pi$; $M = \int_0^{2\pi} \int_1^2 \int_0^4 dz \, r \, dr \, d\theta$

$= \int_0^{2\pi} \int_1^2 4r \, dr \, d\theta = \int_0^{2\pi} 6 \, d\theta = 12\pi \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{2}}$

74. (a) $I_z = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\, dz\, dr\, d\theta = 2\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$

(b) $I_x = \int_0^{2\pi}\int_0^1\int_{-1}^1 \left(r^2\sin^2\theta + z^2\right) dz\; r\; dr\, d\theta = 2\int_0^{2\pi}\int_0^1 \left(2r^3\sin^2\theta + \frac{2r}{3}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{2} + \frac{1}{3}\right) d\theta$

$= \left[\frac{\theta}{4} - \frac{\sin 2\theta}{8} + \frac{\theta}{3}\right]_0^{2\pi} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$

75. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the the x-axis which is through the vertex and parallel to the base of the cone $\Rightarrow I_x = \int_0^{2\pi}\int_0^1\int_r^1 \left(r^2\sin^2\theta + z^2\right) dz\; r\; dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 \left(r^3\sin^2\theta - r^4\sin^2\theta + \frac{r}{3} - \frac{r^4}{4}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{20} + \frac{1}{10}\right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$

76. $I_z = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a 2r^3\sqrt{a^2-r^2}\, dr\, d\theta = 2\int_0^{2\pi}\left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta$

$= 2\int_0^{2\pi} \frac{2}{15}a^5\, d\theta = \frac{8\pi a^5}{15}$

77. $I_z = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^{h} \left(x^2+y^2\right) dz\; r\; dr\, d\theta = \int_0^{2\pi}\int_0^a\int_{\frac{hr}{a}}^{h} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a \left(hr^3 - \frac{hr^4}{a}\right) dr\, d\theta$

$= \int_0^{2\pi} h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta = \int_0^{2\pi} h\left(\frac{a^4}{4} - \frac{a^5}{5a}\right) d\theta = \frac{ha^4}{20}\int_0^{2\pi} d\theta = \frac{\pi h a^4}{10}$

78. (a) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} z\, dz\; r\; dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^5\, dr\, d\theta = \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} z^2\, dz\; r\; dr\, d\theta$

$= \frac{1}{3}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}$, and $\bar{x} = \bar{y} = 0$, by symmetry;

$I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^3\, dz\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{16}\int_0^{2\pi} d\theta = \frac{\pi}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{3}}{2}$

(b) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^4\, dr\, d\theta = \frac{1}{5}\int_0^{2\pi} d\theta = \frac{2\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2\, dz\, dr\, d\theta$

$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{14}\int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^4\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{7}\int_0^{2\pi} d\theta = \frac{2\pi}{7} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{7}}$

79. (a) $M = \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta$

$= \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r - r^4\right) dr\, d\theta = \frac{1}{10}\int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 zr^3\, dz\, dr\, d\theta$

$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r^3 - r^5\right) dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{1}{3}}$

(b) $M = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta = \frac{\pi}{5}$ from part (a); $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^3\, dz\, r\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1 \left(r - r^5\right) dr\, d\theta$

$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 z^2r^3\, dz\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r^3 - r^6\right) dr\, d\theta$

$= \frac{1}{28}\int_0^{2\pi} d\theta = \frac{\pi}{14} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{14}}$

80. (a) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$;

$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^6 \sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{a^7}{7}\int_0^{2\pi}\int_0^{\pi} \left(1 - \cos^2\phi\right)\sin\phi\, d\phi\, d\theta = \frac{a^7}{7}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$

$= \frac{4a^7}{21}\int_0^{2\pi} d\theta = \frac{8a^7\pi}{21} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{10}{21}}\,a$

(b) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi} \frac{(1 - \cos 2\phi)}{2}\, d\phi\, d\theta = \frac{\pi a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}$;

$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^5 \sin^4\phi\, d\rho\, d\phi\, d\theta = \frac{a^6}{6}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi\, d\phi\, d\theta$

$$= \frac{a^6}{6}\int_0^{2\pi}\left(\left[\frac{-\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\,d\phi\right)d\theta = \frac{a^6}{8}\int_0^{2\pi}\left[\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right]_0^{\pi}d\theta = \frac{\pi a^6}{16}\int_0^{2\pi}d\theta$$

$$= \frac{a^6\pi^2}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{a}{\sqrt{2}}$$

81. $M = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} dz\ r\ dr\,d\theta = \int_0^{2\pi}\int_0^{a}\frac{h}{a}r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{h}{a}\int_0^{2\pi}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^{a}d\theta$

$$= \frac{h}{a}\int_0^{2\pi}\frac{a^3}{3}\,d\theta = \frac{2ha^2\pi}{3};\ M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} z\ dz\ r\ dr\,d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\left(a^2r-r^3\right)dr\,d\theta$$

$$= \frac{h^2}{2a^2}\int_0^{2\pi}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)d\theta = \frac{a^2h^2\pi}{4} \Rightarrow \bar{z} = \left(\frac{\pi a^2h^2}{4}\right)\left(\frac{3}{2ha^2\pi}\right) = \frac{3}{8}h, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

82. Let the base radius of the cone be a and the height h, and place the cone's axis of symmetry along the z-axis with the vertex at the origin. Then $M = \frac{\pi r^2 h}{3}$ and $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_{\left(\frac{h}{a}\right)r}^{h} z\ dz\ r\ dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{a}\left(h^2r-\frac{h^2}{a^2}r^3\right)dr\,d\theta$

$$= \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4a^2}\right]_0^{a}d\theta = \frac{h^2}{2}\int_0^{2\pi}\left(\frac{a^2}{2}-\frac{a^2}{4}\right)d\theta = \frac{h^2a^2}{8}\int_0^{2\pi}d\theta = \frac{h^2a^2\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2a^2\pi}{4}\right)\left(\frac{3}{\pi a^2 h}\right) = \frac{3}{4}h, \text{ and}$$

$\bar{x} = \bar{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

83. $M = \int_0^{2\pi}\int_0^{a}\int_0^{h}(z+1)\ dz\ r\ dr\,d\theta = \int_0^{2\pi}\int_0^{a}\left(\frac{h^2}{2}+h\right)r\ dr\,d\theta = \frac{a^2\left(h^2+2h\right)}{4}\int_0^{2\pi}d\theta = \frac{\pi a^2\left(h^2+2h\right)}{2};$

$$M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{h}\left(z^2+z\right)dz\ r\ dr\,d\theta = \int_0^{2\pi}\int_0^{a}\left(\frac{h^3}{3}+\frac{h^2}{2}\right)r\ dr\,d\theta = \frac{a^2\left(2h^3+3h^2\right)}{12}\int_0^{2\pi}d\theta = \frac{\pi a^2\left(2h^3+3h^2\right)}{6}$$

$$\Rightarrow \bar{z} = \left[\frac{\pi a^2\left(2h^3+3h^2\right)}{6}\right]\left[\frac{2}{\pi a^2\left(h^2+2h\right)}\right] = \frac{2h^2+3h}{3h+6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry;}$$

$$I_z = \int_0^{2\pi}\int_0^{a}\int_0^{h}(z+1)r^3\,dz\,dr\,d\theta = \left(\frac{h^2+2h}{2}\right)\int_0^{2\pi}\int_0^{a}r^3\,dr\,d\theta = \left(\frac{h^2+2h}{2}\right)\left(\frac{a^4}{4}\right)\int_0^{2\pi}d\theta = \frac{\pi a^4\left(h^2+2h\right)}{4};$$

$$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\pi a^4\left(h^2+2h\right)}{4}\cdot\frac{2}{\pi a^2\left(h^2+2h\right)}} = \frac{a}{\sqrt{2}}$$

84. The mass of the plant's atmosphere to an altitude h above the surface of the planet is the triple integral

$$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^{h} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_R^{h}\int_0^{2\pi}\int_0^{\pi} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_R^{h}\int_0^{2\pi} \left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta\, d\rho = 2\int_R^{h}\int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho}\rho^2 \, d\theta\, d\rho = 4\pi\mu_0 e^{cR}\int_R^{h} e^{-c\rho}\rho^2\, d\rho$$

$$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^h \quad \text{(by parts)}$$

$$= 4\pi\mu_0 e^{cR}\left(-\frac{h^2e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right).$$

The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

85. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant k: $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (k\rho - kR)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[k\frac{\rho^4}{4} - kR\frac{\rho^3}{3}\right]_0^R \sin\phi\, d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi} k\left(\frac{R^4}{4} - \frac{R^4}{3}\right)\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} -\frac{k}{12}R^4[-\cos\phi]_0^{\pi}\, d\theta = \int_0^{2\pi} -\frac{k}{6}R^4\, d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$$

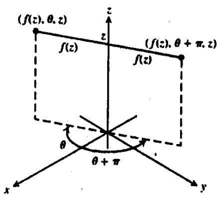
$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4}\rho + \frac{3M}{\pi R^4}R$. At the center of the planet $\rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4}\right)R = \frac{3M}{\pi R^3}$.

86. $x^2 + y^2 = a^2 \Rightarrow (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = a^2 \Rightarrow (\rho^2\sin^2\phi)(\cos^2\theta + \sin^2\theta) = a^2 \Rightarrow \rho^2\sin^2\phi = a^2$ $\Rightarrow \rho\sin\phi = a$ or $\rho\sin\phi = -a \Rightarrow \rho\sin\phi = a$ or $\rho = a\csc\phi$, since $0 \le \phi \le \pi$ and $\rho \ge 0$

87. (a) A plane perpendicular to the x-axis has the form $x = a$ in rectangular coordinates $\Rightarrow r\cos\theta = a$ $\Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r = a\sec\theta$, in cylindrical coordinates.
   (b) A plane perpendicular to the y-axis has the form $y = b$ in rectangular coordinates $\Rightarrow r\sin\theta = b$ $\Rightarrow r = \frac{b}{\sin\theta} \Rightarrow r = b\csc\theta$, in cylindrical coordinates.

88. $ax + by = c \Rightarrow a(r\cos\theta) + b(r\sin\theta) = c \Rightarrow r(a\cos\theta + b\sin\theta) = c \Rightarrow r = \frac{c}{a\cos\theta + b\sin\theta}$

89. The equation $r = f(z)$ implies that the point (r, θ, z) $= (f(z), \theta, z)$ will lie on the surface for all θ. In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does $\Rightarrow$ the surface is symmetric with respect to the z-axis.



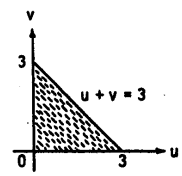
90. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ. In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric with respect to the z-axis.

12.7 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

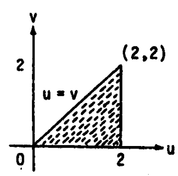
(b) The line segment $y = x$ from $(0,0)$ to $(1,1)$ is $x - y = 0$ $\Rightarrow u = 0$; the line segment $y = -2x$ from $(0,0)$ to $(1,-2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1,1)$ to $(1,-2)$ is $(x - y) + (2x + y) = 3 \Rightarrow u + v = 3$. The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

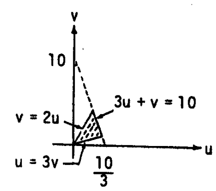
(b) The triangular region in the xy-plane has vertices $(0,0)$, $(2,0)$, and $\left(\frac{2}{3}, \frac{2}{3}\right)$. The line segment $y = x$ from $(0,0)$ to $\left(\frac{2}{3}, \frac{2}{3}\right)$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0,0)$ to $(2,0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $\left(\frac{2}{3}, \frac{2}{3}\right)$ to $(2,0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

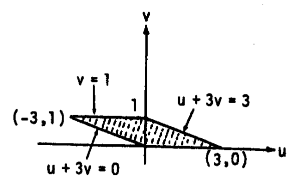
(b) The x-axis $y = 0 \Rightarrow u = 3v$; the y-axis $x = 0 \Rightarrow v = 2u$; the line $x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1$ $\Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The transformed region is sketched at the right.



4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

(b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$; $x = 0 \Rightarrow u + 3v = 0$; $y = x \Rightarrow v = 0$; $y = x + 1$ $\Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.



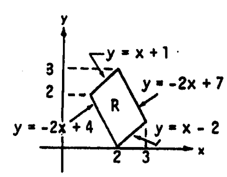
5. $$\int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right) dx\,dy = \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy = \frac{1}{2}\int_0^4 \left[\left(\frac{y}{2}+1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}+1\right)y + \left(\frac{y}{2}\right)y\right] dy$$

$$= \frac{1}{2}\int_0^4 (y + 1 - y)\,dy = \frac{1}{2}\int_0^4 dy = \frac{1}{2}(4) = 2$$

6. $$\iint_R (2x^2 - xy - y^2)\,dx\,dy = \iint_R (x - y)(2x + y)\,dx\,dy$$

$$= \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{3}\iint_G uv\,du\,dv;$$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



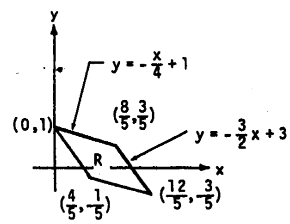
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -2x + 4$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 4$	$v = 4$
$y = -2x + 7$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 7$	$v = 7$
$y = x - 2$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) - 2$	$u = 2$
$y = x + 1$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) + 1$	$u = -1$

$$\Rightarrow \frac{1}{3}\iint_G uv\,du\,dv = \frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,dv\,du = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du = \frac{11}{2}\int_{-1}^{2} u\,du = \left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^{2} = \left(\frac{11}{4}\right)(4-1) = \frac{33}{4}$$

7. $\iint_R (3x^2 + 14xy + 8y^2)\,dx\,dy = \iint_R (3x + 2y)(x + 4y)\,dx\,dy$

$$= \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{10}\iint_G uv\,du\,dv;$$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$	$v = 4$

$$\Rightarrow \frac{1}{10}\iint_G uv\,du\,dv = \frac{1}{10}\int_{2}^{6}\int_{0}^{4} uv\,dv\,du = \frac{1}{10}\int_{2}^{6} u\left[\frac{v^2}{2}\right]_0^4 du = \frac{4}{5}\int_{2}^{6} u\,du = \left(\frac{4}{5}\right)\left[\frac{u^2}{2}\right]_2^6 = \left(\frac{4}{5}\right)(18-2) = \frac{64}{5}$$

8. $\iint_R 2(x - y)\,dx\,dy = \iint_G -2v\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_G -2v\,du\,dv$; the region G is sketched in Exercise 4

$$\Rightarrow \iint_G -2v\,du\,dv = \int_0^1\int_{-3v}^{3-3v} -2v\,du\,dv = \int_0^1 -2v(3 - 3v + 3v)\,dv = \int_0^1 -6v\,dv = \left[-3v^2\right]_0^1 = -3$$

9. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

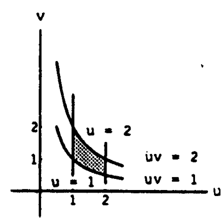
$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus

$$\iint_R \left(\sqrt{\tfrac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3 \int_1^2 (v+u)\left(\tfrac{2u}{v}\right) dv\,du = \int_1^3 \int_1^2 \left(2u + \tfrac{2u^2}{v}\right) dv\,du = \int_1^3 \left[2uv + 2u^2 \ln v\right]_1^2 du$$

$$= \int_1^3 \left(2u + 2u^2 \ln 2\right) du = \left[u^2 + \tfrac{2}{3}u^3 \ln 2\right]_1^3 = 8 + \tfrac{2}{3}(26)(\ln 2) = 8 + \tfrac{52}{3}(\ln 2)$$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and

the region G is sketched at the right



(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,

$$\int_1^2 \int_1^2 \tfrac{y}{x}\,dy\,dx = \int_1^2 \int_{1/u}^{2/u} \left(\tfrac{uv}{u}\right) u\,dv\,du = \int_1^2 \int_{1/u}^{2/u} uv\,dv\,du = \int_1^2 u\left[\tfrac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u\left(\tfrac{2}{u^2} - \tfrac{1}{2u^2}\right) du$$

$$= \tfrac{3}{2}\int_1^2 u\left(\tfrac{1}{u^2}\right) du = \tfrac{3}{2}[\ln u]_1^2 = \tfrac{3}{2}\ln 2$$

11. $x = ar\cos\theta$ and $y = br\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr$;

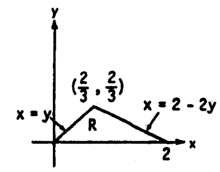
$$I_0 = \iint_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)|J(r,\theta)|\,dr\,d\theta = \int_0^{2\pi}\int_0^1 abr^3\left(a^2\cos^2\theta + b^2\sin^2\theta\right) dr\,d\theta$$

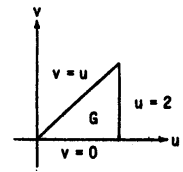
$$= \tfrac{ab}{4}\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\theta = \tfrac{ab}{4}\left[\tfrac{a^2\theta}{2} + \tfrac{a^2\sin 2\theta}{4} + \tfrac{b^2\theta}{2} - \tfrac{b^2\sin 2\theta}{4}\right]_0^{2\pi} = \frac{ab\pi\left(a^2+b^2\right)}{4}$$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint_R dy\,dx = \iint_G ab\,du\,dv = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du$

$$= 2ab\int_{-1}^{1}\sqrt{1-u^2}\,du = 2ab\left[\tfrac{u}{2}\sqrt{1-u^2} + \tfrac{1}{2}\sin^{-1}u\right]_{-1}^{1} = ab\left[\sin^{-1}1 - \sin^{-1}(-1)\right] = ab\left[\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right] = ab\pi$$

13. The region of integration R in the xy-plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:





xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y$	$\frac{1}{3}(u+2v) = \frac{1}{3}(u-v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u+2v) = 2 - \frac{2}{3}(u-v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u-v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3}\int_y^{2-2y} (x+2y)\,e^{(y-x)}\,dx\,dy = \int_0^2\int_0^u ue^{-v}\left|-\frac{1}{3}\right| dv\,du$

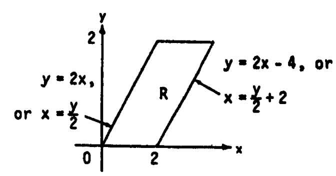
$= \frac{1}{3}\int_0^2 u\left[-e^{-v}\right]_0^u du = \frac{1}{3}\int_0^2 u\left(1-e^{-u}\right) du = \frac{1}{3}\left[u\left(u+e^{-u}\right) - \frac{u^2}{2} + e^{-u}\right]_0^2 = \frac{1}{3}\left[2\left(2+e^{-2}\right) - 2 + e^{-2} - 1\right]$

$= \frac{1}{3}\left(3e^{-2}+1\right) \approx 0.4687$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u+v) - v = 2u$ and

$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next, $u = x - \frac{v}{2}$

$= x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x-y)\,e^{(2x-y)^2}\,dx\,dy = \int_0^2 \int_0^2 v^3(2u)\,e^{4u^2}\,du\,dv = \int_0^2 v^3\left[\frac{1}{4}e^{4u^2}\right]_0^2 dv = \frac{1}{4}\int_0^2 v^3\left(e^{16}-1\right)dv$$
$$= \frac{1}{4}\left(e^{16}-1\right)\left[\frac{v^4}{4}\right]_0^2 = e^{16}-1$$

15. (a) $x = u\cos v$ and $y = u\sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = u\sin v$ and $y = u\cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix} = -u\sin^2 v - u\cos^2 v = -u$

16. (a) $x = u\cos v,\ y = u\sin v,\ z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = 2u-1,\ y = 3v-4,\ z = \frac{1}{2}(w-4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$

17. $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

$$= (\cos\phi)\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$
$$= \left(\rho^2\cos\phi\right)\left(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta\right) + \left(\rho^2\sin\phi\right)\left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\right)$$
$$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi = \left(\rho^2\sin\phi\right)\left(\cos^2\phi + \sin^2\phi\right) = \rho^2\sin\phi$$

18. Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u)\,du = \int_{g(a)}^{g(b)} f(g(x))g'(x)\,dx$ in accordance with formula (1) in Section 4.8. Note that $g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$. Several examples are presented in Section 4.6.

19. $\int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3}\right]_{y/2}^{1+(y/2)} dy\,dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3}\right] dy\,dz$

$= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3}\right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4}\right) dz = \int_0^3 \left(2 + \frac{4z}{3}\right) dz = \left[2z + \frac{2z^2}{3}\right]_0^3 = 12$

20. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in xyz-space into the spherical region $u^2 + v^2 + w^2 \le 1$ in uvw-space $\left(\text{which has volume } V = \frac{4}{3}\pi\right)$

$\Rightarrow V = \iiint_R dx\,dy\,dz = \iiint_G abc\,du\,dv\,dw = \frac{4\pi abc}{3}$

21. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; for R and G as in Exercise 20, $\iiint_R |xyz|\,dx\,dy\,dz$

$= \iiint_G a^2b^2c^2uvw\,dw\,dv\,du = 8a^2b^2c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)\left(\rho^2 \sin\phi\right) d\rho\,d\phi\,d\theta$

$= \frac{4a^2b^2c^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \cos\theta \sin^3\phi \cos\phi\,d\phi\,d\theta = \frac{a^2b^2c^2}{3} \int_0^{\pi/2} \sin\theta \cos\theta\,d\theta = \frac{a^2b^2c^2}{6}$

22. $u = x$, $v = xy$, and $w = 3z \Rightarrow x = u$, $y = \frac{v}{u}$, and $z = \frac{1}{3}w \Rightarrow J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$;

$\iiint_R \left(x^2y + 3xyz\right) dx\,dy\,dz = \iiint_G \left[u^2\left(\frac{v}{u}\right) + 3u\left(\frac{v}{u}\right)\left(\frac{w}{3}\right)\right] |J(u,v,w)|\,du\,dv\,dw = \frac{1}{3} \int_0^3 \int_0^2 \int_1^2 \left(v + \frac{vw}{u}\right) du\,dv\,dw$

$= \frac{1}{3} \int_0^3 \int_0^2 (v + vw \ln 2)\,dv\,dw = \frac{1}{3} \int_0^3 (1 + w \ln 2) \left[\frac{v^2}{2}\right]_0^2 dw = \frac{2}{3} \int_0^3 (1 + w \ln 2)\,dw = \frac{2}{3}\left[w + \frac{w^2}{2} \ln 2\right]_0^3$

$= \frac{2}{3}\left(3 + \frac{9}{2} \ln 2\right) = 2 + 3 \ln 2 = 2 + \ln 8$

23. The first moment about the xy-coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation in Exercise 21 is, $M_{xy} = \iiint_D z \, dz\, dy\, dx = \iiint_G cw \,|J(u, v, w)|\, du\, dv\, dw$

$= abc^2 \iiint_G w \, du\, dv\, dw = (abc^2)\cdot\left(M_{xy} \text{ of the hemisphere } x^2 + y^2 + z^2 = 1,\ z \geq 0\right) = \frac{abc^2\pi}{4}$;

the mass of the semi-ellipsoid is $\frac{2abc\pi}{3} \Rightarrow \bar{z} = \left(\frac{abc^2\pi}{4}\right)\left(\frac{3}{2abc\pi}\right) = \frac{3}{8}c$

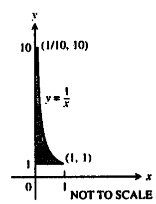
24. A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a a function of r. That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r \cos \theta$, $y = y$ and $z = r \sin \theta$, we have $V = \iiint_G r \, dy\, d\theta\, dy = \int_a^b \int_0^{2\pi} \int_0^{f(r)} r \, dy\, d\theta\, dr = \int_a^b \int_0^{2\pi} [ry]_0^{f(r)} \, d\theta\, dr = \int_a^b \int_0^{2\pi} r f(r) \, d\theta\, dr = \int_a^b [r\theta f(r)]_0^{2\pi} \, dr$

$= \int_a^b 2\pi r f(r) \, dr$. In the last integral, r is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing x instead of r we have $V = \int_a^b 2\pi x f(x) \, dx$, which is the same result obtained using the shell method.

CHAPTER 12 PRACTICE EXERCISES

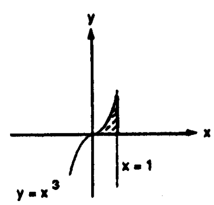
1. $\int_1^{10} \int_0^{1/y} y e^{xy} \, dx\, dy = \int_1^{10} \left[e^{xy}\right]_0^{1/y} dy$

$= \int_1^{10} (e - 1) \, dy = 9e - 9$



2. $\int_0^1 \int_0^{x^3} e^{y/x} \, dy\, dx = \int_1^1 x\left[e^{y/x}\right]_0^{x^3} dx$

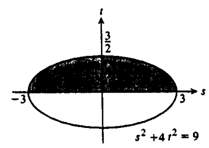
$= \int_0^1 \left(xe^{x^2} - x\right) dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt = \int_0^{3/2} [ts]\Big|_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}}\,dt$

$\displaystyle = \int_0^{3/2} 2t\sqrt{9-4t^2}\,dt = \left[-\tfrac{1}{6}\left(9-4t^2\right)^{3/2}\right]_0^{3/2}$

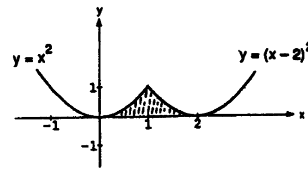
$\displaystyle = -\tfrac{1}{6}\left(0^{3/2}-9^{3/2}\right) = \tfrac{27}{6} = \tfrac{9}{2}$



4. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_{\sqrt{y}}^{2-\sqrt{y}}\,dy$

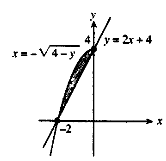
$\displaystyle = \tfrac{1}{2}\int_0^1 y\left(4-4\sqrt{y}+y-y\right)dy$

$\displaystyle = \int_0^1 \left(2y-2y^{3/2}\right)dy = \left[y^2-\frac{4y^{5/2}}{5}\right]_0^1 = \tfrac{1}{5}$



5. $\displaystyle\int_{-2}^{0}\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^{0}\left(-x^2-2x\right)dx$

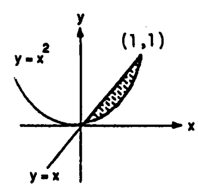
$\displaystyle = \left[-\frac{x^3}{3}-x^2\right]_{-2}^{0} = -\left(\tfrac{8}{3}-4\right) = \tfrac{4}{3}$



6. $\displaystyle\int_0^1\int_y^{\sqrt{y}} \sqrt{x}\,dx\,dy = \int_0^1 \left[\tfrac{2}{3}x^{3/2}\right]_y^{\sqrt{y}}\,dy$

$\displaystyle = \tfrac{2}{3}\int_0^1 \left(y^{3/4}-y^{3/2}\right)dy = \tfrac{2}{3}\left[\tfrac{4}{7}y^{7/4}-\tfrac{2}{5}y^{5/2}\right]_0^1$

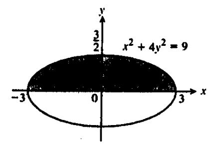
$\displaystyle = \tfrac{2}{3}\left(\tfrac{4}{7}-\tfrac{2}{5}\right) = \tfrac{4}{35}$



7. $\int_{-3}^{3}\int_{0}^{(1/2)\sqrt{9-x^2}} y\,dy\,dx = \int_{-3}^{3}\left[\frac{y^2}{2}\right]_0^{(1/2)\sqrt{9-x^2}} dx$

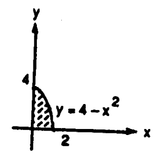
$= \int_{-3}^{3} \frac{1}{8}(9-x^2)\,dx = \left[\frac{9x}{8}-\frac{x^3}{24}\right]_{-3}^{3}$

$= \left(\frac{27}{8}-\frac{27}{24}\right)-\left(-\frac{27}{8}+\frac{27}{24}\right) = \frac{27}{6} = \frac{9}{2}$



8. $\int_{0}^{4}\int_{0}^{\sqrt{4-y}} 2x\,dx\,dy = \int_{0}^{4}\left[x^2\right]_0^{\sqrt{4-y}} dy$

$= \int_{0}^{4} (4-y)\,dy = \left[4y-\frac{y^2}{2}\right]_0^4 = 8$



9. $\int_{0}^{1}\int_{2y}^{2} 4\cos(x^2)\,dx\,dy = \int_{0}^{2}\int_{0}^{\pi/2} 4\cos(x^2)\,dy\,dx = \int_{0}^{2} 2x\cos(x^2)\,dx = \left[\sin(x^2)\right]_0^2 = \sin 4$

10. $\int_{0}^{2}\int_{y/2}^{1} e^{x^2}\,dx\,dy = \int_{0}^{1}\int_{0}^{2x} e^{x^2}\,dy\,dx = \int_{0}^{1} 2xe^{x^2}\,dx = \left[e^{x^2}\right]_0^1 = e-1$

11. $\int_{0}^{8}\int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\,dy\,dx = \int_{0}^{2}\int_{0}^{y^3} \frac{1}{y^4+1}\,dx\,dy = \frac{1}{4}\int_{0}^{2} \frac{4y^3}{y^4+1}\,dy = \frac{\ln 17}{4}$

12. $\int_{0}^{1}\int_{\sqrt[3]{y}}^{1} \frac{2\pi\sin(\pi x^2)}{x^2}\,dx\,dy = \int_{0}^{1}\int_{0}^{x^3} \frac{2\pi\sin(\pi x^2)}{x^2}\,dy\,dx = \int_{0}^{1} 2\pi x\sin(\pi x^2)\,dx = \left[-\cos(\pi x^2)\right]_0^1 = -(-1)-(-1) = 2$

13. $A = \int_{-2}^{0}\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^{0} (-x^2-2x)\,dx = \frac{4}{3}$

14. $A = \int_{1}^{4}\int_{2-y}^{\sqrt{y}} dx\,dy = \int_{1}^{4} (\sqrt{y}-2+y)\,dy = \frac{37}{6}$

15. $V = \int_{0}^{1}\int_{x}^{2-x} (x^2+y^2)\,dy\,dx = \int_{0}^{1}\left[x^2y+\frac{y^3}{3}\right]_x^{2-x} dx = \int_{0}^{1}\left[2x^2+\frac{(2-x)^3}{3}-\frac{7x^3}{3}\right]dx = \left[\frac{2x^3}{3}-\frac{(2-x)^4}{12}-\frac{7x^4}{12}\right]_0^1$

$= \left(\frac{2}{3}-\frac{1}{12}-\frac{7}{12}\right)+\frac{2^4}{12} = \frac{4}{3}$

16. $V = \int_{-3}^{2} \int_{x}^{6-x^2} x^2 \, dy \, dx = \int_{-3}^{2} \left[x^2 y\right]_{x}^{6-x^2} dx = \int_{-3}^{2} \left(6x^2 - x^4 - x^3\right) dx = \frac{125}{4}$

17. average value $= \int_{0}^{1} \int_{0}^{1} xy \, dy \, dx = \int_{0}^{1} \left[\frac{xy^2}{2}\right]_{0}^{1} dx = \int_{0}^{1} \frac{x}{2} \, dx = \frac{1}{4}$

18. average value $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} xy \, dy \, dx = \frac{4}{\pi} \int_{0}^{1} \left[\frac{xy^2}{2}\right]_{0}^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_{0}^{1} \left(x - x^3\right) dx = \frac{1}{2\pi}$

19. $M = \int_{1}^{2} \int_{2/x}^{2} dy \, dx = \int_{1}^{2} \left(2 - \frac{2}{x}\right) dx = 2 - \ln 4$; $M_y = \int_{1}^{2} \int_{2/x}^{2} x \, dy \, dx = \int_{1}^{2} x\left(2 - \frac{2}{x}\right) dx = 1$;
$M_x = \int_{1}^{2} \int_{2/x}^{2} y \, dy \, dx = \int_{1}^{2} \left(2 - \frac{2}{x^2}\right) dx = 1 \Rightarrow \overline{x} = \overline{y} = \frac{1}{2 - \ln 4}$

20. $M = \int_{0}^{4} \int_{-2y}^{2y-y^2} dx \, dy = \int_{0}^{4} \left(4y - y^2\right) dy = \frac{32}{3}$; $M_x = \int_{0}^{4} \int_{-2y}^{2y-y^2} y \, dx \, dy = \int_{0}^{4} \left(4y^2 - y^3\right) dy = \left[\frac{4y^3}{3} - \frac{y^4}{4}\right]_{0}^{4} = \frac{64}{3}$;
$M_y = \int_{0}^{4} \int_{-2y}^{2y-y^2} x \, dx \, dy = \int_{0}^{4} \left[\frac{\left(2y - y^2\right)^2}{2} - 2y^2\right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2}\right]_{0}^{4} = -\frac{128}{5} \Rightarrow \overline{x} = \frac{M_y}{M} = -\frac{12}{5}$ and $\overline{y} = \frac{M_x}{M} = 2$

21. $I_o = \int_{0}^{2} \int_{2x}^{4} \left(x^2 + y^2\right)(3) \, dy \, dx = 3 \int_{0}^{2} \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3}\right) dx = 104$

22. (a) $I_o = \int_{-2}^{2} \int_{-1}^{1} \left(x^2 + y^2\right) dy \, dx = \int_{-2}^{2} \left(2x^2 + \frac{2}{3}\right) dx = \frac{40}{3}$

(b) $I_x = \int_{-a}^{a} \int_{-b}^{b} y^2 \, dy \, dx = \int_{-a}^{a} \frac{2b^3}{3} \, dx = \frac{4ab^3}{3}$; $I_y = \int_{-a}^{a} \int_{-b}^{b} x^2 \, dx \, dy = \int_{-b}^{b} \frac{2a^3}{3} \, dy = \frac{4a^3 b}{3} \Rightarrow I_o = I_x + I_y$
$= \frac{4ab^3}{3} + \frac{4a^3 b}{3} = \frac{4ab\left(b^2 + a^2\right)}{3}$

23. $M = \delta \int_{0}^{3} \int_{0}^{2x/3} dy \, dx = \delta \int_{0}^{3} \frac{2x}{3} \, dx = 3\delta$; $I_x = \delta \int_{0}^{3} \int_{0}^{2x/3} y^2 \, dy \, dx = \frac{8\delta}{81} \int_{0}^{3} x^3 \, dx = \left(\frac{8\delta}{81}\right)\left(\frac{3^4}{4}\right) = 2\delta \Rightarrow R_x = \sqrt{\frac{2}{3}}$

24. $M = \int_{0}^{1} \int_{x^2}^{x} (x+1) \, dy \, dx = \int_{0}^{1} \left(x - x^3\right) dx = \frac{1}{4}$; $M_x = \int_{0}^{1} \int_{x^2}^{x} y(x+1) \, dy \, dx = \frac{1}{2} \int_{0}^{1} \left(x^3 - x^5 + x^2 - x^4\right) dx = \frac{13}{120}$;

$M_y = \int_0^1 \int_{x^2}^{x} x(x+1)\,dy\,dx = \int_0^1 (x^2 - x^4)\,dx = \frac{2}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{13}{30}$; $I_x = \int_0^1 \int_{x^2}^{x} y^2(x+1)\,dy\,dx$

$= \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6)\,dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}$; $I_y = \int_0^1 \int_{x^2}^{x} x^2(x+1)\,dy\,dx = \int_0^1 (x^3 - x^5)\,dx$

$= \frac{1}{12} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{1}{3}}$

25. $M = \int_{-1}^{1} \int_{-1}^{1} \left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^{1} \left(2x^2 + \frac{4}{3}\right) dx = 4$; $M_x = \int_{-1}^{1} \int_{-1}^{1} y\left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^{1} 0\,dx = 0$;

$M_y = \int_{-1}^{1} \int_{-1}^{1} x\left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^{1} \left(2x^3 + \frac{4}{3}x\right) dx = 0$

26. Place the ΔABC with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points A and C is $y = \frac{h}{a}x$; the line through the points C and B is $y = \frac{h}{a-b}(x-b)$. Thus, $M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta\,dx\,dy$

$= b\delta \int_0^h \left(1 - \frac{y}{h}\right) dy = \frac{\delta bh}{2}$; $I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2\delta\,dx\,dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = \frac{\delta bh^3}{12}$; $R_x = \sqrt{\frac{I_x}{M}} = \frac{h}{\sqrt{6}}$

27. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$

28. $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy = \int_0^{2\pi} \int_0^1 r \ln(r^2+1)\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u\,du\,d\theta = \frac{1}{2} \int_0^{2\pi} [u \ln u - u]_1^2\,d\theta$

$= \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1)\,d\theta = [\ln(4) - 1]\,\pi$

29. $M = \int_{-\pi/3}^{\pi/3} \int_0^3 r\,dr\,d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi$; $M_y = \int_{-\pi/3}^{\pi/3} \int_0^3 r^2 \cos\theta\,dr\,d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos\theta\,d\theta = 9\sqrt{3} \Rightarrow \overline{x} = \frac{3\sqrt{3}}{\pi}$,

and $\overline{y} = 0$ by symmetry

30. $M = \int_0^{\pi/2} \int_1^3 r\,dr\,d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$; $M_y = \int_0^{\pi/2} \int_1^3 r^2 \cos\theta\,dr\,d\theta = \frac{26}{3} \int_0^{\pi/2} \cos\theta\,d\theta = \frac{26}{3} \Rightarrow \overline{x} = \frac{13}{3\pi}$, and

$\overline{y} = \frac{13}{3\pi}$ by symmetry

31. (a) $M = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta$ (b)

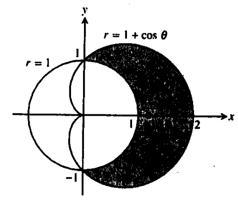
$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{8+\pi}{4};$

$M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r\cos\theta)\, r\, dr\, d\theta$

$= \int_{-\pi/2}^{\pi/2} \left(\cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3}\right) d\theta$

$= \frac{32+15\pi}{24} \Rightarrow \overline{x} = \frac{15\pi+32}{6\pi+48}$, and

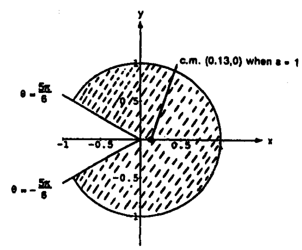
$\overline{y} = 0$ by symmetry



32. (a) $M = \int_{-\alpha}^{\alpha} \int_0^{a} r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2}\, d\theta = a^2\alpha$; $M_y = \int_{-\alpha}^{\alpha} \int_0^{a} (r\cos\theta)\, r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^3\cos\theta}{3}\, d\theta = \frac{2a^3\sin\alpha}{3}$

$\Rightarrow \overline{x} = \frac{2a\sin\alpha}{3\alpha}$, and $\overline{y} = 0$ by symmetry; $\lim\limits_{\alpha\to\pi^-} \overline{x} = \lim\limits_{\alpha\to\pi^-} \frac{2a\sin\alpha}{3\alpha} = 0$

(b) $\overline{x} = \frac{2a}{5\pi}$ and $\overline{y} = 0$



33. $(x^2+y^2)^2 - (x^2-y^2) = 0 \Rightarrow r^4 - r^2\cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta$

$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta - \frac{\tan\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$

34. (a) $\iint_R \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/3}\int_0^{\sec\theta} \frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\pi/3}\left[-\frac{1}{2(1+r^2)}\right]_0^{\sec\theta} d\theta$

$= \int_0^{\pi/3}\left[\frac{1}{2}-\frac{1}{2(1+\sec^2\theta)}\right]d\theta = \frac{1}{2}\int_0^{\pi/3}\frac{\sec^2\theta}{1+\sec^2\theta}\,d\theta;\left[\begin{array}{c} u=\tan\theta \\ du=\sec^2\theta\,d\theta\end{array}\right]\to\frac{1}{2}\int_0^{\sqrt{3}}\frac{du}{2+u^2}$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}\right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4}\tan^{-1}\sqrt{\frac{3}{2}}$

(b) $\iint_R \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty}\frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\pi/2}\lim_{b\to\infty}\left[-\frac{1}{2(1+r^2)}\right]d\theta$

$= \int_0^{\pi/2}\lim_{b\to\infty}\left[\frac{1}{2}-\frac{1}{2(1+b^2)}\right]d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4}$

35. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(x+y+z)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{\pi}[\sin(z+y+\pi)-\sin(z+y)]\,dy\,dz$

$= \int_0^{\pi}[-\cos(z+2\pi)+\cos(z+\pi)+\cos z-\cos(z+\pi)]\,dz = 0$

36. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dz\,dy\,dx = \int_{\ln 6}^{\ln 7}\int_0^{\ln 2} e^{(x+y)}\,dy\,dx = \int_{\ln 6}^{\ln 7} e^x\,dx = 1$

37. $\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dz\,dy\,dx = \int_0^1\int_0^{x^2}\left(\frac{3x^2}{2}-\frac{3y^2}{2}\right)dy\,dx = \int_0^1\left(\frac{3x^4}{2}-\frac{x^6}{2}\right)dx = \frac{8}{35}$

38. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\,dy\,dz\,dx = \int_1^e\int_1^x \frac{1}{z}\,dz\,dx = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = 1$

39. $V = 2\int_0^{\pi/2}\int_{-\cos y}^{0}\int_0^{-2x} dz\,dx\,dy = 2\int_0^{\pi/2}\int_{-\cos y}^{0} -2x\,dx\,dy = 2\int_0^{\pi/2}\cos^2 y\,dy = 2\left[\frac{y}{2}+\frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi}{2}$

40. $V = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}(4-x^2)\,dy\,dx = 4\int_0^2(4-x^2)^{3/2}\,dx$

$= \left[x(4-x^2)^{3/2}+6x\sqrt{4-x^2}+24\sin^{-1}\frac{x}{2}\right]_0^2 = 24\sin^{-1}1 = 12\pi$

41. average $= \frac{1}{3}\int_0^1\int_0^3\int_0^1 30xz\sqrt{x^2+y}\,dz\,dy\,dx = \frac{1}{3}\int_0^1\int_0^3 15x\sqrt{x^2+y}\,dy\,dx = \frac{1}{3}\int_0^3\int_0^1 15x\sqrt{x^2+y}\,dx\,dy$

$= \frac{1}{3}\int_0^3 \left[5\left(x^2+y\right)^{3/2}\right]_0^1 dy = \frac{1}{3}\int_0^3 \left[5(1+y)^{3/2} - 5y^{3/2}\right] dy = \frac{1}{3}\left[2(1+y)^{5/2} - 2y^{5/2}\right]_0^3 = \frac{1}{3}\left[2(4)^{5/2} - 2(3)^{5/2} - 2\right]$

$= \frac{1}{3}\left[2\left(31 - 3^{5/2}\right)\right]$

42. average $= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin\phi\; d\rho\, d\phi\, d\theta = \frac{3a}{16\pi}\int_0^{2\pi}\int_0^{\pi} \sin\phi\; d\phi\, d\theta = \frac{3a}{8\pi}\int_0^{2\pi} d\theta = \frac{3a}{4}$

43. (a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\; dz\,dx\,dy$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2 \sin\phi\; d\rho\, d\phi\, d\theta$

(c) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\; dz\; r\; dr\, d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{2}} \left[r\left(4-r^2\right)^{1/2} - r^2\right] dr\, d\theta = 3\int_0^{2\pi} \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$

$= \int_0^{2\pi} \left(-2^{3/2} - 2^{3/2} + 4^{3/2}\right) d\theta = \left(8 - 4\sqrt{2}\right)\int_0^{2\pi} d\theta = 2\pi\left(8 - 4\sqrt{2}\right)$

44. (a) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2\; dz\; r\; dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\; dz\; r\; dr\, d\theta$

(b) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\; dz\; r\; dr\, d\theta = 84\int_0^{\pi/2}\int_0^1 r^6\sin^2\theta\cos\theta\; dr\, d\theta = 12\int_0^{\pi/2} \sin^2\theta\cos\theta\; d\theta = 4$

45. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\; d\rho\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} (\sec\phi)(\sec\phi\tan\phi)\; d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi} \left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{\pi}{3}$

46. (a) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} (6+4y)\; dz\,dy\,dx$ (b) $\int_0^{\pi/2}\int_0^1\int_0^{r} (6+4r\sin\theta)\; dz\; r\; dr\, d\theta$

(c) $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} (6+4\rho\sin\phi\sin\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

(d) $\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (6r^2+4r^3\sin\theta)\,dr\,d\theta = \int_0^{\pi/2} \left[2r^3+r^4\sin\theta\right]_0^1 d\theta$

$= \int_0^{\pi/2} (2+\sin\theta)\,d\theta = [2\theta-\cos\theta]_0^{\pi/2} = \pi+1$

47. $\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\,dz\,dy\,dx + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\,dz\,dy\,dx$

48. (a) Bounded on the top and bottom by the sphere $x^2+y^2+z^2=4$, on the right by the right circular cylinder $(x-1)^2+y^2=1$, on the left by the plane $y=0$

(b) $\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta$

49. (a) $V = \int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(r\sqrt{8-r^2}-2r\right)dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(8-r^2)^{3/2}-r^2\right]_0^2 d\theta$

$= \int_0^{2\pi}\left[-\frac{1}{3}(4)^{3/2}-4+\frac{1}{3}(8)^{3/2}\right]d\theta = \int_0^{2\pi}\frac{4}{3}\left(-2-3+2\sqrt{8}\right)d\theta = \frac{4}{3}(4\sqrt{2}-5)\int_0^{2\pi} d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$

(b) $V = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\left(2\sqrt{2}\sin\phi-\sec^3\phi\sin\phi\right)d\phi\,d\theta$

$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\left(2\sqrt{2}\sin\phi-\tan\phi\sec^2\phi\right)d\phi\,d\theta = \frac{8}{3}\int_0^{2\pi}\left[-2\sqrt{2}\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{8}{3}\int_0^{2\pi}\left(-2-\frac{1}{2}+2\sqrt{2}\right)d\theta = \frac{8}{3}\int_0^{2\pi}\left(\frac{-5+4\sqrt{2}}{2}\right)d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$

50. $I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^2 (\rho\sin\phi)^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^2 \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$

$= \frac{32}{5}\int_0^{2\pi}\int_0^{\pi/3}\left(\sin\phi-\cos^2\phi\sin\phi\right)d\phi\,d\theta = \frac{32}{5}\int_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

51. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi}\int_0^{\pi}\int_a^b \delta(\rho\sin\phi)^2\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta$

$$= \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta = \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\int_0^{\pi}\left(\sin\phi-\cos^2\phi\sin\phi\right)d\phi\,d\theta$$

$$= \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi}d\theta = \frac{4\delta\left(b^5-a^5\right)}{15}\int_0^{2\pi}d\theta = \frac{8\pi\delta\left(b^5-a^5\right)}{15}$$

52. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}(\rho\sin\phi)^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^5\sin^3\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^6(1+\cos\phi)\sin\phi\,d\phi\,d\theta;$$

$$\begin{bmatrix} u = 1-\cos\phi \\ du = \sin\phi\,d\phi \end{bmatrix} \to \frac{1}{5}\int_0^{2\pi}\int_0^{2}u^6(2-u)\,du\,d\theta = \frac{1}{5}\int_0^{2\pi}\left[\frac{2u^7}{7}-\frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5}\int_0^{2\pi}\left(\frac{1}{7}-\frac{1}{8}\right)2^8\,d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\frac{2^3\cdot 2^5}{56}\,d\theta = \frac{32}{35}\int_0^{2\pi}d\theta = \frac{64\pi}{35}$$

CHAPTER 12 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\,dy\,dx$ (b) $V = \int_{-3}^{2}\int_{x}^{6-x^2}\int_0^{x^2} dz\,dy\,dx$

(c) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\,dy\,dx = \int_{-3}^{2}\left(6x^2-x^4-x^3\right)dx = \left[2x^3-\frac{x^5}{5}-\frac{x^4}{4}\right]_{-3}^{2} = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z=-3$. Then $9 = 25-x^2-y^2 \Rightarrow x^2+y^2 = 16$ is the projection of the volume of water onto the xy-plane

$$\Rightarrow V = \int_0^{2\pi}\int_0^{4}\int_{-\sqrt{25-r^2}}^{-3} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{4}\left(r\sqrt{25-r^2}-3r\right)dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(25-r^2\right)^{3/2}-\frac{3}{2}r^2\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}(9)^{3/2}-24+\frac{1}{3}(25)^{3/2}\right]d\theta = \int_0^{2\pi}\frac{26}{3}\,d\theta = \frac{52\pi}{3}$$

3. Using cylindrical coordinates, $V = \int_0^{2\pi}\int_0^{1}\int_0^{2-r(\cos\theta+\sin\theta)} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1}\left(2r-r^2\cos\theta-r^2\sin\theta\right)dr\,d\theta$

$$= \int_0^{2\pi} \left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta = \left[\theta + \frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta\right]_0^{2\pi} = 2\pi$$

4. $$V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(r\sqrt{2-r^2} - r^3\right) dr\, d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{3}\left(2-r^2\right)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= 4 \int_0^{\pi/2} \left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3}\right) d\theta = \left(\frac{8\sqrt{2}-7}{3}\right) \int_0^{\pi/2} d\theta = \frac{\pi\left(8\sqrt{2}-7\right)}{6}$$

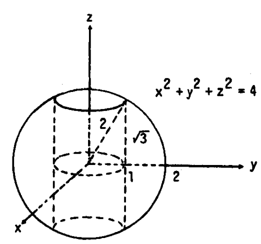
5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 1$. Thus the volume is

$$V = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} dz\, dy\, dx = 4 \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{3-r^2} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(3r - 3r^3\right) dr\, d\theta = 3 \int_0^{\pi/2} d\theta = \frac{3\pi}{2}$$

6. $$V = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{64}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4\phi\, d\phi\, d\theta$$

$$= \frac{64}{3} \int_0^{\pi/2} \left[-\frac{\sin^3\phi\cos\phi}{4}\Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2\phi\, d\phi\right] d\theta = 16 \int_0^{\pi/2} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi/2} d\theta = 4\pi \int_0^{\pi/2} d\theta = 2\pi^2$$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



(b) $$V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \left(3 - z^2\right) dz\, d\theta = 2\sqrt{3} \int_0^{2\pi} d\theta = 4\sqrt{3}\pi$$

8. $$V = \int_0^{\pi} \int_0^{3\sin\theta} \int_0^{\sqrt{9-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi} \int_0^{3\sin\theta} r\sqrt{9-r^2}\, dr\, d\theta = \int_0^{\pi} \left[-\frac{1}{3}\left(9-r^2\right)^{3/2}\right]_0^{3\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[-\frac{1}{3}\left(9 - 9\sin^2\theta\right)^{3/2} + \frac{1}{3}(9)^{3/2}\right] d\theta = 9 \int_0^{\pi} \left[1 - \left(1 - \sin^2\theta\right)^{3/2}\right] d\theta = 9 \int_0^{\pi} \left(1 - \cos^3\theta\right) d\theta$$

$$= \int_0^{\pi} \left(1 - \cos\theta + \sin^2\theta\cos\theta\right) d\theta = 9\left[\theta + \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{\pi} = 9\pi$$

9. The surfaces intersect when $x^2+y^2=\frac{x^2+y^2+1}{2} \Rightarrow x^2+y^2=1$. Thus the volume in cylindrical coordinates is $V=4\int_0^{\pi/2}\int_0^1\int_{r^2}^{(r^2+1)/2} dz\ r\ dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 \left(\frac{r}{2}-\frac{r^3}{2}\right) dr\,d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{4}-\frac{r^4}{8}\right]_0^1 d\theta$

$=\frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

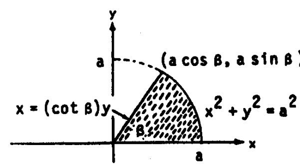
10. $V=\int_0^{\pi/2}\int_1^2\int_0^{r^2\sin\theta\cos\theta} dz\ r\ dr\,d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin\theta\cos\theta\ dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_1^2 \sin\theta\cos\theta\ d\theta$

$=\frac{15}{4}\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{15}{8}$

11. $\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}dx = \int_0^{\infty}\int_a^b e^{-xy}\,dy\,dx = \int_a^b\int_0^{\infty} e^{-xy}\,dx\,dy = \int_a^b\left(\lim_{t\to\infty}\int_0^t e^{-xy}\,dx\right)dy$

$=\int_a^b \lim_{t\to\infty}\left[-\frac{e^{-xy}}{y}\right]_0^t dy = \int_a^b \lim_{t\to\infty}\left(\frac{1}{y}-\frac{e^{-yt}}{y}\right)dy = \int_a^b \frac{1}{y}\,dy = [\ln y]_a^b = \ln\left(\frac{b}{a}\right)$

12. (a) The region of integration is sketched at the right



$\Rightarrow \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = \int_0^{\beta}\int_0^{a} r\ln(r^2)\,dr\,d\theta;$

$\begin{bmatrix} u=r^2 \\ du=2r\ dr\end{bmatrix} \to \frac{1}{2}\int_0^{\beta}\int_0^{a^2}\ln u\ du\,d\theta = \frac{1}{2}\int_0^{\beta}[u\ln u-u]_0^{a^2}\,d\theta$

$=\frac{1}{2}\int_0^{\beta}\left[2a^2\ln a - a^2 - \lim_{t\to 0} t\ln t\right]d\theta = \frac{a^2}{2}\int_0^{\beta}(2\ln a-1)\,d\theta = a^2\beta\left(\ln a-\frac{1}{2}\right)$

(b) $\int_0^{a\cos\beta}\int_0^{(\tan\beta)x}\ln(x^2+y^2)\,dy\,dx + \int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}}\ln(x^2+y^2)\,dy\,dx$

13. $\int_0^x\int_0^u e^{m(x-t)}f(t)\,dt\,du = \int_0^x\int_1^x e^{m(x-t)}f(t)\,du\,dt = \int_0^x (x-t)e^{m(x-t)}f(t)\,dt$; also

$\int_0^x\int_0^v\int_0^u e^{m(x-t)}f(t)\,dt\,du\,dv = \int_0^x\int_t^x\int_t^v e^{m(x-t)}f(t)\,du\,dv\,dt = \int_0^x\int_t^x (v-t)e^{m(x-t)}f(t)\,dv\,dt$

$$= \int_0^x \left[\frac{1}{2}(v-t)^2 e^{m(x-t)} f(t)\right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\, dt$$

14. $$\int_0^1 f(x)\left(\int_0^x g(x-y)f(y)\,dy\right)dx = \int_0^1\int_0^x g(x-y)f(x)f(y)\,dy\,dx$$

$$= \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy = \int_0^1 f(y)\left(\int_y^1 g(x-y)f(x)\,dx\right)dy;$$

$$\int_0^1\int_0^1 g(|x-y|)f(x)f(y)\,dx\,dy = \int_0^1\int_0^x g(x-y)f(x)f(y)\,dy\,dx + \int_0^1\int_x^1 g(y-x)f(x)f(y)\,dy\,dx$$

$$= \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \int_0^1\int_x^1 g(y-x)f(x)f(y)\,dy\,dx$$

$$= \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \underbrace{\int_0^1\int_y^1 g(x-y)f(y)f(x)\,dx\,dy}_{\text{simply interchange x and y variable names}}$$

$= 2\int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy$, and the statement now follows.

15. $$I_o(a) = \int_0^a\int_0^{x/a^2} (x^2+y^2)\,dy\,dx = \int_0^a \left[x^2y + \frac{y^3}{3}\right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6}\right)dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6}\right]_0^a$$

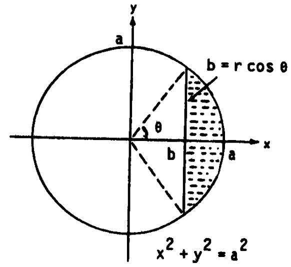
$= \frac{a^2}{4} + \frac{1}{12}a^{-2}$; $I_o'(a) = \frac{1}{2}a - \frac{1}{6}a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}$. Since $I_o''(a) = \frac{1}{2} + \frac{1}{2}a^{-4} > 0$, the value of a does provide a minimum for the polar moment of inertia $I_o(a)$.

16. $$I_o = \int_0^2\int_{2x}^4 (x^2+y^2)(3)\,dy\,dx = 3\int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3}\right)dx = 104$$

17. $M = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta}\left(\frac{a^2}{2}-\frac{b^2}{2}\sec^2\theta\right)d\theta$

$= a^2\theta - b^2\tan\theta = a^2\cos^{-1}\left(\frac{b}{a}\right) - b^2\left(\frac{\sqrt{a^2-b^2}}{b}\right)$

$= a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2}$; $I_o = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r^3\,dr\,d\theta$



$= \frac{1}{4}\int_{-\theta}^{\theta}\left(a^4 - b^4\sec^4\theta\right)d\theta = \frac{1}{4}\int_{-\theta}^{\theta}\left[a^4 - b^4(1+\tan^2\theta)(\sec^2\theta)\right]d\theta = \frac{1}{4}\left[a^4\theta - b^4\tan\theta - \frac{b^4\tan^3\theta}{3}\right]_{-\theta}^{\theta}$

$= \frac{a^4\theta}{2} - \frac{b^4\tan\theta}{2} - \frac{b^4\tan^3\theta}{6} = \frac{1}{2}a^4\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}b^3\sqrt{a^2-b^2} - \frac{1}{6}b\left(a^2-b^2\right)^{3/2}$

18. $M = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} dx\,dy = \int_{-2}^{2}\left(1-\frac{y^2}{4}\right)dy = \left[y - \frac{y^3}{12}\right]_{-2}^{2} = \frac{8}{3}$; $M_y = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} x\,dx\,dy$

$= \int_{-2}^{2}\left[\frac{x^2}{2}\right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \frac{3}{32}\int_{-2}^{2}\left(16 - 8y^2 + y^4\right)dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$

$= \frac{3}{16}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \left(\frac{3}{16}\right)\left(\frac{32\cdot 8}{15}\right) = \frac{48}{15} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{48}{15}\right)\left(\frac{3}{8}\right) = \frac{6}{5}$, and $\bar{y} = 0$ by symmetry

19. $\int_0^a\int_0^b e^{\max\left(b^2x^2,\,a^2y^2\right)}\,dy\,dx = \int_0^a\int_0^{bx/a} e^{b^2x^2}\,dy\,dx + \int_0^b\int_0^{ay/b} e^{a^2y^2}\,dx\,dy$

$= \int_0^a\left(\frac{b}{a}x\right)e^{b^2x^2}\,dx + \int_0^b\left(\frac{a}{b}y\right)e^{a^2y^2}\,dy = \left[\frac{1}{2ab}e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba}e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{b^2a^2}-1\right) + \frac{1}{2ab}\left(e^{a^2b^2}-1\right)$

$= \frac{1}{ab}\left(e^{a^2b^2}-1\right)$

20. $\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{\partial^2F(x,y)}{\partial x\,\partial y}\,dx\,dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1}dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right]dx = \left[F(x_1,y) - F(x_0,y)\right]_{y_0}^{y_1}$

$= F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$

21. (a) (i) Fubini's Theorem
(ii) Treating $G(y)$ as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

(b) $\int_0^{\ln 2}\int_0^{\pi/2} e^x \cos y \, dy\, dx = \left(\int_0^{\ln 2} e^x\, dx\right)\left(\int_0^{\pi/2} \cos y\, dy\right) = \left(e^{\ln 2} - e^0\right)\left(\sin \frac{\pi}{2} - \sin 0\right) = (1)(1) = 1$

(c) $\int_1^2\int_{-1}^1 \frac{x}{y^2}\, dx\, dy = \left(\int_1^2 \frac{1}{y^2}\, dy\right)\left(\int_{-1}^1 x\, dx\right) = \left[-\frac{1}{y}\right]_1^2\left[\frac{x^2}{2}\right]_{-1}^1 = \left(-\frac{1}{2}+1\right)\left(\frac{1}{2}-\frac{1}{2}\right) = 0$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1 x + u_2 y$; the area of the region of integration is $\frac{1}{2}$

$\Rightarrow$ average $= 2\int_0^1\int_0^{1-x} (u_1 x + u_2 y)\, dy\, dx = 2\int_0^1 \left[u_1 x(1-x) + \frac{1}{2}u_2(1-x)^2\right] dx$

$= 2\left[u_1\left(\frac{x^2}{2} - \frac{x^3}{3}\right) - \left(\frac{1}{2}u_2\right)\frac{(1-x)^3}{3}\right]_0^1 = 2\left(\frac{1}{6}u_1 + \frac{1}{6}u_2\right) = \frac{1}{3}(u_1 + u_2)$

(b) average $= \frac{1}{\text{area}}\iint_R (u_1 x + u_2 y)\, dA = \frac{u_1}{\text{area}}\iint_R x\, dA + \frac{u_2}{\text{area}}\iint_R y\, dA = u_1\left(\frac{M_y}{M}\right) + u_2\left(\frac{M_x}{M}\right) = u_1\overline{x} + u_2\overline{y}$

23. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\, dx\, dy = \int_0^{\pi/2}\int_0^\infty \left(e^{-r^2}\right) r\, dr\, d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\, dr\right] d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \lim_{b\to\infty}\left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\, dt = \int_0^\infty \left(y^2\right)^{-1/2}e^{-y^2}(2y)\, dy = 2\int_0^\infty e^{-y^2}\, dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$, where $y = \sqrt{t}$

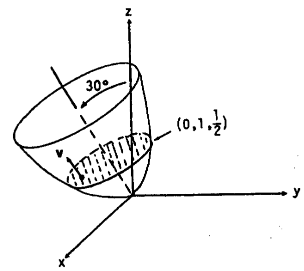
24. $Q = \int_0^{2\pi}\int_0^R kr^2(1 - \sin\theta)\, dr\, d\theta = \frac{kR^3}{3}\int_0^{2\pi}(1 - \sin\theta)\, d\theta = \frac{kR^3}{3}\left[\theta + \cos\theta\right]_0^{2\pi} = \frac{2\pi kR^3}{3}$

25. For a height h in the bowl the volume of water is $V = \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}\int_{x^2+y^2}^{h} dz\, dy\, dx$

$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \left(h - x^2 - y^2\right) dy\, dx = \int_0^{2\pi}\int_0^{\sqrt{h}} \left(h - r^2\right) r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{hr^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi}\frac{h^2}{4}\, d\theta = \frac{h^2\pi}{2}.$

Since the top of the bowl has area 10π, then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.

26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30°, a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies $b = \mathbf{v} \cdot \mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\Rightarrow a = -\sqrt{1-b^2} = -\frac{1}{2} \Rightarrow \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$ $\Rightarrow -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) = 0 \Rightarrow z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$ is an equation of the plane of the water level. Therefore the volume of water is $V = \iint_R \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} dz\,dy\,dx$, where R is the interior of the ellipse



$x^2 + y^2 - \frac{2}{3}y - 1 + \frac{2}{\sqrt{3}} = 0$. When $x = 0$, then $y = \alpha$ or $y = \beta$, where $\alpha = \dfrac{\frac{2}{3} + \sqrt{\frac{4}{9} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$

and $\beta = \dfrac{\frac{2}{3} - \sqrt{\frac{4}{9} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2} \Rightarrow V = \displaystyle\int_\alpha^\beta \int_{-\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}}^{\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}} \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} 1\,dz\,dx\,dy$

(b) $x = 0 \Rightarrow z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \Rightarrow \frac{dz}{dy} = 1 \Rightarrow$ the tangent line has slope 1 or a 45° slant $\Rightarrow$ at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty \Rightarrow \iiint_D z\left(r^2+z^2\right)^{-5/2}\,dV$

$$= \int_0^{2\pi}\int_0^1\int_1^\infty \frac{z}{\left(r^2+z^2\right)^{5/2}}\,dz\,r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1\int_1^a \frac{rz}{\left(r^2+z^2\right)^{5/2}}\,dz\,dr\,d\theta$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{\left(r^2+z^2\right)^{3/2}}\right]_1^a dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{\left(r^2+a^2\right)^{3/2}} + \left(\frac{1}{3}\right)\frac{r}{\left(r^2+1\right)^{3/2}}\right] dr\,d\theta$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}\left(r^2+a^2\right)^{-1/2} - \frac{1}{3}\left(r^2+1\right)^{-1/2}\right]_0^1 d\theta = \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(2^{-1/2}\right) - \frac{1}{3}\left(a^2\right)^{-1/2} + \frac{1}{3}\right] d\theta$$

$$= \lim_{a\to\infty} 2\pi\left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3}\left(\frac{1}{a}\right) + \frac{1}{3}\right] = 2\pi\left[\frac{1}{3} - \left(\frac{1}{3}\right)\frac{\sqrt{2}}{2}\right].$$

28. Let's see?

The length of the "unit" line segment is: $L = 2\int_0^1 dx = 2$

The area of the unit circle is: $A = 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = \pi.$

The volume of the unit sphere is: $V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi.$

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$V_{\text{hyper}} = 16\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx = 16\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\,dz\,dy\,dx$$

$$= 16\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2}\sqrt{1-\frac{z^2}{1-x^2-y^2}}\,dz\,dy\,dx = \left[\begin{array}{c} \frac{z}{\sqrt{1-x^2-y^2}} = \cos\theta \\ dz = -\sqrt{1-x^2-y^2}\sin\theta\,d\theta \end{array}\right]$$

$$= 16\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1-x^2-y^2\right) \int_{\pi/2}^{0} -\sqrt{1-\cos^2\theta}\,\sin\theta\,d\theta\,dy\,dx = 16\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1-x^2-y^2\right) \int_{\pi/2}^{0} -\sin^2\theta\,d\theta\,dy\,dx$$

$$= 16\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{4}\left(1-x^2-y^2\right) dy\,dx = 4\pi\int_0^1 \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}\left(1-x^2\right)^{3/2} dx$$

$$= 4\pi\int_0^1 \sqrt{1-x^2}\left[\left(1-x^2\right) - \frac{1-x^2}{3}\right] dx = \frac{8}{3}\pi\int_0^1 \left(1-x^2\right)^{3/2} dx = \left[\begin{array}{c} x = \cos\theta \\ dx = -\sin\theta\,d\theta \end{array}\right] = -\frac{8}{3}\pi\int_{\pi/2}^{0} \sin^4\theta\,d\theta$$

$$= -\frac{8}{3}\pi\int_{\pi/2}^{0} \left[\frac{1-\cos 2\theta}{2}\right]^2 = -\frac{2}{3}\pi\int_{\pi/2}^{0} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta = -\frac{2}{3}\pi\int_{\pi/2}^{0} \left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta = \frac{\pi^2}{2}$$

NOTES: