22. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4 \sin 4t)\mathbf{i} + (4 \cos 4t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4 \sin 4t)^2 + (4 \cos 4t)^2 + 4^2}$

$= \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2}\, dt = \left[4\sqrt{2}\, t\right]_0^{\pi/2} = 2\pi\sqrt{2}$

(b) $\mathbf{r} = \left(\cos \frac{t}{2}\right)\mathbf{i} + \left(\sin \frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2} \sin \frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2} \cos \frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2} \sin \frac{t}{2}\right)^2 + \left(\frac{1}{2} \cos \frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2}\, dt = \left[\frac{\sqrt{2}}{2} t\right]_0^{4\pi} = 2\pi\sqrt{2}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1+1}$

$= \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^{0} \sqrt{2}\, dt = \left[\sqrt{2}\, t\right]_{-2\pi}^{0} = 2\pi\sqrt{2}$

23. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \cos^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(1 + \cos^2 t\right)^{-1/2}\mathbf{i} + \cos t\left(1 + \cos^2 t\right)^{-1/2}\mathbf{j}$

$\frac{d\mathbf{T}}{dt} = \sin t \cos t\left(1 + \cos^2 t\right)^{-3/2}\mathbf{i} + \left[\sin t \cos^2 t\left(1 - \cos^2 t\right)^{-3/2} - \sin t\left(1 + \cos^2 t\right)^{-1/2}\right]\mathbf{j}$

$\kappa(0) = \frac{1}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|}\left|\frac{d\mathbf{T}}{dt}\left(\frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{1}}|0\mathbf{i} - \mathbf{j}| = (1)\sqrt{0^2 + 1^2} = 1 \Rightarrow \rho(0) = \frac{1}{\kappa(0)} = 1$

Since $\frac{d\mathbf{T}}{dt}(0) = -\mathbf{j}$, the curve is concave down at $\left(\frac{\pi}{2}, 1\right)$ and the center of the circle of curvature is at $\left(\frac{\pi}{2}, 0\right) \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + (y-0)^2 = 1 \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$ is an equation of the circle of curvature.

24. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t}\right)\mathbf{i} - \left(1 - \frac{1}{t^2}\right)\mathbf{j} = \left(\frac{2}{t}\right)\mathbf{i} + \left(\frac{1-t^2}{t^2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left(\frac{2}{t}\right)^2 + \left(\frac{1-t^2}{t^2}\right)^2} = \frac{1+t^2}{t^2}$

$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{2t}{1+t^2}\right)\mathbf{i} + \left(\frac{1-t^2}{1+t^2}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{2-2t^2}{\left(1+t^2\right)^2}\right)\mathbf{i} - \left(\frac{4t}{\left(1+t^2\right)^2}\right)\mathbf{j}$

$\kappa(0) = \frac{1}{|\mathbf{v}(1)|}\left|\frac{d\mathbf{T}}{dt}(1)\right| = \frac{1}{2}|0\mathbf{i} - \mathbf{j}| = \left(\frac{1}{2}\right)\sqrt{(0)^2 + 1^2} = \frac{1}{2} \Rightarrow \rho(0) = \frac{1}{\kappa(0)} = 2$

Since $\frac{d\mathbf{T}}{dt}(1) = -\mathbf{j}$, the curve is concave down at $(0, -2)$ and the center of the circle of curvature is at $(0, -4) \Rightarrow x^2 + (y+4)^2 = 4$ is an equation of the circle of curvature.

10.7 THE TNB FRAME: TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

1. $\mathbf{r} = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3 \cos t)\mathbf{i} + (-3 \sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3 \cos t)^2 + (-3 \sin t)^2 + 4^2}$

$= \sqrt{25} = 5 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5} \cos t\right)\mathbf{i} - \left(\frac{3}{5} \sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5} \sin t\right)\mathbf{i} - \left(\frac{3}{5} \cos t\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(-\frac{3}{5}\cos t\right)^2} = \frac{3}{5} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$; $\mathbf{a} = (-3\sin t)\mathbf{i} + (-3\cos t)\mathbf{j}$

$\Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} = (12\cos t)\mathbf{i} - (12\sin t)\mathbf{j} - 9\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|$

$= \sqrt{(12\cos t)^2 + (-12\sin t)^2 + (-9)^2} = \sqrt{225} = 15 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{15}{5^3} = \frac{3}{25}$; $\mathbf{B} = \mathbf{T}\times\mathbf{N}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}$; $\frac{d\mathbf{a}}{dt} = (-3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$

$$\Rightarrow \tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\sin t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{-36\sin^2 t - 36\cos^2 t}{15^2} = -\frac{4}{25}$$

2. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2}$

$= |t| = t$, if $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$

$\Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix}$

$= [(t\cos t)(\sin t + t\cos t) - (t\sin t)(\cos t - t\sin t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{(t^2)^2} = t^2$

$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{t^3} = \frac{1}{t}$; $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t)\mathbf{k} = \mathbf{k}$;

$$\frac{d\mathbf{a}}{dt} = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j} \Rightarrow \tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}$$

$= \frac{0}{|\mathbf{v}\times\mathbf{a}|^2} = 0$

3. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \Rightarrow$
$|\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2e^{2t}} = e^t\sqrt{2}$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$;

$\mathbf{a} = (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix} = 2e^{2t}\mathbf{k}$

$\Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{(2e^{2t})^2} = 2e^{2t} \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2e^{2t}}{(e^t\sqrt{2})^3} = \frac{1}{e^t\sqrt{2}}$;

$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix}$

$= \left[\frac{1}{2}(\cos t - \sin t)(-\sin t + \cos t) - \frac{1}{2}(-\cos t - \sin t)(\sin t + \cos t)\right]\mathbf{k}$

$= \left[\frac{1}{2}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + \frac{1}{2}(\cos^2 t + 2\sin t \cos t + \sin^2 t)\right]\mathbf{k} = \mathbf{k}$;

$\frac{d\mathbf{a}}{dt} = (-2e^t \sin t - 2e^t \cos t)\mathbf{i} + (2e^t \cos t - 2e^t \sin t)\mathbf{j}$

$\Rightarrow \tau = \frac{\begin{vmatrix} e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \\ -2e^t \sin t - 2e^t \cos t & 2e^t \cos t - 2e^t \sin t & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$

4. $\mathbf{r} = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} - (12 \sin 2t)\mathbf{j} + 5\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13} \sin 2t\right)\mathbf{i} - \left(\frac{24}{13} \cos 2t\right)\mathbf{j}$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13}\sin 2t\right)^2 + \left(-\frac{24}{13}\cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j};$$

$$\mathbf{a} = (-24\sin 2t)\mathbf{i} - (24\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix}$$

$$= (120\cos 2t)\mathbf{i} - (120\sin 2t)\mathbf{j} - 288\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{(120\cos 2t)^2 + (-120\sin 2t)^2 + (-288)^2} = 312$$

$$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{312}{13^3} = \frac{24}{169};\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$= \left(\frac{5}{13}\cos 2t\right)\mathbf{i} - \left(\frac{5}{13}\sin 2t\right)\mathbf{j} - \frac{12}{13}\mathbf{k};\ \frac{d\mathbf{a}}{dt} = (-48\cos 2t)\mathbf{i} + (48\sin 2t)\mathbf{j}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = -\frac{(5)(24)(48)}{(312)^2} = -\frac{5\cdot 1\cdot 2}{13\cdot 13} = -\frac{10}{169}$$

5. $\mathbf{r} = \left(\frac{t^3}{3}\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j},\ t>0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4+t^2} = t\sqrt{t^2+1}$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$$= \frac{t}{\sqrt{t^2+t}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2+1)^{3/2}}\mathbf{i} - \frac{t}{(t^2+1)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2+1)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{1+t^2}{(t^2+1)^3}} = \frac{1}{t^2+1} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1}{\sqrt{t^2+1}}\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\mathbf{j};\ \mathbf{a} = 2t\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & t & 0 \\ 2t & 1 & 0 \end{vmatrix} = -t^2\mathbf{k}$$

$$\Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{(-t^2)^2} = t^2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{\left(t\sqrt{t^2+1}\right)^3} = \frac{1}{t(t^2+1)^{3/2}};$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = -\mathbf{k};\ \frac{d\mathbf{a}}{dt} = 2\mathbf{i} \Rightarrow \tau = \frac{\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

6. $\mathbf{r} = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j},\ 0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\mathbf{i} + (3\sin^2 t \cos t)\mathbf{j}$

$\Rightarrow |\mathbf{v}| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\cos t \sin t$, since $0 < t < \frac{\pi}{2}$

$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$

$= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a} = (6\cos t \sin^2 t - 3\cos^3 t)\mathbf{i} + (6\sin t \cos^2 t - 3\sin^3 t)\mathbf{j}$

$$\Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos^2 t \sin t & 3\sin^2 t \cos t & 0 \\ 6\cos t \sin^2 t - 3\cos^3 t & 6\sin t \cos^2 t - 3\sin^3 t & 0 \end{vmatrix}$$

$= (-18\sin^2 t \cos^4 t + 9\cos^2 t \sin^4 t - 18\sin^4 t \cos^2 t + 9\sin^2 t \cos^4 t)\mathbf{k} = (-9\sin^2 t \cos^4 t - 9\cos^2 t \sin^4 t)\mathbf{k}$

$= (-9\sin^2 t \cos^2 t)(\cos^2 t + \sin^2 t)\mathbf{k} = (-9\sin^2 t \cos^2 t)\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 9\sin^2 t \cos^2 t \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

$$= \frac{9\cos^2 t \sin^2 t}{(3\cos t \sin t)^3} = \frac{1}{3\cos t \sin t};\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k};\ \frac{d\mathbf{a}}{dt} = f(t)\mathbf{i} + g(t)\mathbf{j} \text{ where}$$

$f(t) = \frac{d}{dt}(6\cos t \sin^2 t - 3\cos^3 t)$ and $g(t) = \frac{d}{dt}(6\sin t \cos^2 t - 3\sin^3 t)$

$$\Rightarrow \tau = \frac{\begin{vmatrix} -3\cos^2 t \sin t & 3\sin^2 t \cos t & 0 \\ 6\cos t \sin^2 t - 3\cos^3 t & 6\sin t \cos^2 t - 3\sin^3 t & 0 \\ f(t) & g(t) & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

7. $\mathbf{r} = t\mathbf{i} + \left(a\cosh\frac{t}{a}\right)\mathbf{j},\ a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh\frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2\left(\frac{t}{a}\right)} = \sqrt{\cosh^2\left(\frac{t}{a}\right)} = \cosh\frac{t}{a}$

$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i} + \left(\tanh\frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a}\operatorname{sech}\frac{t}{a}\tanh\frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a}\operatorname{sech}^2\frac{t}{a}\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{a^2}\operatorname{sech}^2\left(\frac{t}{a}\right)\tanh^2\left(\frac{t}{a}\right) + \frac{1}{a^2}\operatorname{sech}^4\left(\frac{t}{a}\right)} = \frac{1}{a}\operatorname{sech}\left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\tanh\frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j};$

$$\mathbf{a} = \left(\frac{1}{a}\cosh\frac{t}{a}\right)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a}\cosh\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \left(\frac{1}{a}\cosh\frac{t}{a}\right)\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \frac{1}{a}\cosh\left(\frac{t}{a}\right) \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

$$= \frac{\frac{1}{a}\cosh\left(\frac{t}{a}\right)}{\cosh^3\left(\frac{t}{a}\right)} = \frac{1}{a}\operatorname{sech}^2\left(\frac{t}{a}\right);\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0 \\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \mathbf{k};\ \frac{d\mathbf{a}}{dt} = \frac{1}{a^2}\sinh\left(\frac{t}{a}\right)\mathbf{j}$$

$$\tau = \frac{\begin{vmatrix} 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a}\cosh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a^2}\sinh\left(\frac{t}{a}\right) & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

8. $\mathbf{r} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2}\cosh t$

$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)\mathbf{k}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t\tanh^2 t} = \frac{1}{\sqrt{2}}\operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k};$

$$\mathbf{a} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} - \mathbf{k}$$

$\Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2}\cosh t \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{2}\cosh t}{\left(\sqrt{2}\right)^3\cosh^3 t} = \frac{1}{2}\operatorname{sech}^2 t;$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\tanh t & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k};$$

$$\frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} \Rightarrow \tau = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{-1}{2\cosh^2 t} = -\frac{1}{2}\operatorname{sech}^2 t$$

9. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$

$= \sqrt{a^2 + b^2} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0;\ \mathbf{a} = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \sqrt{a^2} = |a|$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\mathbf{N} = |a|\mathbf{N}$

10. $\mathbf{r} = (1+3t)\mathbf{i} + (t-2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0$; $\mathbf{a} = \mathbf{0}$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$

11. $\mathbf{r} = (t+1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \Rightarrow a_T = \frac{1}{2}(5+4t^2)^{-1/2}(8t)$
$= 4t(5+4t^2)^{-1/2} \Rightarrow a_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}$; $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2 \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2}$
$= \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$

12. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + 2t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2} = \sqrt{5t^2+1} \Rightarrow a_T = \frac{1}{2}(5t^2+1)^{-1/2}(10t)$
$= \frac{5t}{\sqrt{5t^2+1}} \Rightarrow a_T(0) = 0$; $\mathbf{a} = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)|$
$= \sqrt{2^2 + 2^2} = 2\sqrt{2} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(2\sqrt{2})^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$

13. $\mathbf{r} = t^2\mathbf{i} + \left(t + \frac{1}{3}t^3\right)\mathbf{j} + \left(t - \frac{1}{3}t^3\right)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + (1+t^2)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + (1+t^2)^2 + (1-t^2)^2}$
$= \sqrt{2(t^4 + 2t^2 + 1)} = \sqrt{2}(1+t^2) \Rightarrow a_T = 2t\sqrt{2} \Rightarrow a_T(0) = 0$; $\mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$

14. $\mathbf{r} = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (\sqrt{2}e^t)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2$;
$\mathbf{a} = (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{i} + (e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$= (-2e^t\sin t)\mathbf{i} + (2e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6}$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

15. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2}$
$= 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}$
$\Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$, the normal to the osculating plane; $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$ lies on the

osculating plane $\Rightarrow 0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+(z-(-1))=0 \Rightarrow z=-1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0 \Rightarrow -\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0$ $\Rightarrow -x+y=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0 \Rightarrow -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y=-1 \Rightarrow x+y=\sqrt{2}$ is the rectifying plane

16. $\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+t\mathbf{k} \Rightarrow \mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+\mathbf{k} \Rightarrow |\mathbf{v}|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2} \Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}$ $=\left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i}+\left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}=\left(-\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i}+\left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|$ $=\sqrt{\frac{1}{2}\cos^2 t+\frac{1}{2}\sin^2 t}=\frac{1}{\sqrt{2}} \Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}$; thus $\mathbf{T}(0)=\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$ and $\mathbf{N}(0)=-\mathbf{i}$

$$\Rightarrow \mathbf{B}(0)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -1 & 0 & 0\end{vmatrix}=-\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k},$$

the normal to the osculating plane; $\mathbf{r}(0)=\mathbf{i} \Rightarrow P(1,0,0)$ lies on the osculating plane $\Rightarrow 0(x-1)-\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \Rightarrow y-z=0$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow 0(x-1)+\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \Rightarrow y+z=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow -1(x-1)+0(y-0)+0(z-0)=0 \Rightarrow x=1$ is the rectifying plane

17. Yes. If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N=\kappa|\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a}=a_T\mathbf{T}+a_N\mathbf{N} \neq \mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T=\frac{d}{dt}|\mathbf{v}|=0 \Rightarrow \mathbf{a}=a_N\mathbf{N}$ is orthogonal to $\mathbf{T} \Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a}\perp\mathbf{v} \Rightarrow \mathbf{a}\perp\mathbf{T} \Rightarrow a_T=0 \Rightarrow \frac{d}{dt}|\mathbf{v}|=0 \Rightarrow |\mathbf{v}|$ is constant

20. $\mathbf{r}(t)=x(t)\mathbf{i}+(x(t))^2\mathbf{j} \Rightarrow \mathbf{v}(t)=\frac{dx}{dt}\mathbf{i}+2x(t)\frac{dx}{dt}\mathbf{j} \Rightarrow |\mathbf{v}(t)|=\left|\frac{dx}{dt}\right|\sqrt{1+4x^2}=10$ $\Rightarrow \left|\frac{dx}{dt}\right|=10\left(1+4x^2\right)^{-1/2} \Rightarrow \frac{dx}{dt}=\pm 10\left(1+4x^2\right)^{-1/2}$

$\mathbf{a}(t)=\frac{d}{dt}\left(\frac{dx}{dt}\mathbf{i}+2x\frac{dx}{dt}\mathbf{j}\right)=\frac{d^2x}{dt^2}\mathbf{i}+\left[2\left(\frac{dx}{dt}\right)^2+2x\frac{d^2x}{dt^2}\right]\mathbf{j};\ \frac{d^2x}{dt^2}=\mp 40x\left(1+4x^2\right)^{-3/2}$

$\Rightarrow \mathbf{a}(t)=\mp\frac{40x}{\left(1+4x^2\right)^{3/2}}\mathbf{i}+\left[\frac{200}{1+4x^2}\mp\frac{80x^2}{\left(1+4x^2\right)^{3/2}}\right]\mathbf{j}$; At $x=0$, $\mathbf{a}(t)=200\mathbf{j} \Rightarrow \mathbf{F}=m\mathbf{a}=200m\mathbf{j}$;

At $x=\sqrt{2}$, $\mathbf{a}(t)=\mp\frac{40\sqrt{2}}{27}\mathbf{i}+\left(\frac{200}{9}\mp\frac{160}{27}\right)\mathbf{j}=\mp\frac{40\sqrt{2}}{27}\mathbf{i}+\frac{600\mp 160}{27}\mathbf{j} \Rightarrow \mathbf{F}=m\mathbf{a}=\frac{40m}{27}\left[\mp\sqrt{2}\mathbf{i}+(15\mp 3)\mathbf{j}\right]$

21. $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(\text{constant}) = 0$ and $a_N = \kappa|\mathbf{v}|^2 \Rightarrow \mathbf{F} = m\mathbf{a} = m\kappa|\mathbf{v}|^2\mathbf{N} \Rightarrow |\mathbf{F}| = m\kappa|\mathbf{v}|^2$ $= (m|\mathbf{v}|^2)\kappa$, a constant multiple of the curvature κ of the trajectory

22. $a_N = 0 \Rightarrow \kappa|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

23. (a) $\mathbf{r} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow \mathbf{a} = f''(x)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\mathbf{k}$

$\Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{(f''(x))^2} = |f''(x)|$ and $|\mathbf{v}| = \sqrt{1^2 + [f'(x)]^2} = \sqrt{1 + [f'(x)]^2} \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

$= \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$

(b) $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|-\sec^2 x|}{[1 + (-\tan x)^2]^{3/2}} = \frac{\sec^2 x}{|\sec^3 x|}$ $= \frac{1}{\sec x} = \cos x$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $x = x_0$ gives a point of inflection $\Rightarrow f''(x_0) = 0$ (since f is twice differentiable) $\Rightarrow \kappa = 0$

24. (a) $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow \mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix} = (\dot{x}\ddot{y} - \dot{y}\ddot{x})\mathbf{k}$

$\Rightarrow |\mathbf{v}\times\mathbf{a}| = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$ and $|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$

(b) $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}$, $0 < t < \pi \Rightarrow x = t$ and $y = \ln(\sin t) \Rightarrow \dot{x} = 1$, $\ddot{x} = 0$; $\dot{y} = \frac{\cos t}{\sin t} = \cot t$, $\ddot{y} = -\csc^2 t$

$\Rightarrow \kappa = \frac{|-\csc^2 t - 0|}{(1 + \cot^2 t)^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$

(c) $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$ and $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1 + \sinh^2 t} = \frac{1}{\cosh t}$

$= \text{sech } t$, $\ddot{x} = -\text{sech } t \tanh t$; $\dot{y} = \frac{\sinh t}{\cosh t} = \tanh t$, $\ddot{y} = \text{sech}^2 t \Rightarrow \kappa = \frac{|\text{sech}^3 t + \text{sech } t \tanh^2 t|}{(\text{sech}^2 t + \tanh^2 t)} = |\text{sech } t|$

$= \text{sech } t$

25. (a) $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is tangent to the curve at the point $(f(t), g(t))$;
$\mathbf{n}\cdot\mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}]\cdot[f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0$; $-\mathbf{n}\cdot\mathbf{v} = -(\mathbf{n}\cdot\mathbf{v}) = 0$; thus,
$\mathbf{n}$ and $-\mathbf{n}$ are both normal to the curve at the point

(b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{4e^{4t}+1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$

(c) $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4-t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{1+\frac{t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2}\left(\sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j}\right)$

26. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}(t^2\mathbf{i} - \mathbf{j})$ for $t < 0$ and $\mathbf{N} = \frac{1}{\sqrt{1+t^4}}(-t^2\mathbf{i} + \mathbf{j})$ for $t > 0$

(b) From part (a), $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}(\mathbf{i} + t^2\mathbf{j}) \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1+t^4)^{3/2}}(\mathbf{i} + t^2\mathbf{j}) + \frac{1}{\sqrt{1+t^4}}(2t\mathbf{j})$

$= \frac{-2t^3}{(1+t^4)^{3/2}}\left[\mathbf{i} + t^2\mathbf{j} - \left(\frac{1+t^4}{t^2}\right)\mathbf{j}\right] = \frac{-2t^3}{(1+t^4)^{3/2}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \frac{2|t|^3}{(1+t^4)^{3/2}}\sqrt{1+\frac{1}{t^4}} = \frac{2|t|}{(1+t^4)^{3/2}}\sqrt{1+t^4}$

$= \frac{2|t|}{1+t^4}$; $\mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{1+t^4}{2|t|}\right)\left[\frac{2t^3}{(1+t^4)^{3/2}}\right]\left(-\mathbf{i} + \frac{1}{t^2}\mathbf{j}\right) = \frac{t}{|t|}\left(-\frac{t^2}{\sqrt{1+t^4}}\mathbf{i} + \frac{1}{\sqrt{1+t^4}}\mathbf{j}\right)$, $t \neq 0$. The normal

$\mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$ so the curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$

$= \left|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right| = 0$ at $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ is undefined. Since $x = t$ and $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$, the curve is the cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.

27. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a$; from Exercise 23(a), $\kappa(x) = \frac{|2a|}{(1+4a^2x^2)^{3/2}} = |2a|\left(1+4a^2x^2\right)^{-3/2}$ $\Rightarrow \kappa'(x) = -\frac{3}{2}|2a|\left(1+4a^2x^2\right)^{-5/2}(8a^2x)$; thus, $\kappa'(x) = 0 \Rightarrow x = 0$. Now, $\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$ so that $\kappa(x)$ has an absolute maximum at $x = 0$ which is the vertex of the parabola. Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.

28. $\mathbf{r} = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (b\cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a\cos t)\mathbf{i} - (b\sin t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = |ab| = ab, \text{ since } a > b > 0;\ \kappa(t) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

$= ab\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-3/2}$; $\kappa'(t) = -\frac{3}{2}(ab)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}\left(2a^2\sin t\cos t - 2b^2\sin t\cos t\right)$

$= -\frac{3}{2}(ab)\left(a^2 - b^2\right)(\sin 2t)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}$; thus, $\kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi$ identifying points on the major axis, or $t = \frac{\pi}{2}, \frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t) < 0$ for

$0 < t < \frac{\pi}{2}$ and for $\pi < t < \frac{3\pi}{2}$; $\kappa'(t) > 0$ for $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$. Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.

29. $\kappa = \frac{a}{a^2+b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2+b^2}{(a^2+b^2)^2}$; $\frac{d\kappa}{da} = 0 \Rightarrow a^2+b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b$ since $a, b > 0$. Now, $\frac{d\kappa}{da} > 0$ if $a < b$ and $\frac{d\kappa}{da} < 0$ if $a > b \Rightarrow \kappa$ is at a maximum for $a = b$ and $\kappa(b) = \frac{b}{b^2+b^2} = \frac{1}{2b}$ is the maximum value.

30. From Example 3, $|\mathbf{v}| = t$ and $a_N = t$ so that $a_N = \kappa|\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}$, $t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$

31. $\mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v}\times\mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0$. Since the curve is a plane curve, $\tau = 0$.

32. From Example 4, the curvature of the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}$, $a, b \geq 0$ is $\kappa = \frac{a}{a^2+b^2}$; also $|\mathbf{v}| = \sqrt{a^2+b^2}$. For the helix $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 4\pi$, $a = 3$ and $b = 1 \Rightarrow \kappa = \frac{3}{3^2+1^2} = \frac{3}{10}$ and $|\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10}\sqrt{10}\,dt = \left[\frac{3}{\sqrt{10}}t\right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$

33. (a) From Exercise 30, $\kappa = \frac{1}{t}$ and $|\mathbf{v}| = t \Rightarrow K = \int_a^b \left(\frac{1}{t}\right)(t)\,dt = b - a$

(b) $y = x^2 \Rightarrow x = t$ and $y = t^2$, $-\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2}$; also $\mathbf{a} = 2\mathbf{j}$

$$\Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{\left(\sqrt{1+4t^2}\right)^3}.$$ Then

$$K = \int_{-\infty}^{\infty} \frac{2}{\left(\sqrt{1+4t^2}\right)^3}\left(\sqrt{1+4t^2}\right)dt = \int_{-\infty}^{\infty} \frac{2}{1+4t^2}\,dt = \lim_{a\to-\infty} \int_a^0 \frac{2}{1+4t^2}\,dt + \lim_{b\to\infty} \int_0^b \frac{2}{1+4t^2}\,dt$$

$$= \lim_{a\to-\infty} \left[\tan^{-1} 2t\right]_a^0 + \lim_{b\to\infty} \left[\tan^{-1} 2t\right]_0^b = \lim_{a\to-\infty} \left(-\tan^{-1} 2a\right) + \lim_{b\to\infty} \left(\tan^{-1} 2b\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

34. From Example 4, $\tau = \frac{b}{a^2+b^2} \Rightarrow \tau'(b) = \frac{a^2-b^2}{(a^2+b^2)^2}$; $\tau'(b) = 0 \Rightarrow \frac{a^2-b^2}{(a^2+b^2)^2} = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow b = \pm a$ $\Rightarrow b = a$ since $a, b > 0$. Also $b < a \Rightarrow \tau' > 0$ and $b > a \Rightarrow \tau' < 0$ so τ_{max} occurs when $b = a \Rightarrow \tau_{max} = \frac{a}{a^2+a^2}$ $= \frac{1}{2a}$

35. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$; $\mathbf{v}\cdot\mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$
$\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k}$ and $\mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0$, $g(a) = 0$ and $C = 0 \Rightarrow h(t) = 0$.

36. From Example 4, $\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2+b^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{1}{\sqrt{a^2+b^2}}[-(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}]$; $\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[-(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}] \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$

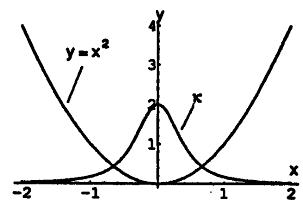
$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j};\ \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a \sin t}{\sqrt{a^2+b^2}} & \frac{a \cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{b \sin t}{\sqrt{a^2+b^2}}\mathbf{i} - \frac{b \cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2+b^2}}\mathbf{k} \Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[(b \cos t)\mathbf{i} + (b \sin t)\mathbf{j}] \Rightarrow \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = -\frac{b}{\sqrt{a^2+b^2}}$$

$\Rightarrow \tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right) = \left(-\frac{1}{\sqrt{a^2+b^2}}\right)\left(-\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{b}{a^2+b^2}$, which is consistent with the result in Example 4.

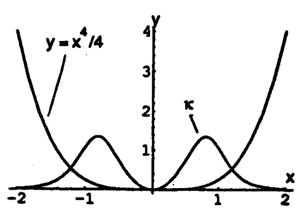
37. $y = x^2 \Rightarrow f'(x) = 2x$ and $f''(x) = 2$

$$\Rightarrow \kappa = \frac{|2|}{\left(1+(2x)^2\right)^{3/2}} = \frac{2}{\left(1+4x^2\right)^{3/2}}$$



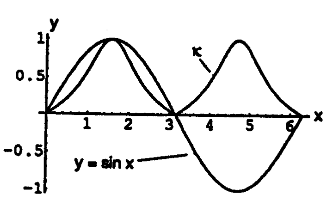
38. $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$ and $f''(x) = 3x^2$

$$\Rightarrow \kappa = \frac{|3x^2|}{\left(1+(x^3)^2\right)^{3/2}} = \frac{3x^2}{\left(1+x^6\right)^{3/2}}$$



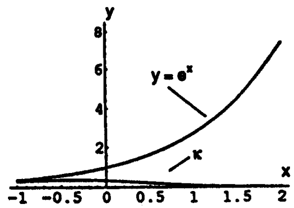
39. $y = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$

$$\Rightarrow \kappa = \frac{|-\sin x|}{\left(1+\cos^2 x\right)^{3/2}} = \frac{|\sin x|}{\left(1+\cos^2 x\right)^{3/2}}$$



40. $y = e^x \Rightarrow f'(x) = e^x$ and $f''(x) = e^x$

$$\Rightarrow \kappa = \frac{|e^x|}{\left(1+(e^x)^2\right)^{3/2}} = \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}$$



41-48. Example CAS commands:

Maple:

```
with(plots):
x:= t -> t^3 - 2*t^2 - t;
y:= t -> 3*t/sqrt(1 + t^2);
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
ds:= t -> sqrt((dx^2)(t) + (dy^2)(t));
d2x:= t -> D(dx)(t);
d2y:= t -> D(dy)(t);
kap:= t -> abs(dx(t)*d2y(t) - dy(t)*d2x(t))/((ds)(t))^3;
a:= t -> x0 - (1/kap(t))*(dy(t)/ds(t));
b:= t -> x0 + (1/kap(t))*(dx(t)/ds(t));
s1:= plot([x(t),y(t), t = -2..5], -15..5, -10..4, scaling=CONSTRAINED):
display(s1);
t0:=1: x0:= x(t0): y0:= y(t0):
circle:= ((x-a(t0))^2 + (y-b(t0))^2 = (1/kap(t0))^2);
s2:=implicitplot(circle, x=-15..6,y=-10..5,scaling=CONSTRAINED):
s3:=plot([a(t0),b(t0),x0,y0]):
display({s1,s2,s3});
```

Mathematica:

```
Clear[x,y,t]
r[t_] = {x[t],y[t]}
x[t_] = t^3 - 2 t^2 - t
y[t_] = 3 t / Sqrt[1+t^2]
{a,b} = {-2,5};
t0 = 1;
p1 = ParametricPlot[ {x[t],y[t]}, {t,a,b},
 AspectRatio -> Automatic ]
v0 = r'[t0]
s0 = Sqrt[ v0 . v0 ]
k0 = Abs[ x' [t0] y''[t0] - y' [t0] x'' [t0] ]/s0^3
N[%]
n0 = { -y' [t0], x' [t0] } / s0
r0 = r[t0]
c0 = r0 + 1/k0 n0
```

Note: Plot the circle parametrically rather than implicitly:

```
circ = ParametricPlot[ Evaluate[c0 + 1/k0 {Cos[t],Sin[t]}],
 {t,0,2Pi}, AspectRatio -> Automatic ]
line = Graphics[{Line[{c0,r0}]}]
Show[ p1, circ, line ]
```

10.8 PLANETARY MOTION AND SATELLITES

1. $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow T^2 = \frac{4\pi^2}{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(5.975\times10^{24}\ \text{kg})}(6{,}808{,}000\ \text{m})^3$

   $\approx 3.125\times10^7\ \text{sec}^2 \Rightarrow T \approx \sqrt{3125\times10^4\ \text{sec}^2} \approx 55.90\times10^2\ \text{sec} \approx 93.2\ \text{min}$

2. $e = 0.0167$ and perihelion distance $= 149{,}577{,}000$ km and $e = \frac{r_0v_0^2}{GM} - 1$

   $\Rightarrow 0.0167 = \frac{(149{,}577{,}000{,}000\ \text{m})v_0^2}{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(1.99\times10^{30}\ \text{kg})} - 1 \Rightarrow v_0^2 \approx 9.02\times10^8\ \text{m}^2/\text{sec}^2$

   $\Rightarrow v_0 \approx \sqrt{9.02\times10^8\ \text{m}^2/\text{sec}^2} \approx 3.00\times10^4\ \text{m/sec}$

3. 92.25 min $= 5535$ sec and $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow a^3 = \frac{GM}{4\pi^2}T^2$

   $\Rightarrow a^3 = \frac{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(5.975\times10^{24}\ \text{kg})}{4\pi^2}(5535\ \text{sec})^2 = 3.094\times10^{20}\ \text{m}^3 \Rightarrow a \approx \sqrt[3]{3.094\times10^{20}\ \text{m}^3}$

   $= 6.763\times10^6\ \text{m} \approx 6763$ km; the mean distance from center of the Earth $= \frac{12{,}757\ \text{km} + 183\ \text{km} + 589\ \text{km}}{2}$

   $= 6765$ km

4. $T = 1639\ \text{min} = 98{,}340$ sec and mass of Mars $= 6.418\times10^{23}\ \text{kg} \Rightarrow a^3 = \frac{GM}{4\pi^2}T^2$

   $= \frac{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(6.418\times10^{23}\ \text{kg})(98{,}340\ \text{sec})^2}{4\pi^2} \approx 1.049\times10^{22}\ \text{m}^3 \Rightarrow a \approx \sqrt[3]{1.049\times10^{22}\ \text{m}^3}$

   $= 2.189\times10^7\ \text{m} = 21{,}890$ km

5. $2a$ = diameter of Mars + perigee height + apogee height $= D + 1499\ \text{km} + 35{,}800$ km

   $\Rightarrow 2(21{,}890)\ \text{km} = D + 37{,}300\ \text{km} \Rightarrow D = 6480$ km

6. $a = 22{,}030\ \text{km} = 2.203\times10^7$ m and $T^2 = \frac{4\pi^2}{GM}a^3$

   $\Rightarrow T^2 = \frac{4\pi^2}{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(6.418\times10^{23}\ \text{kg})}(2.203\times10^7\ \text{sec})^3 \approx 9.857\times10^9\ \text{sec}^2$

   $\Rightarrow T \approx \sqrt{9.857\times10^8\ \text{sec}^2} \approx 9.928\times10^4\ \text{sec} \approx 1655$ min

7. (a) Period of the satellite = rotational period of the Earth $\Rightarrow$ period of the satellite $= 1436.1$ min

   $= 86{,}166$ sec; $a^3 = \frac{GMT^2}{4\pi^2} \Rightarrow a^3 = \frac{(6.6720\times10^{-11}\ \text{Nm}^2\text{kg}^{-2})(5.975\times10^{24}\ \text{kg})(86{,}166\ \text{sec})^2}{4\pi^2}$

$\approx 7.4973 \times 10^{22}\ \mathrm{m}^3 \Rightarrow a \approx \sqrt[3]{74.973 \times 10^{21}\ \mathrm{m}^3} \approx 4.2167 \times 10^7\ \mathrm{m} = 42{,}167\ \mathrm{km}$

(b) The radius of the Earth is approximately 6379 km $\Rightarrow$ the height of the orbit is $42{,}167 - 6379 = 35{,}788$ km

(c) Symcom 3, GOES 4, and Intelsat 5

8. $T = 1477.4\ \mathrm{min} = 88{,}644\ \mathrm{sec} \Rightarrow a^3 = \frac{GMT^2}{4\pi^2}$

$= \frac{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(6.418 \times 10^{23}\ \mathrm{kg})(88{,}644\ \mathrm{sec})^2}{4\pi^2} = 8.523 \times 10^{21}\ \mathrm{m}^3 \Rightarrow a \approx \sqrt[3]{8.523 \times 10^{21}\ \mathrm{m}^3}$

$\approx 2.043 \times 10^7\ \mathrm{m} = 20{,}430\ \mathrm{km}$

9. Period of the Moon $= 2.36055 \times 10^6\ \mathrm{sec} \Rightarrow a^3 = \frac{GMT^2}{4\pi^2}$

$= \frac{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(5.975 \times 10^{24}\ \mathrm{kg})(2.36055 \times 10^6\ \mathrm{sec})^2}{4\pi^2} \approx 5.627 \times 10^{25}\ \mathrm{m}^3 \Rightarrow a \approx \sqrt[3]{5.627 \times 10^{25}\ \mathrm{m}^3}$

$\approx 3.832 \times 10^8\ \mathrm{m} = 383{,}200$ km from the center of the Earth, or about 376,821 km from the surface

10. $r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow |v| = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(5.975 \times 10^{24}\ \mathrm{kg})}{r}} \approx 1.9966 \times 10^7 r^{-1/2}\ \mathrm{m/sec}$

11. Solar System: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(1.99 \times 10^{30}\ \mathrm{kg})} \approx 2.97 \times 10^{-19}\ \mathrm{sec^2/m^3}$;

Earth: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(5.975 \times 10^{24}\ \mathrm{kg})} \approx 9.903 \times 10^{-14}\ \mathrm{sec^2/m^3}$;

Moon: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}})(7.354 \times 10^{22}\ \mathrm{kg})} \approx 8.046 \times 10^{-12}\ \mathrm{sec^2/m^3}$;

12. $e = \frac{r_0 v_0^2}{GM} - 1 \Rightarrow v_0^2 = \frac{GM(e+1)}{r_0} \Rightarrow v_0 = \sqrt{\frac{GM(e+1)}{r_0}}$;

Circle: $e = 0 \Rightarrow v_0 = \sqrt{\frac{GM}{r_0}}$

Ellipse: $0 < e < 1 \Rightarrow \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$

Parabola: $e = 1 \Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$

Hyperbola: $e > 1 \Rightarrow v_0 > \sqrt{\frac{2GM}{r_0}}$

13. $r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$ which is constant since G, M, and r (the radius of orbit) are constant

14. $\Delta A = \frac{1}{2}\left|\mathbf{r}(t + \Delta t) \times \mathbf{r}(t)\right| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\frac{\mathbf{r}(t + \Delta t)}{\Delta t} \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t) + \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right|$

$= \frac{1}{2}\left|\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) + \frac{1}{\Delta t}\mathbf{r}(t) \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{1}{2}\left|\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right|$

$= \frac{1}{2}\left|\frac{d\mathbf{r}}{dt} \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\mathbf{r}(t) \times \frac{d\mathbf{r}}{dt}\right| = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}|$

15. $T = \left(\frac{2\pi a^2}{r_0 v_0}\right)\sqrt{1-e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)(1-e^2) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[1-\left(\frac{r_0 v_0^2}{GM}-1\right)^2\right]$ (from Equation 34)

$= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[-\frac{r_0^2 v_0^4}{G^2M^2} + 2\left(\frac{r_0 v_0^2}{GM}\right)\right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[\frac{2GMr_0 v_0^2 - r_0^2 v_0^4}{G^2M^2}\right] = \frac{(4\pi^2 a^4)\left(2GM - r_0 v_0^2\right)}{r_0 G^2 M^2}$

$= (4\pi^2 a^4)\left(\frac{2GM - r_0 v_0^2}{2r_0 GM}\right)\left(\frac{2}{GM}\right) = (4\pi^2 a^4)\left(\frac{1}{2a}\right)\left(\frac{2}{GM}\right)$ (from Equation 35) $\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$

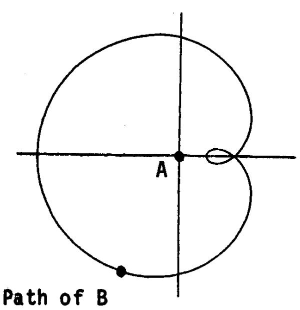
16. Let $\mathbf{r}_{AB}(t)$ denote the vector from planet A to planet B at time t. Then $\mathbf{r}_{AB}(t) = \mathbf{r}_B(t) - \mathbf{r}_A(t)$

$= [3\cos(\pi t) - 2\cos(2\pi t)]\mathbf{i} + [3\sin(\pi t) - 2\sin(2\pi t)]\mathbf{j}$

$= \left[3\cos(\pi t) - 2\left(\cos^2(\pi t) - \sin^2(\pi t)\right)\right]\mathbf{i} + [3\sin(\pi t) - 4\sin(\pi t)\cos(\pi t)]\mathbf{j}$

$= \left[3\cos(\pi t) - 4\cos^2(\pi t) + 2\right]\mathbf{i} + [(3 - 4\cos(\pi t))\sin(\pi t)]\mathbf{j} \Rightarrow$ parametric equations for the path are $x(t) = 2 + [3 - 4\cos(\pi t)]\cos(\pi t)$ and $y(t) = [3 - 4\cos(\pi t)]\sin(\pi t)$

17. Setting $\theta = \pi t$ and $r = 3 - 4\cos\theta$, we see that $x - 2 = r\cos\theta$ and $y = r\sin\theta \Rightarrow$ the graph of the path of planet B is the limaçon $r = 3 - 4\cos\theta$ shown at the right. The planet A is located at $x = -2$.



18. (i) Perihelion is the time t such that $|\mathbf{r}(t)|$ is a minimum.
(ii) Aphelion is the time t such that $|\mathbf{r}(t)|$ is a maximum.
(iii) Equinox is the time t such that $\mathbf{r}(t) \cdot \mathbf{w} = 0$.
(iv) Summer solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a maximum.
(v) Winter solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a minimum.

CHAPTER 10 PRACTICE EXERCISES

1. length $= |2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}| = \sqrt{4+9+36} = 7$, $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} = 7\left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

2. length $= |\mathbf{i}+2\mathbf{j}-\mathbf{k}| = \sqrt{1+4+1} = \sqrt{6}$, $\mathbf{i}+2\mathbf{j}-\mathbf{k} = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}$

3. $2\frac{\mathbf{v}}{|\mathbf{v}|} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{4^2+(-1)^2+4^2}} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{33}} = \frac{8}{\sqrt{33}}\mathbf{i}-\frac{2}{\sqrt{33}}\mathbf{j}+\frac{8}{\sqrt{33}}\mathbf{k}$

4. $-5\frac{\mathbf{v}}{|\mathbf{v}|} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{j}}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{j}}{\sqrt{\frac{9}{25}+\frac{16}{25}}} = -3\mathbf{i}-4\mathbf{j}$

5. $|\mathbf{v}| = \sqrt{1+1} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{4+1+4} = 3$, $\mathbf{v}\cdot\mathbf{u} = 3$, $\mathbf{u}\cdot\mathbf{v} = 3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = -2\mathbf{i}+2\mathbf{j}-\mathbf{k}$,

$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $|\mathbf{v}\times\mathbf{u}| = \sqrt{4+4+1} = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$,

$|\mathbf{u}|\cos\theta = \frac{3}{\sqrt{2}}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{3}{2}(\mathbf{i}+\mathbf{j})$

6. $|\mathbf{v}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$, $|\mathbf{u}| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$, $\mathbf{v}\cdot\mathbf{u} = (1)(-1)+(1)(0)+(2)(-1) = -3$,

$\mathbf{u}\cdot\mathbf{v} = -3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = \mathbf{i}+\mathbf{j}-\mathbf{k}$,

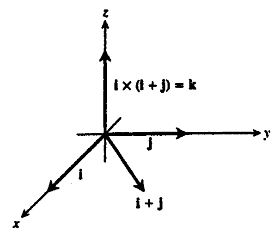
$|\mathbf{v}\times\mathbf{u}| = \sqrt{(-1)^2+(-1)^2+1^2} = \sqrt{3}$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{6}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, $|\mathbf{u}|\cos\theta = \frac{-3}{\sqrt{6}} = -\sqrt{\frac{9}{6}} = -\sqrt{\frac{3}{2}}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{-3}{6}(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = -\frac{1}{2}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

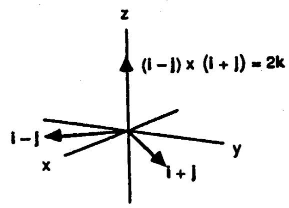
7. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left[\mathbf{u}-\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\right] = \frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k}) + \left[(\mathbf{i}+\mathbf{j}-5\mathbf{k})-\frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})\right] = \frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})-\frac{1}{3}(5\mathbf{i}+\mathbf{j}+11\mathbf{k})$,
where $\mathbf{v}\cdot\mathbf{u} = 8$ and $\mathbf{v}\cdot\mathbf{v} = 6$

8. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left[\mathbf{u}-\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\right] = -\frac{1}{5}(\mathbf{i}-2\mathbf{j}) + \left[(\mathbf{i}+\mathbf{j}+\mathbf{k})-\left(\frac{-1}{5}\right)(\mathbf{i}-2\mathbf{j})\right] = -\frac{1}{5}(\mathbf{i}-2\mathbf{j}) + \left(\frac{6}{5}\mathbf{i}+\frac{3}{5}\mathbf{j}+\mathbf{k}\right)$,
where $\mathbf{v}\cdot\mathbf{u} = -1$ and $\mathbf{v}\cdot\mathbf{v} = 5$

9. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 1 & 1 & 0\end{vmatrix}=\mathbf{k}$



10. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 1 & 1 & 0\end{vmatrix}=2\mathbf{k}$



11. Let $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ and $\mathbf{w}=w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$. Then $|\mathbf{v}-2\mathbf{w}|^2=\left|(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})-2(w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k})\right|^2$
$=\left|(v_1-2w_1)\mathbf{i}+(v_2-2w_2)\mathbf{j}+(v_3-2w_3)\mathbf{k}\right|^2=\left(\sqrt{(v_1-2w_1)^2+(v_2-2w_2)^2+(v_3-2w_3)^2}\right)^2$
$=\left(v_1^2+v_2^2+v_3^2\right)-4(v_1w_1+v_2w_2+v_3w_3)+4\left(w_1^2+w_2^2+w_3^2\right)=|\mathbf{v}|^2-4\mathbf{v}\cdot\mathbf{w}+4|\mathbf{w}|^2$
$=|\mathbf{v}|^2-4|\mathbf{u}||\mathbf{w}|\cos\theta+4|\mathbf{w}|^2=4-4(2)(3)\left(\cos\frac{\pi}{3}\right)+36=40-24\left(\frac{1}{2}\right)=40-12=28\Rightarrow|\mathbf{v}-2\mathbf{w}|=\sqrt{28}$
$=2\sqrt{7}$

12. $\mathbf{u}$ and $\mathbf{v}$ are parallel when $\mathbf{u}\times\mathbf{v}=\mathbf{0}\Rightarrow\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 4 & -5\\ -4 & -8 & a\end{vmatrix}=\mathbf{0}\Rightarrow(4a-40)\mathbf{i}+(20-2a)\mathbf{j}+(0)\mathbf{k}=\mathbf{0}$

$\Rightarrow 4a-40=0$ and $20-2a=0\Rightarrow a=10$

13. (a) area $=|\mathbf{u}\times\mathbf{v}|=\text{abs}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -1\\ 2 & 1 & 1\end{vmatrix}=|2\mathbf{i}-3\mathbf{j}-\mathbf{k}|=\sqrt{4+9+1}=\sqrt{14}$

(b) volume $=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}1 & 1 & -1\\ 2 & 1 & 1\\ -1 & -2 & 3\end{vmatrix}=1(3+2)+1(-1-6)-1(-4+1)=1$

14. (a) area $= |\mathbf{u}\times\mathbf{v}| = \text{abs}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = |\mathbf{k}| = 1$

(b) volume $= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-0)+1(0-0)+0 = 1$

15. The desired vector is $\mathbf{n}\times\mathbf{v}$ or $\mathbf{v}\times\mathbf{n}$ since $\mathbf{n}\times\mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

16. If $a=0$ and $b\neq 0$, then the line $by=c$ and $\mathbf{i}$ are parallel. If $a\neq 0$ and $b=0$, then the line $ax=c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax+by=c$ contains the points $\left(\frac{c}{a},0\right)$ and $\left(0,\frac{c}{b}\right) \Rightarrow$ the vector $ab\left(\frac{c}{a}\mathbf{i}-\frac{c}{b}\mathbf{j}\right) = c(b\mathbf{i}-a\mathbf{j})$ and the line are parallel. Therefore, the vector $b\mathbf{i}-a\mathbf{j}$ is parallel to the line $ax+by=c$ in every case.

17. Parametric equations for the line are $x = 1-3t$, $y = 2$, $z = 3+7t$.

18. The line is parallel to $\overrightarrow{PQ} = 0\mathbf{i}+\mathbf{j}-\mathbf{k}$ and contains the point $P(1,2,0) \Rightarrow$ parametric equations are $x=1$, $y=2+t$, $z=-t$ for $0\le t\le 1$.

19. $P(3,-2,1)$ and $\mathbf{n} = 2\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow (2)(x-3)+(1)(y-(-2))+(1)(z-1)=0 \Rightarrow 2x+y+z=5$

20. $P(-1,6,0)$ and $\mathbf{n} = \mathbf{i}-2\mathbf{j}+3\mathbf{k} \Rightarrow (1)(x-(-1))+(-2)(y-6)+(3)(z-0)=0 \Rightarrow x-2y+3z=-13$

21. $P(1,-1,2)$, $Q(2,1,3)$ and $R(-1,2,-1) \Rightarrow \overrightarrow{PQ} = \mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\overrightarrow{PR} = -2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$ and $\overrightarrow{PQ}\times\overrightarrow{PR}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i}+\mathbf{j}+7\mathbf{k}$ is normal to the plane $\Rightarrow (-9)(x-1)+(1)(y+1)+(7)(z-2)=0$

$\Rightarrow -9x+y+7z=4$

22. $P(1,0,0)$, $Q(0,1,0)$ and $R(0,0,1) \Rightarrow \overrightarrow{PQ} = -\mathbf{i}+\mathbf{j}$, $\overrightarrow{PR} = -\mathbf{i}+\mathbf{k}$ and $\overrightarrow{PQ}\times\overrightarrow{PR}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ is normal to the plane $\Rightarrow (1)(x-1)+(1)(y-0)+(1)(z-0)=0$

$\Rightarrow x+y+z=1$

23. $\left(0,-\frac{1}{2},-\frac{3}{2}\right)$, since $t=-\frac{1}{2}$, $y=-\frac{1}{2}$ and $z=-\frac{3}{2}$ when $x=0$; $(-1,0,-3)$, since $t=-1$, $x=-1$ and $z=-3$ when $y=0$; $(1,-1,0)$, since $t=0$, $x=1$ and $y=-1$ when $z=0$

24. $x = 2t$, $y = -t$, $z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6$ $\Rightarrow t = \frac{2}{3} \Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ is the point of intersection

25. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

26. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on both planes, the desired line is $x = -5 + 5t$, $y = 3 - t$, $z = -3t$.

27. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and is the same as the direction of the given line.

28. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the planes are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0$ and $y = \frac{19}{12}$ $\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right)$ is a point on the line we seek. Therefore, the line is $x = -12t$, $y = \frac{19}{12} + 15t$ and $z = \frac{1}{6} + 6t$.

29. A vector in the direction of the plane's normal is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $P(1, 2, 3)$ on the plane $\Rightarrow 7(x - 1) - 3(y - 2) - 5(z - 3) = 0 \Rightarrow 7x - 3y - 5z = -14$.

30. $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane $\Rightarrow \mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = -3\mathbf{j} + 3\mathbf{k}$ is orthogonal to $\mathbf{v}$ and parallel to the plane

31. The vector $\mathbf{v}\times\mathbf{w}$ is normal to the plane of $\mathbf{v}$ and $\mathbf{w} \Rightarrow \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is orthogonal to $\mathbf{u}$ and parallel to the plane:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\mathbf{i}+3\mathbf{j}-\mathbf{k} \text{ and } \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\mathbf{i}-3\mathbf{j}+\mathbf{k}$$

$\Rightarrow |\mathbf{u}\times(\mathbf{v}\times\mathbf{w})| = \sqrt{4+9+1} = \sqrt{14}$ and $\frac{1}{\sqrt{14}}(-2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ is the desired unit vector.

32. A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i}-\mathbf{j}-3\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{25+1+9} = \sqrt{35} \Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{2}{\sqrt{35}}(5\mathbf{i}-\mathbf{j}-3\mathbf{k})$ is the desired vector.

33. The line containing $(0,0,0)$ normal to the plane is represented by $x = 2t$, $y = -t$, and $z = -t$. This line intersects the plane $3x-5y+2z=6$ when $3(2t)-5(-t)+2(-t)=6 \Rightarrow t = \frac{2}{3} \Rightarrow$ the point is $\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

34. The line is represented by $x = 3+2t$, $y = 2-t$, and $z = 1+2t$. It meets the plane $2x-y+2z=-2$ when $2(3+2t)-(2-t)+2(1+2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$.

35. The vector $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i}-11\mathbf{j}-3\mathbf{k}$ is normal to the plane.

(a) No, the plane is not orthogonal to $\overrightarrow{PQ}\times\overrightarrow{PR}$.
(b) No, these equations represent a line, not a plane.
(c) No, the plane $(x+2)+11(y-1)-3z=0$ has normal $\mathbf{i}+11\mathbf{j}-3\mathbf{k}$ which is not parallel to $\overrightarrow{PQ}\times\overrightarrow{PR}$.
(d) No, this vector equation is equivalent to the equations $3y+3z=3$, $3x-2z=-6$, and $3x+2y=-4$ $\Rightarrow x = -\frac{4}{3}-\frac{2}{3}t$, $y = t$, $z = 1-t$, which represents a line, not a plane.
(e) Yes, this is a plane containing the point $R(-2,1,0)$ with normal $\overrightarrow{PQ}\times\overrightarrow{PR}$.

36. (a) The line through A and B is $x = 1+t$, $y = -t$, $z = -1+5t$; the line through C and D must be parallel and is L_1: $x = 1+t$, $y = 2-t$, $z = 3+5t$. The line through B and C is $x = 1$, $y = 2+2s$, $z = 3+4s$; the line through A and D must be parallel and is L_2: $x = 2$, $y = -1+2s$, $z = 4+4s$. The lines L_1 and L_2 intersect at $D(2,1,8)$ where $t = 1$ and $s = 1$.

(b) $\cos\theta = \frac{(2\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+5\mathbf{k})}{\sqrt{20}\sqrt{27}} = \frac{3}{\sqrt{15}}$

(c) $\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\overrightarrow{BC}\cdot\overrightarrow{BC}}\right)\overrightarrow{BC} = \frac{18}{20}\overrightarrow{BC} = \frac{9}{5}(\mathbf{j}+2\mathbf{k})$ where $\overrightarrow{BA} = \mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{j}+4\mathbf{k}$

(d) area $= |(2\mathbf{j}+4\mathbf{k})\times(\mathbf{i}-\mathbf{j}+5\mathbf{k})| = |14\mathbf{i}+4\mathbf{j}-2\mathbf{k}| = 6\sqrt{6}$

(e) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x-1) + 4(y-0) - 2(z+1) = 0$ $\Rightarrow 7x + 2y - z = 8$.

(f) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow$ the area of the projection on the yz-plane is $|\mathbf{n}\cdot\mathbf{i}| = 14$; the area of the projection on the xy-plane is $|\mathbf{n}\cdot\mathbf{j}| = 4$; and the area of the projection on the xy-plane is $|\mathbf{n}\cdot\mathbf{k}| = 2$.

37. The line L passes through the point $P(0,0,-1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\overrightarrow{PS}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (-1-2)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k},$$ we find the distance

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

38. The line L passes through the point $P(2,2,0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\overrightarrow{PS}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (1+2)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k},$$ we find the distance

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

39. The point $P(4,0,0)$ lies on the plane $x - y = 4$, and $\overrightarrow{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$

$$\Rightarrow d = \frac{|\mathbf{n}\cdot\overrightarrow{PS}|}{|\mathbf{n}|} = \left|\frac{2+0+0}{\sqrt{1+1+0}}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

40. The point $P(0,0,2)$ lies on the plane $2x + 3y + z = 2$, and $\overrightarrow{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10+2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{|\mathbf{n}\cdot\overrightarrow{PS}|}{|\mathbf{n}|} = \left|\frac{6+0+8}{\sqrt{4+9+1}}\right| = \frac{14}{\sqrt{14}} = \sqrt{14}.$

41. A normal to the plane is $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left|\frac{\overrightarrow{AP}\cdot\mathbf{n}}{\mathbf{n}}\right|$

$$= \left|\frac{(\mathbf{i}+4\mathbf{j})\cdot(-\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{1+4+4}}\right| = \left|\frac{-1-8+0}{3}\right| = 3$$

42. $P(0,0,0)$ lies on the plane $2x + 3y + 5z = 0$, and $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \Rightarrow$

$$d = \left|\frac{\mathbf{n}\cdot\overrightarrow{PS}}{|\mathbf{n}|}\right| = \left|\frac{4+6+15}{\sqrt{4+9+25}}\right| = \frac{25}{\sqrt{38}}$$

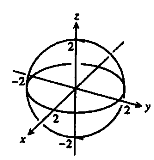
43. $\overrightarrow{AB} = -2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\overrightarrow{CD} = \mathbf{i}+4\mathbf{j}-\mathbf{k}$, and $\overrightarrow{AC} = 2\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i}-\mathbf{j}-9\mathbf{k} \Rightarrow$ the distance is

$$d = \left|\frac{(2\mathbf{i}+\mathbf{j})\cdot(-5\mathbf{i}-\mathbf{j}-9\mathbf{k})}{\sqrt{25+1+81}}\right| = \frac{11}{\sqrt{107}}$$

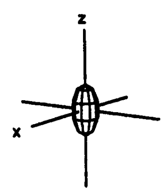
44. $\overrightarrow{AB} = -2\mathbf{i}+4\mathbf{j}-\mathbf{k}$, $\overrightarrow{CD} = \mathbf{i}-\mathbf{j}+2\mathbf{k}$, and $\overrightarrow{AC} = -3\mathbf{i}+3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i}+3\mathbf{j}-2\mathbf{k} \Rightarrow$ the distance

is $d = \left|\frac{(-3\mathbf{i}+3\mathbf{j})\cdot(7\mathbf{i}+3\mathbf{j}-2\mathbf{k})}{\sqrt{49+9+4}}\right| = \frac{12}{\sqrt{62}}$

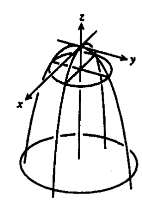
45. $x^2+y^2+z^2 = 4$



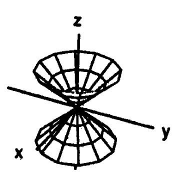
46. $4x^2+4y^2+z^2 = 4$



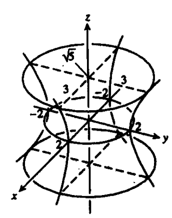
47. $z = -(x^2+y^2)$



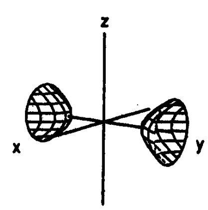
48. $x^2+y^2 = z^2$



49. $x^2+y^2-z^2 = 4$



50. $y^2-x^2-z^2 = 1$



51. $\mathbf{r} = (2\cos t)\mathbf{i}+(2\sin t)\mathbf{j}+t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i}+(2\cos t)\mathbf{j}+2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2+(2\cos t)^2+(2t)^2}$

$$= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1+t^2}\,dt = \left[t\sqrt{1+t^2}+\ln\left|t+\sqrt{1+t^2}\right|\right]_0^{\pi/4} = \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}}+\ln\left(\frac{\pi}{4}+\sqrt{1+\frac{\pi^2}{16}}\right)$$

52. $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 3t^{1/2}\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + \left(3t^{1/2}\right)^2} = \sqrt{9+9t} = 3\sqrt{1+t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1+t}\,dt = \left[2(1+t)^{3/2}\right]_0^3$

$= 14$

53. $\mathbf{r} = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left[\frac{2}{3}(1+t)^{1/2}\right]^2 + \left[-\frac{2}{3}(1-t)^{1/2}\right]^2 + \left(\frac{1}{3}\right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{\sqrt{2}}{3}$

$$\Rightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j};\ \mathbf{B}(0) = \mathbf{T}(0)\times\mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k};$$

$\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$ and $\mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0)\times\mathbf{a}(0)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = -\frac{1}{9}\mathbf{i} + \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3};$$

$$\dot{\mathbf{a}} = -\frac{1}{6}(1+t)^{-3/2}\mathbf{i} + \frac{1}{6}(1-t)^{-3/2}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6};$$

$t = 0 \Rightarrow \left(\frac{4}{9}, \frac{4}{9}, 0\right)$ is the point on the curve

54. $\mathbf{r} = \left(e^t \sin 2t\right)\mathbf{i} + \left(e^t \cos 2t\right)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{v} = \left(e^t \sin 2t + 2e^t \cos 2t\right)\mathbf{i} + \left(e^t \cos 2t - 2e^t \sin 2t\right)\mathbf{j} + 2e^t\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left(e^t \sin 2t + 2e^t \cos 2t\right)^2 + \left(e^t \cos 2t - 2e^t \sin 2t\right)^2 + \left(2e^t\right)^2} = 3e^t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{1}{3}\sin 2t + \frac{2}{3}\cos 2t\right)\mathbf{i} + \left(\frac{1}{3}\cos 2t - \frac{2}{3}\sin 2t\right)\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k};$

$\frac{d\mathbf{T}}{dt} = \left(\frac{2}{3}\cos 2t - \frac{4}{3}\sin 2t\right)\mathbf{i} + \left(-\frac{2}{3}\sin 2t - \frac{4}{3}\cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{2}{3}\sqrt{5}$

$$\Rightarrow \mathbf{N}(0) = \frac{\left(\frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j};\ \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}}\mathbf{i} + \frac{2}{3\sqrt{5}}\mathbf{j} - \frac{5}{3\sqrt{5}}\mathbf{k};$$

$\mathbf{a} = \left(4e^t \cos 2t - 3e^t \sin 2t\right)\mathbf{i} + \left(-3e^t \cos 2t - 4e^t \sin 2t\right)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$$\Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{64 + 16 + 100} = 6\sqrt{5} \text{ and } |\mathbf{v}(0)| = 3$$

$$\Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9};$$

$$\dot{\mathbf{a}} = \left(4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t\right)\mathbf{i} + \left(-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t\right)\mathbf{j} + 2e^t\mathbf{k}$$

$$= \left(-2e^t \cos 2t - 11e^t \sin 2t\right)\mathbf{i} + \left(-11\, e^t \cos 2t + 2e^t \sin 2t\right)\mathbf{j}\ + 2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i}\ - 11\mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow \tau(0) = \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9};\ t = 0 \Rightarrow (0, 1, 2) \text{ is on the curve}$$

55. $\mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1 + e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1 + e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j};$

$$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{\left(1 + e^{4t}\right)^{3/2}}\mathbf{i} + \frac{2e^{2t}}{\left(1 + e^{4t}\right)^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j};$$

$$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k};\ \mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 8 \text{ and } |\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}};\ \dot{\mathbf{a}} = 4e^{2t}\mathbf{j}$$

$$\Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0;\ t = \ln 2 \Rightarrow (\ln 2, 2, 0) \text{ is on the curve}$$

56. $\mathbf{r} = (3\cosh 2t)\mathbf{i} + (3\sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6\sinh 2t)\mathbf{i} + (6\cosh 2t)\mathbf{j} + 6\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{36\sinh^2 2t + 36\cosh^2 2t + 36} = 6\sqrt{2}\cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} 2t\right)\mathbf{k}$

$\Rightarrow \mathbf{T}(\ln 2) = \frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}}\operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}}\operatorname{sech} 2t\tanh 2t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2)$

$= \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$

$$\Rightarrow \mathbf{N}(\ln 2) = \tfrac{8}{17}\mathbf{i} - \tfrac{15}{17}\mathbf{k};\ \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{8}{17\sqrt{2}}\mathbf{k};$$

$\mathbf{a} = (12\cosh 2t)\mathbf{i} + (12\sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\mathbf{i} + \frac{45}{2}\mathbf{j}$ and

$$\mathbf{v}(\ln 2) = 6\left(\tfrac{15}{8}\right)\mathbf{i} + 6\left(\tfrac{17}{8}\right)\mathbf{j} + 6\mathbf{k} = \tfrac{45}{4}\mathbf{i} + \tfrac{51}{4}\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix}$$

$= -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2}$ and $|\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2} \Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867}$;

$$\dot{\mathbf{a}} = (24\sinh 2t)\mathbf{i} + (24\cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{32}{867};$$

$t = \ln 2 \Rightarrow \left(\frac{51}{8}, \frac{45}{8}, 6\ln 2\right)$ is on the curve

57. $\mathbf{r} = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6\cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6\sin t)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6\sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36\sin^2 t}$

$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}\left(25 + 100t + 100t^2 + 36\sin^2 t\right)^{-1/2}(100 + 200t + 72\sin t\cos t) \Rightarrow a_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10$;

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (t\cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6\cos t)^2} = \sqrt{100 + 36\cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$

58. $\mathbf{r} = (2+t)\mathbf{i} + (t+2t^2)\mathbf{j} + (1+t^2)\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1+4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1+4t)^2 + (2t)^2}$
$= \sqrt{2+8t+20t^2} \Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(2+8t+20t^2)^{-1/2}(8+40t) \Rightarrow a_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2};\ \mathbf{a} = 4\mathbf{j} + 2\mathbf{k}$
$\Rightarrow |\mathbf{a}| = \sqrt{4^2+2^2} = \sqrt{20} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - (2\sqrt{2})^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$

59. $\mathbf{r} = (\sin t)\mathbf{i} + (\sqrt{2}\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - (\sqrt{2}\sin t)\mathbf{j} + (\cos t)\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + (-\sqrt{2}\sin t)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{k};$
$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}}\sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}}\sin t\right)^2} = 1$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k};\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\cos t & -\sin t & \frac{1}{\sqrt{2}}\cos t \\ -\frac{1}{\sqrt{2}}\sin t & -\cos t & -\frac{1}{\sqrt{2}}\sin t \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k};\ \mathbf{a} = (-\sin t)\mathbf{i} - (\sqrt{2}\cos t)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix}$$

$= \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}};\ \dot{\mathbf{a}} = (-\cos t)\mathbf{i} + (\sqrt{2}\sin t)\mathbf{j} - (\cos t)\mathbf{k}$

$$\Rightarrow \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\cos t)(\sqrt{2}) - (\sqrt{2}\sin t)(0) + (\cos t)(-\sqrt{2})}{4} = 0$$

60. $\mathbf{r} = \mathbf{i} + (5\cos t)\mathbf{j} + (3\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5\sin t)\mathbf{j} + (3\cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5\cos t)\mathbf{j} - (3\sin t)\mathbf{k}$
$\Rightarrow \mathbf{v}\cdot\mathbf{a} = 25\sin t\cos t - 9\sin t\cos t = 16\sin t\cos t;\ \mathbf{v}\cdot\mathbf{a} = 0 \Rightarrow 16\sin t\cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$
$\Rightarrow t = 0, \frac{\pi}{2}$ or π

61. $\mathbf{r} = 2\mathbf{i} + \left(4\sin\frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r}\cdot(\mathbf{i}-\mathbf{j}) = 2(1) + \left(4\sin\frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4\sin\frac{t}{2} \Rightarrow \sin\frac{t}{2} = \frac{1}{2} \Rightarrow \frac{t}{2} = \frac{\pi}{6}$
$\Rightarrow t = \frac{\pi}{3}$ (for the first time)

62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2+9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14}$

$\Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$, which is normal to the normal plane

$\Rightarrow \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$ or $x + 2y + 3z = 6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} = \frac{\sqrt{19}}{7\sqrt{14}};\ \frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left.\frac{d^2s}{dt^2}\right|_{t=1}$

$= \frac{1}{2}\left(1 + 4t^2 + 9t^4\right)^{-1/2}\left(8t + 36t^3\right)\Big|_{t=1} = \frac{22}{\sqrt{14}}$, so $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \Rightarrow 2\mathbf{j} + 6\mathbf{k}$

$= \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2\mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k}\right) \Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1)$
$= 0$ or $11x + 8y - 9z = 10$ is an equation of the rectifying plane. Finally, $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$

$= \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0$ or $3x - 3y + z$

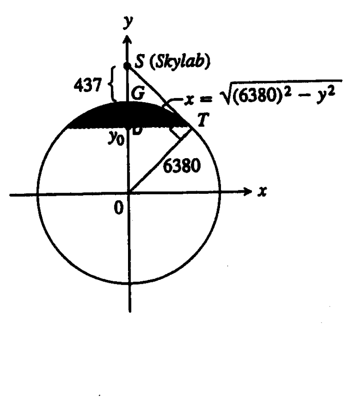
$= 1$ is an equation of the osculating plane.

63. $\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k};\ \mathbf{r}(0) = \mathbf{i} \Rightarrow (1,0,0)$ is on the line $\Rightarrow x = 1 + t,\ y = t$, and $z = -t$ are parametric equations of the line

64. $\mathbf{r} = \left(\sqrt{2}\cos t\right)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \left(-\sqrt{2}\sin t\right)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right)$
$= \left(-\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector tangent to the helix when $t = \frac{\pi}{4} \Rightarrow$ the tangent line is parallel to $\mathbf{v}\left(\frac{\pi}{4}\right)$; also $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow$ the point $\left(1, 1, \frac{\pi}{4}\right)$ is on the line $\Rightarrow x = 1 - t,\ y = 1 + t$, and $z = \frac{\pi}{4} + t$ are parametric equations of the line

65. $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO} \Rightarrow \frac{y_0}{6380} = \frac{6380}{6380 + 437}$
$\Rightarrow y_0 = \frac{6380^2}{6817} \Rightarrow y_0 \approx 5971$ km;

$$VA = \int_{5971}^{6380} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$
$$= 2\pi\int_{5971}^{6817} \sqrt{6380^2 - y^2}\left(\frac{6380}{\sqrt{6380^2 - y^2}}\right)dy$$
$$= 2\pi\int_{5971}^{6817} 6380\,dy = 2\pi\left[6380y\right]_{5971}^{6817}$$



$= 16{,}395{,}469\ \text{km}^2 \approx 1.639 \times 10^7\ \text{km}^2$;

percentage visible $\approx \dfrac{16{,}395{,}469\ \text{km}^2}{4\pi(6380\ \text{km})^2} \approx 3.21\%$

66. $\ddot{s} = \frac{d}{dt}\sqrt{\dot{x}^2+\dot{y}^2} = \dfrac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \Rightarrow \ddot{x}^2+\ddot{y}^2-\ddot{s}^2 = \ddot{x}^2+\ddot{y}^2-\dfrac{(\dot{x}\ddot{x}+\dot{y}\ddot{y})^2}{\dot{x}^2+\dot{y}^2}$

$= \dfrac{(\ddot{x}^2+\ddot{y}^2)(\dot{x}^2+\dot{y}^2)-(\dot{x}^2\ddot{x}^2+2\dot{x}\ddot{x}\dot{y}\ddot{y}+\dot{y}^2\ddot{y}^2)}{\dot{x}^2+\dot{y}^2} = \dfrac{\dot{x}^2\ddot{y}^2+\dot{y}^2\ddot{x}^2-2\dot{x}\ddot{x}\dot{y}\ddot{y}}{\dot{x}^2+\dot{y}^2} = \dfrac{(\dot{x}\ddot{y}-\dot{y}\ddot{x})^2}{\dot{x}^2+\dot{y}^2}$

$\Rightarrow \sqrt{\ddot{x}^2+\ddot{y}^2-\ddot{s}^2} = \dfrac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\sqrt{\dot{x}^2+\dot{y}^2}} \Rightarrow \dfrac{\dot{x}^2+\dot{y}^2}{\sqrt{\ddot{x}^2+\ddot{y}^2-\ddot{s}^2}} = \dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|} = \frac{1}{\kappa} = \rho$

CHAPTER 10 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. Information from ship A indicates the submarine is now on the line L_1: $x = 4+2t$, $y = 3t$, $z = -\frac{1}{3}t$; information from ship B indicates the submarine is now on the line L_2: $x = 18s$, $y = 5-6s$, $z = -s$. The current position of the sub is $\left(6, 3, -\frac{1}{3}\right)$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both points $P\left(2, -1, -\frac{1}{3}\right)$ and $Q\left(6, 3, -\frac{1}{3}\right)$; the line representing this path is L: $x = 2+4t$, $y = -1+4t$, $z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\dfrac{|\overrightarrow{PQ}|}{4} = \dfrac{\sqrt{32}}{4} = \sqrt{2}$ thousand ft/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand ft from Q along the line L $\Rightarrow 20\sqrt{2} = \sqrt{(2+4t-6)^2+(-1+4t-3)^2+0^2} \Rightarrow 800 = 16(t-1)^2+16(t-1)^2 = 32(t-1)^2$ $\Rightarrow (t-1)^2 = \frac{800}{32} = 25 \Rightarrow t = 6 \Rightarrow$ the submarine will be located at $\left(26, 23, -\frac{1}{3}\right)$ in 20 minutes.

2. H_2 stops its flight when $6+110t = 446 \Rightarrow t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246-446)^2+(57-13)^2+(9-0)^2} \approx 204.98$ miles. At 150 mph, it would take about 1.37 hours for H_1 to reach H_2.

3. Torque $= |\overrightarrow{PQ} \times \mathbf{F}| \Rightarrow 15$ ft-lb $= |\overrightarrow{PQ}|\,|\mathbf{F}| \sin\frac{\pi}{2} = \frac{3}{4}$ ft $\cdot |\mathbf{F}| \Rightarrow |\mathbf{F}| = 20$ lb

4. Let $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i}+3\mathbf{j}+2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b} \Rightarrow \mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is clockwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i}+\mathbf{j}-2\mathbf{k}$; $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \Rightarrow (2, 2, 2)$ is the center of the circular path $(1, 3, 2)$ takes $\Rightarrow$ radius $= \sqrt{1^2+(-1)^2+0^2} = \sqrt{2} \Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/sec $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\dfrac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|} = \frac{1}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}-\frac{2}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v} = |\mathbf{v}|\left(\dfrac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}\right)$ $= \frac{3}{2}\sqrt{2}\left(\frac{1}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}-\frac{2}{\sqrt{6}}\mathbf{k}\right) = \frac{\sqrt{3}}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}-\sqrt{3}\mathbf{k}$

5. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\overrightarrow{PP_1}$, $\overrightarrow{PP_2}$ and $\overrightarrow{PP_3}$ all lie in the plane. Thus $\overrightarrow{PP_1} \cdot (\overrightarrow{PP_2} \times \overrightarrow{PP_3}) = 0$

$$\Rightarrow \begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$$ by the determinant formula for the triple scalar product in Section 10.2.

(b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

6. Let L_1: $x = a_1 s + b_1$, $y = a_2 s + b_2$, $z = a_3 s + b_3$ and L_2: $x = c_1 t + d_1$, $y = c_2 t + d_2$, $z = c_3 t + d_3$. If $L_1 \parallel L_2$,

then for some k, $a_i = kc_i$, i = 1, 2, 3 and the determinant $\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1 - d_1 \\ kc_2 & c_2 & b_2 - d_2 \\ kc_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0$,

since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the

system $\begin{cases} a_1 s - c_1 t + (b_1 - d_1) = 0 \\ a_2 s - c_2 t + (b_2 - d_2) = 0 \\ a_3 s - c_3 t + (b_3 - d_3) = 0 \end{cases}$ has a nontrivial solution $\Leftrightarrow$ the determinant of the coefficients is zero.

7. Let (x, y, z) be any point on $Ax + By + Cz - D = 0$. Let $\overrightarrow{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and

$\mathbf{n} = \dfrac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}$. The distance is $\left|\text{proj}_{\mathbf{n}}\, \overrightarrow{QP_1}\right| = \left|((x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}) \cdot \left(\dfrac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}\right)\right|$

$= \dfrac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$.

8. Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the sphere, i.e., $r = \frac{1}{2}\, \dfrac{|3 - 9|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ (see also Exercise 9a). Clearly, the points $(1, 2, 3)$ and $(-1, -2, -3)$ are on the line containing the sphere's center. Hence, the line containing the center is $x = 1 + 2t$, $y = 2 + 4t$, $z = 3 + 6t$. The distance from the plane $x + y + z - 3 = 0$ to the center is $\sqrt{3}$

$\Rightarrow \dfrac{|(1 + 2t) + (2 + 4t) + (3 + 6t) - 3|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ from part (a) $\Rightarrow t = 0 \Rightarrow$ the center is at $(1, 2, 3)$. Therefore an equation of the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$.

9. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is

$d = \dfrac{|Ax_1 + By_1 + Cz_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \dfrac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}$, since $Ax_1 + By_1 + Cz_1 = D_1$, by Exercise 7.

(b) $d = \dfrac{|12 - 6|}{\sqrt{4 + 9 + 1}} = \dfrac{6}{\sqrt{14}}$

10. $\frac{|2(3)+(-1)(2)+2(-1)+4|}{\sqrt{14}} = \frac{|2(3)+(-1)(2)+2(-1)+D|}{\sqrt{14}} \Rightarrow D = -8 \text{ or } 4 \Rightarrow$ the desired plane is $2x - y + 2x = 8$

11. Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC}$ and $P(x, y, x)$ be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\overrightarrow{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

12. (a) $\mathbf{u} \times \mathbf{v} = 4\mathbf{i} \times \mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$; $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}$; $\mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$; $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

(c) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k}$;

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 4 & -4 & -4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$;

$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 10(-\mathbf{i} - \mathbf{k}) - 1(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

13. (a) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}+(\mathbf{v}\cdot\mathbf{u})\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}+(\mathbf{w}\cdot\mathbf{v})\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\mathbf{v}=\mathbf{0}$

(b) $[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{i})]\mathbf{i}+[(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{j})]\mathbf{j}+[(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{k})]\mathbf{k}=[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{i}]\mathbf{i}+[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{j}]\mathbf{j}+[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{k}]\mathbf{k}=\mathbf{u}\times\mathbf{v}$

(c) $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{r})=\mathbf{u}\cdot[\mathbf{v}\times(\mathbf{w}\times\mathbf{r})]=\mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{r})\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\mathbf{r}]=(\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{r})-(\mathbf{u}\cdot\mathbf{r})(\mathbf{v}\cdot\mathbf{w})$

$$=\begin{vmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{w}\\ \mathbf{u}\cdot\mathbf{r} & \mathbf{v}\cdot\mathbf{r}\end{vmatrix}$$

14. The formula is always true; $\mathbf{u}\times[\mathbf{u}\times(\mathbf{u}\times\mathbf{v})]\cdot\mathbf{w}=\mathbf{u}\times[(\mathbf{u}\cdot\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{v}]\cdot\mathbf{w}$
$=[(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\times\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{u}\times\mathbf{v}]\cdot\mathbf{w}=-|\mathbf{u}|^2\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}=-|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$

15. $\mathbf{n}=\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ is normal to the plane $x+2y+6z=6$; $\mathbf{v}\times\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\1&2&6\end{vmatrix}=4\mathbf{i}-5\mathbf{j}+\mathbf{k}$ is parallel to the plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n}\Rightarrow\mathbf{w}=\mathbf{n}\times(\mathbf{v}\times\mathbf{n})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&6\\4&-5&1\end{vmatrix}=32\mathbf{i}+23\mathbf{j}-13\mathbf{k}$ is a vector parallel to the plane $x+2y+6z=6$ in the direction of the projection vector $\text{proj}_P\,\mathbf{v}$. Therefore,

$$\text{proj}_P\,\mathbf{v}=\text{proj}_{\mathbf{w}}\,\mathbf{v}=\left(\mathbf{v}\cdot\frac{\mathbf{w}}{|\mathbf{w}|}\right)\frac{\mathbf{w}}{|\mathbf{w}|}=\left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right)\mathbf{w}=\left(\frac{32+23-13}{32^2+23^2+13^2}\right)\mathbf{w}=\frac{42}{1722}\mathbf{w}=\frac{1}{41}\mathbf{w}=\frac{32}{41}\mathbf{i}+\frac{23}{41}\mathbf{j}-\frac{13}{41}\mathbf{k}$$

16. If $\mathbf{u}=(\cos A)\mathbf{i}+(\sin A)\mathbf{j}$ and $\mathbf{v}=(\cos B)\mathbf{i}+(\sin B)\mathbf{j}$, where $B>A$, then $\mathbf{u}\times\mathbf{v}=[|\mathbf{u}||\mathbf{v}|\sin(B-A)]\mathbf{k}$
$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\cos A&\sin A&0\\\cos B&\sin B&0\end{vmatrix}=(\cos A\sin B-\sin A\cos B)\mathbf{k}\Rightarrow\sin(B-A)=\cos A\sin B-\sin A\cos B$, since $|\mathbf{u}|=1$ and $|\mathbf{v}|=1$.

17. (a) The vector from $(0,d)$ to $(kd,0)$ is $\mathbf{r}_k=kd\mathbf{i}-d\mathbf{j}\Rightarrow|\mathbf{r}_k|^3=\frac{1}{d^3(k^2+1)^{3/2}}\Rightarrow\frac{\mathbf{r}_k}{|\mathbf{r}_k|^3}=\frac{k\mathbf{i}-\mathbf{j}}{d^2(k^2+1)^{3/2}}$. The total force on the mass $(0,d)$ due to the masses Q_k for $k=-n,-n+1,\ldots,n-1,n$ is

$$\mathbf{F}=\frac{GMm}{d^2}(-\mathbf{j})+\frac{GMm}{2d^2}\left(\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right)+\frac{GMm}{5d^2}\left(\frac{2\mathbf{i}-\mathbf{j}}{\sqrt{5}}\right)+\ldots+\frac{GMm}{(n^2+1)d^2}\left(\frac{n\mathbf{i}-\mathbf{j}}{\sqrt{n^2+1}}\right)+\frac{GMm}{2d^2}\left(\frac{-\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right)$$
$$+\frac{GMm}{5d^2}\left(\frac{-2\mathbf{i}-\mathbf{j}}{\sqrt{5}}\right)+\ldots+\frac{GMm}{(n^2+1)d^2}\left(\frac{-n\mathbf{i}-\mathbf{j}}{\sqrt{n^2+1}}\right)$$

The $\mathbf{i}$ components cancel, giving

$$\mathbf{F}=\frac{GMm}{d^2}\left(-1-\frac{2}{2\sqrt{2}}-\frac{2}{5\sqrt{5}}-\ldots-\frac{2}{(n^2+1)(n^2+1)^{1/2}}\right)\mathbf{j}\Rightarrow\text{the magnitude of the force is}$$

$$|\mathbf{F}| = \frac{GMm}{d^2}\left(1+\sum_{i=1}^{n}\frac{2}{(i^2+1)^{3/2}}\right).$$

(b) Yes, it is finite: $\lim_{n\to\infty}|\mathbf{F}| = \frac{GMm}{d^2}\left(1+\sum_{i=1}^{\infty}\frac{2}{(i^2+1)^{3/2}}\right)$ is finite since $\sum_{i=1}^{\infty}\frac{2}{(i^2+1)^{3/2}}$ converges.

18. (a) If $\vec{x}\cdot\vec{y}=0$, then $\vec{x}\times(\vec{x}\times\vec{y}) = (\vec{x}\cdot\vec{y})\vec{x}-(\vec{x}\cdot\vec{x})\vec{y} = -(\vec{x}\cdot\vec{x})\vec{y}$. This means that

$\vec{x}\oplus\vec{y} = \vec{x}+\vec{y}+\frac{1}{c^2}\cdot\frac{1}{1+\sqrt{1-\frac{\vec{x}\cdot\vec{x}}{c^2}}}(-(\vec{x}\cdot\vec{x}))\vec{y} = \vec{x}+\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)\vec{y}$. Since $\vec{x}$ and $\vec{y}$ are orthogonal, then $|\vec{x}\oplus\vec{y}|^2 = |\vec{x}|^2+\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)^2|\vec{y}|^2$. A calculation will show that

$|\vec{x}|^2+\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)^2c^2 = c^2$. Since $|\vec{y}|<c$, then $|\vec{y}|^2<c^2$ so

$\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)^2|\vec{y}|^2 < \left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)c^2$. This means that

$$|\vec{x}\oplus\vec{y}|^2 = |\vec{x}|^2+\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)^2|\vec{y}|^2 < |\vec{x}|^2+\left(1-\frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2|\vec{x}|^2}}\right)^2c^2 = c^2.$$

We now have $|\vec{x}\oplus\vec{y}|^2<c^2$, so $|\vec{x}\oplus\vec{y}|<c$.

(b) If $\vec{x}$ and $\vec{y}$ are parallel, then $\vec{x}\times(\vec{x}\times\vec{y}) = \vec{0}$. This gives $\vec{x}\oplus\vec{y} = \frac{\vec{x}+\vec{y}}{1+\frac{\vec{x}\cdot\vec{y}}{c^2}}$.

(i) If $\vec{x}$ and $\vec{y}$ have the same direction, then $\vec{x}\oplus\vec{y} = \frac{\vec{x}+\vec{y}}{1+\frac{|\vec{x}|}{c}\cdot\frac{|\vec{y}|}{c}}$ and $|\vec{x}\oplus\vec{y}| = \frac{|\vec{x}|+|\vec{y}|}{1+\frac{|\vec{x}|}{c}\cdot\frac{|\vec{y}|}{c}}$.

Since $|\vec{y}|<c$, $|\vec{x}|<c$, we have $|\vec{y}|\left(1-\frac{|\vec{x}|}{c}\right)<c\left(1-\frac{|\vec{x}|}{c}\right)\Rightarrow|\vec{y}|-\frac{|\vec{y}||\vec{x}|}{c}<c-|\vec{x}|$

$\Rightarrow|\vec{x}|+|\vec{y}|<c+\frac{|\vec{x}||\vec{y}|}{c} = c\left(1+\frac{|\vec{x}|}{c}\cdot\frac{|\vec{y}|}{c}\right)\Rightarrow\frac{|\vec{x}|+|\vec{y}|}{1+\frac{|\vec{x}|}{c}\cdot\frac{|\vec{y}|}{c}}<c$. This means that $|\vec{x}\oplus\vec{y}|<c$.

(ii) If $\vec{x}$ and $\vec{y}$ have opposite directions, then $\vec{x}\cdot\vec{y} = -|\vec{x}||\vec{y}|$ and $\vec{x}\oplus\vec{y} = \frac{\vec{x}+\vec{y}}{1-\frac{|\vec{x}||\vec{y}|}{c^2}}$.

Assume $|\vec{x}|\ge|\vec{y}|$, then $|\vec{x}\oplus\vec{y}| = \frac{|\vec{x}|-|\vec{y}|}{1-\frac{|\vec{x}||\vec{y}|}{c^2}}$. Since $|\vec{x}|<c$, we have $|\vec{x}|\left(1+\frac{|\vec{y}|}{c}\right)<c\left(1+\frac{|\vec{y}|}{c}\right)$

$\Rightarrow|\vec{x}|+\frac{|\vec{x}||\vec{y}|}{c}<c+|\vec{y}|\Rightarrow|\vec{x}|-|\vec{y}|<c-\frac{|\vec{x}||\vec{y}|}{c} = c\left(1-\frac{|\vec{x}||\vec{y}|}{c^2}\right)\Rightarrow\frac{|\vec{x}|-|\vec{y}|}{1-\frac{|\vec{x}||\vec{y}|}{c^2}}<c.$

This means that $|\vec{x} \oplus \vec{y}| < c$. A similar argument holds if $|\vec{x}| > |\vec{y}|$.

(c) $\lim\limits_{c \to \infty} \vec{x} \oplus \vec{y} = \vec{x} + \vec{y}$.

19. $r = \frac{(1+e)r_0}{1 + e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2}$; $\frac{dr}{d\theta} = 0 \Rightarrow \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2} = 0 \Rightarrow (1+e)r_0(e\sin\theta) = 0$ $\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π. Note that $\frac{dr}{d\theta} > 0$ when $\sin\theta > 0$ and $\frac{dr}{d\theta} < 0$ when $\sin\theta < 0$. Since $\sin\theta < 0$ on $-\pi < \theta < 0$ and $\sin\theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1+e)r_0}{1+e\cos 0} = r_0$

20. (a) $f(x) = x - 1 - \frac{1}{2}\sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2}\sin 2 \ge \frac{1}{2}$ since $|\sin 2| \le 1$; since f is continuous on $[0,2]$, the Intermediate Value Theorem implies there is a root between 0 and 2

(b) Root ≈ 1.4987011335179

21. (a) $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = \left(\frac{dr}{dt}\right)[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}] + \left(r\frac{d\theta}{dt}\right)[(-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}] \Rightarrow \mathbf{v}\cdot\mathbf{i}\ \frac{dx}{dt}$ and $\mathbf{v}\cdot\mathbf{i} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta \Rightarrow \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta$; $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt}$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta$ $\Rightarrow \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta$

(b) $\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$
$= \left(\frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta\right)(\cos\theta) + \left(\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right)(\sin\theta)$ by part (a),
$\Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \frac{dr}{dt}$; therefore, $\frac{dr}{dt} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$;
$\mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta$
$= \left(\frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta\right)(-\sin\theta) + \left(\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right)(\cos\theta)$ by part (a) $\Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = r\frac{d\theta}{dt}$;
therefore, $r\frac{d\theta}{dt} = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta$

22. $r = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta)\left(\frac{d\theta}{dt}\right)^2 + f'(\theta)\frac{d^2\theta}{dt^2}$; $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta$
$= \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\mathbf{i} + \left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]^{1/2} = \left[(f')^2 + f^2\right]^{1/2}\left(\frac{d\theta}{dt}\right)$;
$|\mathbf{v}\times\mathbf{a}| = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$, where $x = r\cos\theta$ and $y = r\sin\theta$. Then $\frac{dx}{dt} = (-r\sin\theta)\frac{d\theta}{dt} + (\cos\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2x}{dt^2} = (-2\sin\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (r\sin\theta)\frac{d^2\theta}{dt^2} + (\cos\theta)\frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r\cos\theta)\frac{d\theta}{dt} + (\sin\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2y}{dt^2} = (2\cos\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\sin\theta)\left(\frac{d\theta}{dt}\right)^2 + (r\cos\theta)\frac{d^2\theta}{dt^2} + (\sin\theta)\frac{d^2r}{dt^2}$. Then $|\mathbf{v}\times\mathbf{a}|$
$= $ (after <u>much</u> algebra) $r^2\left(\frac{d\theta}{dt}\right)^3 + r\frac{d^2\theta}{dt^2}\frac{dr}{dt} - r\frac{d\theta}{dt}\left(\frac{dr}{dt}\right)^2 \Rightarrow \kappa = \frac{r^2\left(\frac{d\theta}{dt}\right)^3 + r\left(\frac{d^2\theta}{dt^2}\right)\left(\frac{dr}{dt}\right) - r\left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)^2}{\left[(f')^2 + f^2\right]^{3/2}}$

$$=\frac{r^2+r\left(\frac{d^2\theta}{dt^2}\right)(f')\left(\frac{dt}{d\theta}\right)^2-rf''-rf'\left(\frac{d^2\theta}{dt^2}\right)\left(\frac{dt}{d\theta}\right)^2+2(f')^2}{\left[(f')^2+f^2\right]^{3/2}}=\frac{f^2-ff''+2(f')^2}{\left[(f')^2+f^2\right]^{3/2}}$$

23. (a) Let $r=2-t$ and $\theta=3t \Rightarrow \frac{dr}{dt}=-1$ and $\frac{d\theta}{dt}=3 \Rightarrow \frac{d^2r}{dt^2}=\frac{d^2\theta}{dt^2}=0$. The halfway point is $(1,3) \Rightarrow t=1$;

$$\mathbf{v}=\frac{dr}{dt}\mathbf{u}_r+r\frac{d\theta}{dt}\mathbf{u}_\theta \Rightarrow \mathbf{v}(1)=-\mathbf{u}_r+3\mathbf{u}_\theta;\ \mathbf{a}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r+\left[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta \Rightarrow \mathbf{a}(1)=-9\mathbf{u}_r-6\mathbf{u}_\theta$$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians

$$\Rightarrow L=\int_0^6 \sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta=\int_0^6 \sqrt{\left(2-\frac{\theta}{3}\right)^2+\left(-\frac{1}{3}\right)^2}\,d\theta=\int_0^6 \sqrt{4-\frac{4\theta}{3}+\frac{\theta^2}{9}+\frac{1}{9}}\,d\theta$$

$$=\int_0^6 \sqrt{\frac{37-12\theta+\theta^2}{9}}\,d\theta=\frac{1}{3}\int_0^6 \sqrt{(\theta-6)^2+1}\,d\theta=\frac{1}{3}\left[\frac{(\theta-6)}{2}\sqrt{(\theta-6)^2+1}+\frac{1}{2}\ln\left|\theta-6+\sqrt{(\theta-6)^2+1}\right|\right]_0^6$$

$$=\sqrt{37}-\frac{1}{6}\ln\left(\sqrt{37}-6\right)\approx 6.5 \text{ in.}$$

24. $\mathbf{L}(t)=\mathbf{r}(t)+m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt}=\left(\frac{d\mathbf{r}}{dt}\times m\mathbf{v}\right)+\left(\mathbf{r}+m\,\frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt}=(\mathbf{v}\times m\mathbf{v})+(\mathbf{r}\times m\mathbf{a})=\mathbf{r}\times m\mathbf{a}$; $\mathbf{F}=m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3}\mathbf{r}$ $=m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt}=\mathbf{r}\times m\mathbf{a}=\mathbf{r}\times\left(-\frac{c}{|\mathbf{r}|^3}\mathbf{r}\right)=-\frac{c}{|\mathbf{r}|^3}(\mathbf{r}\times\mathbf{r})=\mathbf{0} \Rightarrow \mathbf{L}=$ constant vector