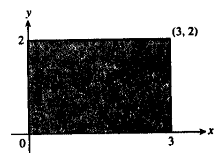
CHAPTER 12 MULTIPLE INTEGRALS

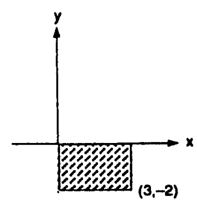
12.1 DOUBLE INTEGRALS

1. $\int\limits_0^3 \int\limits_0^2 (4 - y^2)\, dy\, dx = \int\limits_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \frac{16}{3} \int\limits_0^3 dx = 16$



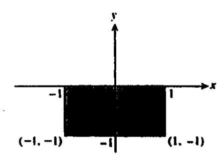
2. $\int\limits_0^3 \int\limits_{-2}^0 ((x^2 y - 2xy)\, dy\, dx = \int\limits_0^3 \left[\frac{x^2 y^2}{2} - xy^2\right]_{-2}^0 dx$

$= \int\limits_0^3 (4x - 2x^2)\, dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$



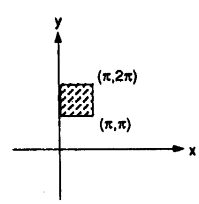
3. $\int\limits_{-1}^0 \int\limits_{-1}^1 (x + y + 1)\, dx\, dy = \int\limits_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^1 dy$

$= \int\limits_{-1}^0 (2y + 2)\, dy = \left[y^2 + 2y\right]_{-1}^0 = 1$

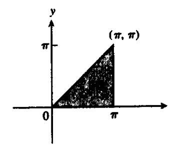


4. $\int\limits_{\pi}^{2\pi} \int\limits_0^{\pi} (\sin x + \cos y)\, dx\, dy = \int\limits_{\pi}^{2\pi} \left[(-\cos x) + (\cos y)x\right]_0^{\pi} dy$

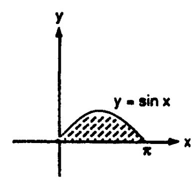
$= \int\limits_{\pi}^{2\pi} (\pi \cos y + 2)\, dy = \left[\pi \sin y + 2y\right]_{\pi}^{2\pi} = 2\pi$



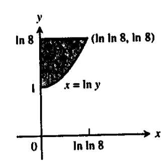
5. $$\int_0^{\pi}\int_0^{x} (x \sin y)\, dy\, dx = \int_0^{\pi} [-x \cos y]_0^x\, dx$$
$$= \int_0^{\pi} (x - x\cos x)\, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x)\right]_0^{\pi} = \frac{\pi^2}{2} + 2$$



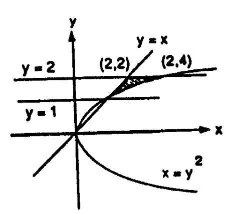
6. $$\int_0^{\pi}\int_0^{\sin x} y\, dy\, dx = \int_0^{\pi} \left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^{\pi} \frac{1}{2}\sin^2 x\, dx$$
$$= \frac{1}{4}\int_0^{\pi} (1 - \cos 2x)\, dx = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}$$



7. $$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\, dx\, dy = \int_1^{\ln 8} \left[e^{x+y}\right]_0^{\ln y} dy = \int_1^{\ln 8} \left(ye^y - e^y\right) dy$$
$$= \left[(y-1)e^y - e^y\right]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e = 8\ln 8 - 16 + e$$

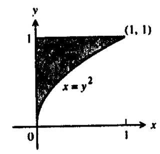


8. $$\int_1^{2}\int_y^{y^2} dx\, dy = \int_1^{2} \left(y^2 - y\right) dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$$
$$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$$



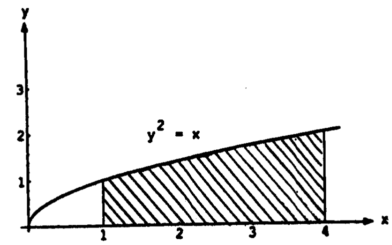
9. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy = \int_0^1 \left[3y^2 e^{xy}\right]_0^{y^2} dy$

$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2\right) dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$



10. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx = \int_1^4 \left[\frac{3}{2}\sqrt{x}\, e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$

$= \frac{3}{2}(e-1) \int_1^4 \sqrt{x}\, dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$



11. $\int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx = \int_1^2 \left[x \ln y\right]_x^{2x} dx = (\ln 2) \int_1^2 x\, dx = \frac{3}{2} \ln 2$

12. $\int_1^2 \int_1^2 \frac{1}{xy}\, dy\, dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln 1)\, dx = (\ln 2) \int_1^2 \frac{1}{x}\, dx = (\ln 2)^2$

13. $\int_0^1 \int_0^{1-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

14. $\int_0^1 \int_0^{\pi} y \cos xy\, dx\, dy = \int_0^1 \left[\sin xy\right]_0^{\pi} dy = \int_0^1 \sin \pi y\, dy = \left[-\frac{1}{\pi} \cos \pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

15. $\int_0^1 \int_0^{1-u} (v - \sqrt{u})\, dv\, du = \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du = \int_0^1 \left[\frac{1 - 2u + u^2}{2} - \sqrt{u}(1-u)\right] du$

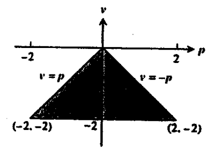
$= \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

16. $\int_1^2 \int_0^{\ln t} e^s \ln t \, ds\, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t)\, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t\right]_1^2$

$= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

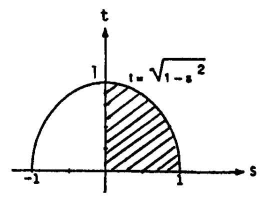
17. $\int_{-2}^0 \int_v^{-v} 2\, dp\, dv = \int_{-2}^0 [p]_v^{-v}\, dv = 2\int_{-2}^0 -2v\, dv$

$= -2\left[v^2\right]_{-2}^0 = 8$



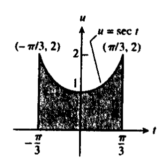
18. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\, dt\, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$

$= \int_0^1 4\left(1 - s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



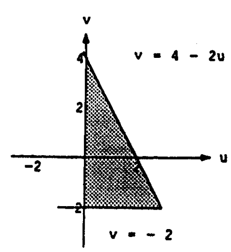
19. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t\, du\, dt = \int_{-\pi/3}^{\pi/3} \left[(3 \cos t)u\right]_0^{\sec t}$

$= \int_{-\pi/3}^{\pi/3} 3\, dt = 2\pi$

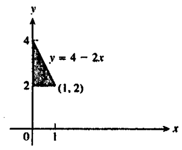


20. $\int_0^3 \int_1^{4-2u} \frac{4-2u}{v^2}\, dv\, du = \int_0^3 \left[\frac{2u-4}{v}\right]_1^{4-2u} du$

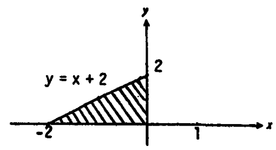
$= \int_0^3 (3 - 2u)\, du = \left[3u - u^2\right]_0^3 = 0$



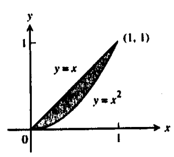
21. $\int_{2}^{4} \int_{0}^{(4-y)/2} dx\,dy$



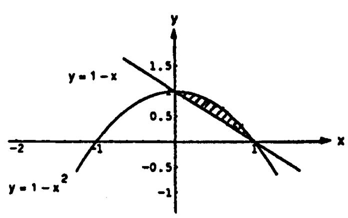
22. $\int_{-2}^{0} \int_{0}^{x+2} dy\,dx$



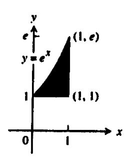
23. $\int_{0}^{1} \int_{x^2}^{x} dy\,dx$



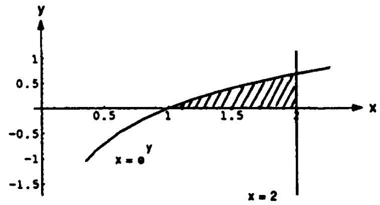
24. $\int_{0}^{1} \int_{1-y}^{\sqrt{1-y}} dx\,dy$



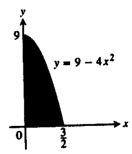
25. $\int_{1}^{e} \int_{\ln y}^{1} dx\,dy$



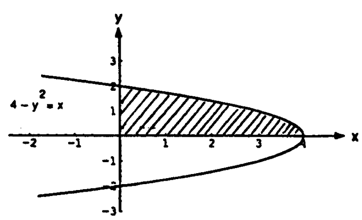
26. $\int_1^2 \int_0^{\ln x} dy\, dx$



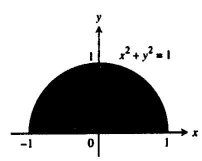
27. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\, dx\, dy$



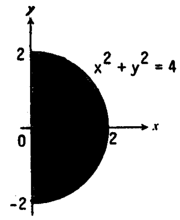
28. $\int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx$



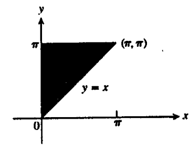
29. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\, dy\, dx$



30. $\int\limits_{-2}^{2}\int\limits_{0}^{\sqrt{4-y^2}} 6x\,dx\,dy$



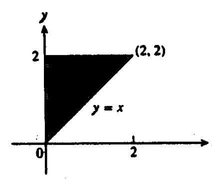
31. $\int\limits_{0}^{\pi}\int\limits_{x}^{\pi} \frac{\sin y}{y}\,dy\,dx = \int\limits_{0}^{\pi}\int\limits_{0}^{y} \frac{\sin y}{y}\,dx\,dy = \int\limits_{0}^{\pi} \sin y\;dy = 2$



32. $\int\limits_{0}^{2}\int\limits_{x}^{2} 2y^2 \sin xy\;dy\,dx = \int\limits_{0}^{2}\int\limits_{0}^{y} 2y^2 \sin xy\;dx\,dy$

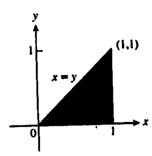
$= \int\limits_{0}^{2} \left[-2y \cos xy\right]_0^y dy = \int\limits_{0}^{2} \left(-2y \cos y^2 + 2y\right) dy$

$= \left[-\sin y^2 + y^2\right]_0^2 = 4 - \sin 4$



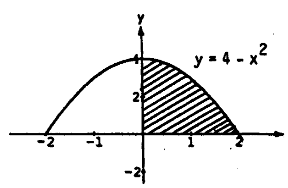
33. $\int\limits_{0}^{1}\int\limits_{y}^{1} x^2 e^{xy}\,dx\,dy = \int\limits_{0}^{1}\int\limits_{0}^{x} x^2 e^{xy}\,dy\,dx = \int\limits_{0}^{1} \left[xe^{xy}\right]_0^x dx$

$= \int\limits_{0}^{1} \left(xe^{x^2} - x\right) dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



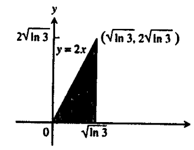
34. $\int\limits_{0}^{2}\int\limits_{0}^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx = \int\limits_{0}^{4}\int\limits_{0}^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy$

$= \int\limits_{0}^{4} \left[\frac{x^2 e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}} dy = \int\limits_{0}^{4} \frac{e^{2y}}{2}\,dy = \left[\frac{e^{2y}}{4}\right]_0^4 = \frac{e^8 - 1}{4}$



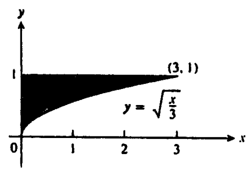
35. $$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\,dy\,dx$$

$$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\,dx = \left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$$



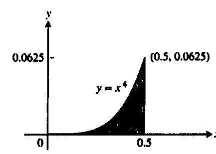
36. $$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx = \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$= \int_0^1 3y^2e^{y^3}\,dy = \left[e^{y^3}\right]_0^1 = e - 1$$



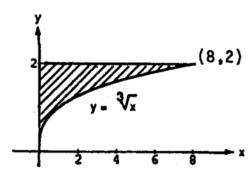
37. $$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy = \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$$

$$= \int_0^{1/2} x^4\cos(16\pi x^5)\,dx = \left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$$



38. $$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$$

$$= \int_0^2 \frac{y^3}{y^4+1}\,dy = \tfrac{1}{4}\left[\ln(y^4+1)\right]_0^2 = \frac{\ln 17}{4}$$

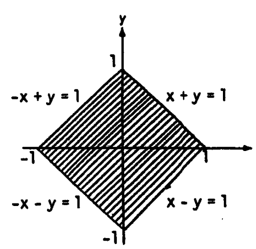


39. $$\iint_R (y - 2x^2)\,dA = \int_{-1}^0 \int_{-x-1}^{x+1} (y - 2x^2)\,dy\,dx + \int_0^1 \int_{x-1}^{1-x} (y - 2x^2)\,dy\,dx$$

$$= \int_{-1}^0 \left[\tfrac{1}{2}y^2 - 2x^2y\right]_{-x-1}^{x+1} dx + \int_0^1 \left[\tfrac{1}{2}y^2 - 2x^2y\right]_{x-1}^{1-x} dx$$

$$= \int_{-1}^0 \left[\tfrac{1}{2}(x+1)^2 - 2x^2(x+1) - \tfrac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$$

$$+ \int_0^1 \left[\tfrac{1}{2}(1-x)^2 - 2x^2(1-x) - \tfrac{1}{2}(x-1)^2 + 2x^2(x-1)\right] dx$$

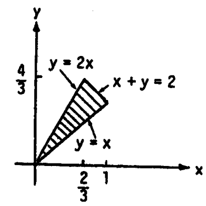


$$= -4\int_{-1}^{0} (x^3+x^2)\,dx + 4\int_{0}^{1} (x^3-x^2)\,dx = -4\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{0} + 4\left[\frac{x^4}{4}-\frac{x^3}{3}\right]_{0}^{1}$$

$$= 4\left[\frac{(-1)^4}{4}+\frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4}-\frac{1}{3}\right) = 8\left(\frac{3}{12}-\frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$$

40. $\displaystyle\iint_R xy\,dA = \int_0^{2/3}\int_0^{2x} xy\,dy\,dx + \int_{2/3}^{1}\int_x^{2-x} xy\,dy\,dx$

$$= \int_0^{2/3} \left[\frac{1}{2}xy^2\right]_x^{2x} dx + \int_{2/3}^{1} \left[\frac{1}{2}xy^2\right]_x^{2-x} dx$$



$$= \int_0^{2/3} \left(2x^3-\frac{1}{2}x^3\right) dx + \int_{2/3}^{1} \left[\frac{1}{2}x(2-x)^2-\frac{1}{2}x^3\right] dx$$

$$= \int_0^{2/3} \frac{3}{2}x^3\,dx + \int_{2/3}^{1} (2x-x^2)\,dx$$

$$= \left[\frac{3}{8}x^4\right]_0^{2/3} + \left[x^2-\frac{2}{3}x^3\right]_{2/3}^{1} = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right)+\left(1-\frac{2}{3}\right)-\left[\frac{4}{9}-\left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81}+\frac{27}{81}-\left(\frac{36}{81}-\frac{16}{81}\right) = \frac{13}{81}$$

41. $\displaystyle V = \int_0^1\int_x^{2-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2y+\frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2-\frac{7x^3}{3}+\frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3}-\frac{7x^4}{12}-\frac{(2-x)^4}{12}\right]_0^1$

$$= \left(\frac{2}{3}-\frac{7}{12}-\frac{1}{12}\right)-\left(0-0-\frac{16}{12}\right) = \frac{4}{3}$$

42. $\displaystyle V = \int_{-2}^{1}\int_x^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} [x^2y]_x^{2-x^2}\,dx = \int_{-2}^{1} (2x^2-x^4-x^3)\,dx = \left[\frac{2}{3}x^3-\frac{1}{5}x^5-\frac{1}{4}x^4\right]_{-2}^{1}$

$$= \left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{32}{5}-\frac{16}{4}\right) = \left(\frac{40}{60}-\frac{12}{60}-\frac{15}{60}\right)-\left(-\frac{320}{60}+\frac{384}{60}-\frac{240}{60}\right) = \frac{189}{60} = \frac{63}{20}$$

43. $\displaystyle V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{-4}^{1} [xy+4y]_{3x}^{4-x^2}\,dx = \int_{-4}^{1} \left[x(4-x^2)+4(4-x^2)-3x^2-12x\right] dx$

$$= \int_{-4}^{1} (-x^3-7x^2-8x+16)\,dx = \left[-\frac{1}{4}x^4-\frac{7}{3}x^3-4x^2+16x\right]_{-4}^{1} = \left(-\frac{1}{4}-\frac{7}{3}+12\right)-\left(\frac{64}{3}-64\right)$$

$$= \frac{157}{3}-\frac{1}{4} = \frac{625}{12}$$

44. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3-y)\,dy\,dx = \int_0^2 \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$

$= \left[\frac{3}{2}x\sqrt{4-x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2 = 6\left(\frac{\pi}{2}\right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$

45. $V = \int_0^2 \int_0^3 (4-y^2)\,dx\,dy = \int_0^2 \left[4x - y^2x\right]_0^3 dy = \int_0^2 (12 - 3y^2)\,dy = \left[12y - y^3\right]_0^2 = 24 - 8 = 16$

46. $V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\,dy\,dx = \int_0^2 \left[(4-x^2)y - \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_0^2 \frac{1}{2}(4-x^2)^2\,dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx$

$= \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480 - 320 + 96}{30} = \frac{128}{15}$

47. $V = \int_0^2 \int_0^{2-x} (12 - 3y^2)\,dy\,dx = \int_0^2 \left[12y - y^3\right]_0^{2-x} dx = \int_0^2 \left[24 - 12x - (2-x)^3\right] dx$

$= \left[24x - 6x^2 + \frac{(2-x)^4}{4}\right]_0^2 = 20$

48. $V = \int_{-1}^0 \int_{-x-1}^{x+1} (3 - 3x)\,dy\,dx + \int_0^1 \int_{x-1}^{1-x} (3-3x)\,dy\,dx = 6\int_{-1}^0 (1-x^2)\,dx + 6\int_0^1 (1-x)^2\,dx = 2 + 4 = 6$

49. $V = \int_1^2 \int_{-1/x}^{1/x} (x+1)\,dy\,dx = \int_1^2 \left[xy + y\right]_{-1/x}^{1/x} dx = \int_1^2 \left[1 + \frac{1}{x} - \left(-1 - \frac{1}{x}\right)\right] = 2\int_1^2 \left(1 + \frac{1}{x}\right) dx$

$= 2\left[x + \ln x\right]_1^2 = 2(1 + \ln 2)$

50. $V = 4\int_0^{\pi/3} \int_0^{\sec x} (1 + y^2)\,dy\,dx = 4\int_0^{\pi/3} \left[y + \frac{y^3}{3}\right]_0^{\sec x} dx = 4\int_0^{\pi/3} \left(\sec x + \frac{\sec^3 x}{3}\right) dx$

$= \frac{2}{3}\left[7\ln|\sec x + \tan x| + \sec x \tan x\right]_0^{\pi/3} = \frac{2}{3}\left[7\ln\left(2 + \sqrt{3}\right) + 2\sqrt{3}\right]$

51. $\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\,dy\,dx = \int_1^\infty \left[\frac{\ln y}{x^3}\right]_{e^{-x}}^1 dx = \int_1^\infty -\left(\frac{-x}{x^3}\right) dx = -\lim_{b\to\infty} \left[\frac{1}{x}\right]_1^b = -\lim_{b\to\infty} \left(\frac{1}{b} - 1\right) = 1$

52. $\int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y+1)\,dy\,dx = \int_{-1}^1 \left[y^2 + y\right]_{-1/(1-x^2)^{1/2}}^{1/(1-x^2)^{1/2}} dx = \int_{-1}^1 \frac{2}{\sqrt{1-x^2}}\,dx = 4\lim_{b\to 1^-} \left[\sin^{-1} x\right]_0^b$

$= 4\lim_{b\to 1^-} \left[\sin^{-1} b - 0\right] = 2\pi$

53. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy = 2\int_0^{\infty} \left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0\right) dy = 2\pi \lim_{b\to\infty} \int_0^b \frac{1}{y^2+1}\,dy$

$= 2\pi\left(\lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0\right) = (2\pi)\left(\frac{\pi}{2}\right) = \pi^2$

54. $\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\,dx\,dy = \int_0^{\infty} e^{-2y} \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b dy = \int_0^{\infty} e^{-2y} \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) dy$

$= \int_0^{\infty} e^{-2y}\,dy = \frac{1}{2}\lim_{b\to\infty} \left(-e^{-2b} + 1\right) = \frac{1}{2}$

55. $\int\int_R f(x,y)\,dA \approx \frac{1}{4} f\left(-\frac{1}{2},0\right) + \frac{1}{8} f(0,0) + \frac{1}{8} f\left(\frac{1}{4},0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0+\frac{1}{4}\right) = -\frac{3}{32}$

56. $\int\int_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{7}{4},\frac{9}{4}\right) + f\left(\frac{9}{4},\frac{9}{4}\right) + f\left(\frac{5}{4},\frac{11}{4}\right) + f\left(\frac{7}{4},\frac{11}{4}\right) + f\left(\frac{9}{4},\frac{11}{4}\right) + f\left(\frac{11}{4},\frac{11}{4}\right) + f\left(\frac{5}{4},\frac{13}{4}\right) + f\left(\frac{7}{4},\frac{13}{4}\right)\right.$

$\left. + f\left(\frac{9}{4},\frac{13}{4}\right) + f\left(\frac{11}{4},\frac{13}{4}\right) + f\left(\frac{7}{4},\frac{15}{4}\right) + f\left(\frac{9}{4},\frac{15}{4}\right)\right]$

$= \frac{1}{16}(25+27+27+29+31+33+31+33+35+37+37+39) = \frac{384}{16} = 24$

57. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2+y^2=4$ at the point $\left(\sqrt{3},1\right) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$.

Thus, $\int\int_R f(x,y)\,dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^{\sqrt{3}} \left[\left(4-x^2\right) - \frac{x}{3}\sqrt{4-x^2}\right] dx = \left[4x - \frac{x^3}{3} + \frac{\left(4-x^2\right)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}}$

$= \frac{20\sqrt{3}}{9}$

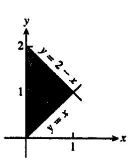
58. $\int_2^{\infty}\int_0^2 \frac{1}{\left(x^2-x\right)(y-1)^{2/3}}\,dy\,dx = \int_2^{\infty} \left[\frac{3(y-1)^{1/3}}{\left(x^2-x\right)}\right]_0^2 dx = \int_2^{\infty} \left(\frac{3}{x^2-x} + \frac{3}{x^2-x}\right) dx = 6\int_2^{\infty} \frac{dx}{x(x-1)}$

$= 6\lim_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = 6\lim_{b\to\infty} \left[\ln(x-1) - \ln x\right]_2^b = 6\lim_{b\to\infty} \left[\ln(b-1) - \ln b - \ln 1 + \ln 2\right]$

$= 6\left[\lim_{b\to\infty} \ln\left(1-\frac{1}{b}\right) + \ln 2\right] = 6 \ln 2$

59. $V = \int_0^1 \int_x^{2-x} \left(x^2+y^2\right) dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$

$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$



$$=\left(\frac{2}{3}-\frac{7}{12}-\frac{1}{12}\right)-\left(0-0-\frac{16}{12}\right)=\left(\frac{2}{3}+\frac{8}{12}\right)=\frac{4}{3}$$

60. $$\int_0^2 \left(\tan^{-1}\pi x-\tan^{-1}x\right)dx=\int_0^2\int_x^{\pi x}\frac{1}{1+y^2}\,dy\,dx=\int_0^2\int_{y/\pi}^{y}\frac{1}{1+y^2}\,dx\,dy+\int_2^{2\pi}\int_{y/\pi}^{2}\frac{1}{1+y^2}\,dx\,dy$$

$$=\int_0^2\frac{\left(1-\frac{1}{\pi}\right)y}{1+y^2}\,dy+\int_2^{2\pi}\frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy=\left(\frac{\pi-1}{2\pi}\right)\left[\ln\left(1+y^2\right)\right]_0^2+\left[2\tan^{-1}y-\frac{1}{2\pi}\ln\left(1+y^2\right)\right]_2^{2\pi}$$

$$=\left(\frac{\pi-1}{2\pi}\right)\ln 5+2\tan^{-1}2\pi-\frac{1}{2\pi}\ln\left(1+4\pi^2\right)-2\tan^{-1}2-\frac{1}{2\pi}\ln 5$$

$$=2\tan^{-1}2\pi-2\tan^{-1}2-\frac{1}{2\pi}\ln\left(1+4\pi^2\right)+\frac{\ln 5}{2}$$

61. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4-x^2-2y^2\geq 0$ or $x^2+2y^2\leq 4$, which is the ellipse $x^2+2y^2=4$ together with its interior.

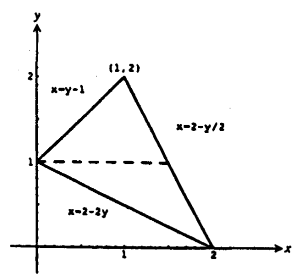
62. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2+y^2-9\leq 0$ or $x^2+y^2\leq 9$, which is the closed disk of radius 3 centered at the origin.

63. No, it is not all right. By Fubini's theorem, the two orders of integration must give the same result.

64. One way would be to partition R into two triangles with the line $y = 1$. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:

$$\iint_R f(x,y)\,dA=\int_0^1\int_{2-2y}^{2-(y/2)} f(x,y)\,dx\,dy+\int_1^2\int_{y-1}^{2-(y/2)} f(x,y)\,dx\,dy.$$

Partitioning R with the line $x = 1$ would let us write the integral of f over R as a sum of iterated integrals with order dy dx.



65. $$\int_{-b}^{b}\int_{-b}^{b}e^{-x^2-y^2}\,dx\,dy=\int_{-b}^{b}\int_{-b}^{b}e^{-y^2}e^{-x^2}\,dx\,dy=\int_{-b}^{b}e^{-y^2}\left(\int_{-b}^{b}e^{-x^2}\,dx\right)dy=\left(\int_{-b}^{b}e^{-x^2}\,dx\right)\left(\int_{-b}^{b}e^{-y^2}\,dy\right)$$

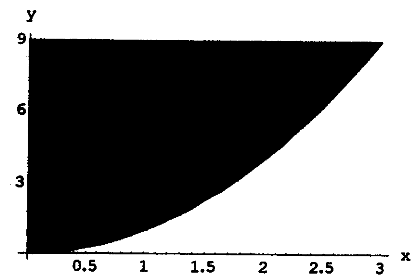
$$=\left(\int_{-b}^{b}e^{-x^2}\,dx\right)^2=\left(2\int_0^{b}e^{-x^2}\,dx\right)^2=4\left(\int_0^{b}e^{-x^2}\,dx\right)^2$$; taking limits as $b\to\infty$ gives the stated result.

```
Show[{p1}, AxesLabel → {x,y}, Ticks → {Automatic, {0,3,6,9}},
  DisplayFunction → $DisplayFunction];
```

Maple:

```
>f:=x->x^2;
>plot([f(x),9], x=0..3, color=[white, blue],
  filled=true, labels=[x,y]);
```

The following graph was generated using Mathematica.



Evaluate the integrals:

$$\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy = \frac{\sin(81)}{4} \approx -0.157472$$

73. Plot the region of integration with the following commands:

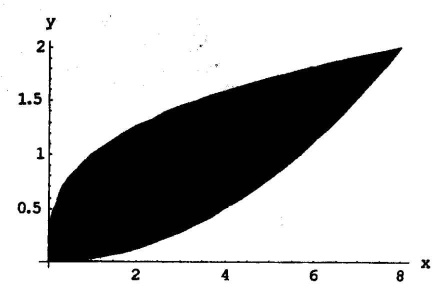
Mathematica:

```
<< Graphics`FilledPlot`
p1 = FilledPlot[{x^2/32,x^(1/3)}, {x,0,8}, DisplayFunction → Identity];
Show[{p1}, AxesLabel → {x,y}, DisplayFunction → $DisplayFunction];
```

Maple:

```
>plot([x^2/32,x^(1/3)], x=0..8,
  color=[white, blue], filled=true, labels=[x,y]);
```

The following graph was generated using Mathematica.



Evaluate the integrals:

$$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx = \frac{67{,}520}{693} \approx 97.4315$$

74. Plot the region of integration with the following commands:

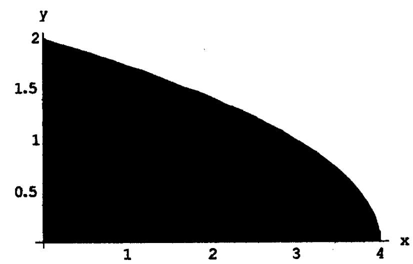
Mathematica:

```
<< Graphics`FilledPlot`;
p1 = FilledPlot[{0,Sqrt[4-x]}, {x,0,4}, DisplayFunction → Identity];
Show[{p1}, AxesLabel → {x,y}, DisplayFunction → $DisplayFunction];
```

Maple:

```
>plot([sqrt(4-x)[, x=0..4,
  color=[blue], filled=true, labels=[x,y]);
```

The following graph was generated using Mathematica.



Evaluate the integrals:

$$\int_0^2 \int_0^{4-y^2} e^{xy}\, dx\, dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy}\, dy\, dx \approx 20.5648$$

75. Plot the region of integration with the following commands:

Mathematica:

```
<< Graphics`FilledPlot`;
p1 = FilledPlot[{0,x^2}, {x,1,2}, PlotRange → {{0,3}, {0,9}}, DisplayFunction → Identity];
p2 = Plot[x^2, {x,0,3}, DisplayFunction → Identity];
Show[{p1,p2}, AxesLabel → {x,y}, Ticks → {Automatic, {1,2,3,4,5,6,7,8,9}},
  DisplayFunction → $DisplayFunction];
```

Maple:

```
>plots[display]([plot([x^2],x = 1..2,
  color=[blue], filled=true, labels=[x,y]),
  plot([x^2], x=0..3)], view=[0..3,0..9]);
```

The following graph was generated using Mathematica.

Evaluate the integrals:

$$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx = \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y}\, dx\, dy = -1 + \ln\left(\frac{27}{4}\right) \approx 0.909543$$

76. Plot the region of integration with the following commands:

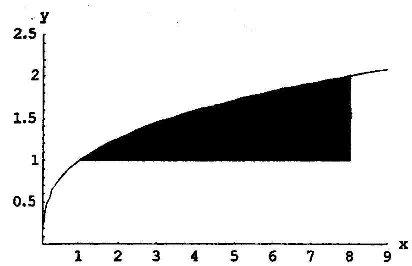
Mathematica:

```
<< Graphics`FilledPlot`;
p2 = Plot[{x^(1/3)}, {x,0,9}, PlotRange → {(0,9), (0,2.5)}, DisplayFunction → Identity];
p3 = FilledPlot[{1,x^(1/3)}, {x,1,8}, DisplayFunction → Identity];
Show[{p2,p3}, AxesLabel → {x,y}, DisplayFunction → $DisplayFunction];
```

Maple:

```
>plots[display]([plot([1,x^(1/3)], x=1..8,
  color=[white, blue], filled=true, labels=[x,y]),
  plot([x^(1/3)], x=0..9)], view=[0..9,0..2.5]);
```

The following graph was generated using Mathematica.



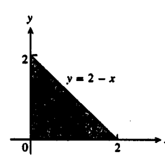
Evaluate the integrals:

$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx \approx 0.866649$$

12.2 AREAS, MOMENTS, AND CENTERS OF MASS

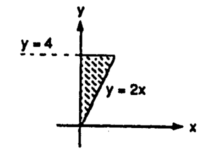
1. $\int_0^2 \int_0^{2-x} dy\, dx = \int_0^2 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$

or $\int_0^2 \int_0^{2-y} dx\, dy = \int_0^2 (2-y)\, dy = 2$



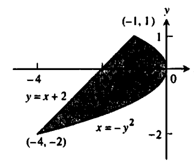
2. $\int_0^2 \int_{2x}^4 dy\,dx = \int_0^2 (4-2x)\,dx = \left[4x - x^2\right]_0^2 = 4,$

or $\int_0^4 \int_0^{y/2} dx\,dy = \int_0^4 \frac{y}{2}\,dy = 4$

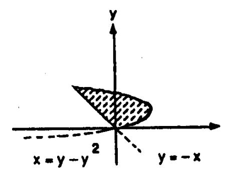


3. $\int_{-2}^{1} \int_{y-2}^{-y^2} dx\,dy = \int_{-2}^{1} \left(-y^2 - y + 2\right) dy = \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1}$

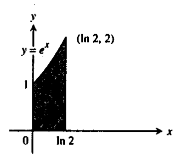
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



4. $\int_0^2 \int_{-y}^{y-y^2} dx\,dy = \int_0^2 \left(2y - y^2\right) dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

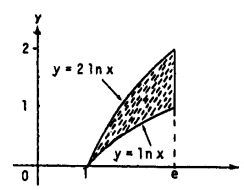


5. $\int_0^{\ln 2} \int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = \left[e^x\right]_0^{\ln 2} = 2 - 1 = 1$

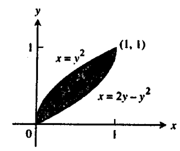


6. $\int_1^e \int_{\ln x}^{2 \ln x} dy\,dx = \int_1^e \ln x\,dx = [x \ln x - x]_1^e$

$= (e - e) - (0 - 1) = 1$

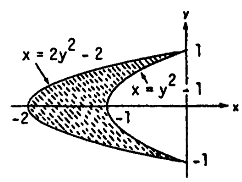


7. $\displaystyle\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 \left(2y - 2y^2\right) dy = \left[y^2 - \tfrac{2}{3}y^3\right]_0^1 = \tfrac{1}{3}$



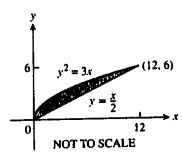
8. $\displaystyle\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^1 \left(y^2 - 1 - 2y^2 + 2\right) dy$

$\displaystyle = \int_{-1}^1 \left(1 - y^2\right) dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



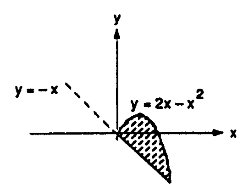
9. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\,dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6$

$\displaystyle = 36 - \frac{216}{9} = 12$



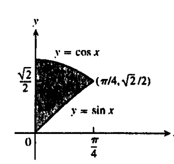
10. $\displaystyle\int_0^3 \int_{-x}^{2x-x^2} dy\,dx = \int_0^3 \left(3x - x^2\right) dx = \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\right]_0^3$

$\displaystyle = \frac{27}{2} - 9 = \frac{9}{2}$



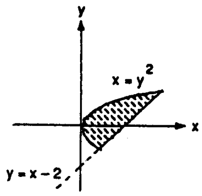
11. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx = \int_0^{\pi/4} (\cos x - \sin x)\,dx = \left[\sin x + \cos x\right]_0^{\pi/4}$

$\displaystyle = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



12. $\int_{-1}^{2}\int_{y^2}^{y+2} dx\,dy = \int_{-1}^{2} \left(y+2-y^2\right) dy = \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}$

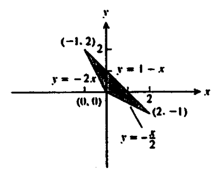
$= \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right) = 5-\frac{1}{2} = \frac{9}{2}$



13. $\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$

$= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{2} \left(1-\frac{x}{2}\right) dx$

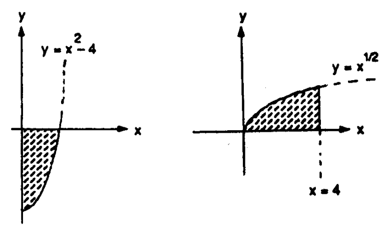
$= \left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{4}\right]_{0}^{2} = -\left(-1+\frac{1}{2}\right)+(2-1) = \frac{3}{2}$



14. $\int_{0}^{2}\int_{x^2-4}^{0} dy\,dx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dy\,dx$

$= \int_{0}^{2} \left(4-x^2\right) dx + \int_{0}^{4} x^{1/2}\,dx$

$= \left[4x-\frac{x^3}{3}\right]_{0}^{2} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{4} = \left(8-\frac{8}{3}\right)+\frac{16}{3} = \frac{32}{3}$



15. (a) average $= \frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi} \sin(x+y)\,dy\,dx = \frac{1}{\pi^2}\int_{0}^{\pi} \left[-\cos(x+y)\right]_{0}^{\pi} dx = \frac{1}{\pi^2}\int_{0}^{\pi} \left[-\cos(x+\pi)+\cos x\right] dx$

$= \frac{1}{\pi^2}\left[-\sin(x+\pi)+\sin x\right]_{0}^{\pi} = \frac{1}{\pi^2}\left[(-\sin 0+\sin \pi)-(-\sin \pi+\sin 0)\right] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)}\int_{0}^{\pi}\int_{0}^{\pi/2} \sin(x+y)\,dy\,dx = \frac{2}{\pi^2}\int_{0}^{\pi} \left[-\cos(x+y)\right]_{0}^{\pi/2} dx = \frac{2}{\pi^2}\int_{0}^{\pi} \left[-\cos\left(x+\frac{\pi}{2}\right)+\cos x\right] dx$

$= \frac{2}{\pi^2}\left[-\sin\left(x+\frac{\pi}{2}\right)+\sin x\right]_{0}^{\pi} = \frac{2}{\pi^2}\left[\left(-\sin\frac{3\pi}{2}+\sin \pi\right)-\left(-\sin\frac{\pi}{2}+\sin 0\right)\right] = \frac{4}{\pi^2}$

16. average value over the square $= \int_0^1 \int_0^1 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4} = 0.25$;

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx$

$= \frac{2}{\pi} \int_0^1 (x - x^3)\,dx = \frac{2}{\pi}\left[\frac{x^2}{2} - \frac{x^2}{4}\right]_0^1 = \frac{1}{2\pi} \approx 0.159$

17. average height $= \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2)\,dy\,dx = \frac{1}{4} \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{4x}{3}\right]_0^2 = \frac{8}{3}$

18. average $= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy}\,dy\,dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2} dx$

$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{x}(\ln 2 + \ln \ln 2 - \ln \ln 2)\,dx = \left(\frac{1}{\ln 2}\right) \int_{\ln 2}^{2\ln 2} \frac{dx}{x} = \left(\frac{1}{\ln 2}\right) [\ln x]_{\ln 2}^{2\ln 2}$

$= \left(\frac{1}{\ln 2}\right)(\ln 2 + \ln \ln 2 - \ln \ln 2) = 1$

19. $M = \int_0^1 \int_x^{2-x^2} 3\,dy\,dx = 3\int_0^1 (2 - x^2 - x)\,dx = \frac{7}{2}$; $M_y = \int_0^1 \int_x^{2-x^2} 3x\,dy\,dx = 3 \int_0^1 [xy]_x^{2-x^2}\,dx$

$= 3 \int_0^1 (2x - x^3 - x^2)\,dx = \frac{5}{4}$; $M_x = \int_0^1 \int_x^{2-x^2} 3y\,dy\,dx = \frac{3}{2} \int_0^1 [y^2]_x^{2-x^2}\,dx = \frac{3}{2} \int_0^1 (4 - 5x^2 + x^4)\,dx = \frac{19}{5}$

$\Rightarrow \bar{x} = \frac{5}{14}$ and $\bar{y} = \frac{38}{35}$

20. $M = \delta \int_0^3 \int_0^3 dy\,dx = \delta \int_0^3 3\,dx = 9\delta$; $I_x = \delta \int_0^3 \int_0^3 y^2\,dy\,dx = \delta \int_0^3 \left[\frac{y^3}{3}\right]_0^3 dx = 27\delta$; $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{3}$;

$I_y = \delta \int_0^3 \int_0^3 x^2\,dy\,dx = \delta \int_0^3 [x^2 y]_0^3\,dx = \delta \int_0^3 3x^2\,dx = 27\delta$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{3}$

21. $M = \int_0^2 \int_{y^2/2}^{4-y} dx\,dy = \int_0^2 \left(4 - y - \frac{y^2}{2}\right) dy = \frac{14}{3}$; $M_y = \int_0^2 \int_{y^2/2}^{4-y} x\,dx\,dy = \frac{1}{2} \int_0^2 [x^2]_{y^2/2}^{4-y}\,dy$

$= \frac{1}{2} \int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4}\right) dy = \frac{128}{15}$; $M_x = \int_0^2 \int_{y^2/2}^{4-y} y\,dx\,dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2}\right) dy = \frac{10}{3}$

$\Rightarrow \bar{x} = \frac{64}{35}$ and $\bar{y} = \frac{5}{7}$

22. $M = \int_0^3 \int_0^{3-x} dy\,dx = \int_0^3 (3-x)\,dx = \frac{9}{2}$; $M_y = \int_0^3 \int_0^{3-x} x\,dy\,dx = \int_0^3 [xy]_0^{3-x}\,dx = \int_0^3 (3x - x^2)\,dx = \frac{9}{2}$
$\Rightarrow \bar{x} = 1$ and $\bar{y} = 1$, by symmetry

23. $M = 2\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$; $M_x = 2\int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 [y^2]_0^{\sqrt{1-x^2}}\,dx$
$= \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} \Rightarrow \bar{y} = \frac{4}{3\pi}$ and $\bar{x} = 0$, by symmetry

24. $M = \frac{125\delta}{6}$; $M_y = \delta \int_0^5 \int_x^{6x-x^2} x\,dy\,dx = \delta \int_0^5 [xy]_x^{6x-x^2}\,dx = \delta \int_0^5 (5x^2 - x^3)\,dx = \frac{625\delta}{12}$;

$M_x = \delta \int_0^5 \int_x^{6x-x^2} y\,dy\,dx = \frac{\delta}{2}\int_0^5 [y^2]_x^{6x-x^2}\,dx = \frac{\delta}{2}\int_0^5 (35x^2 - 12x^3 + x^4)\,dx = \frac{625\delta}{6} \Rightarrow \bar{x} = \frac{5}{2}$ and $\bar{y} = 5$

25. $M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \frac{\pi a^2}{4}$; $M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^a [xy]_0^{\sqrt{a^2-x^2}}\,dx = \int_0^a x\sqrt{a^2-x^2}\,dx = \frac{a^3}{3}$
$\Rightarrow \bar{x} = \bar{y} = \frac{4a}{3\pi}$, by symmetry

26. $M = \int_0^\pi \int_0^{\sin x} dy\,dx = \int_0^\pi \sin x\,dx = 2$; $M_x = \int_0^\pi \int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^\pi [y^2]_0^{\sin x}\,dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx$
$= \frac{1}{4}\int_0^\pi (1 - \cos 2x)\,dx = \frac{\pi}{4} \Rightarrow \bar{x} = \frac{\pi}{2}$ and $\bar{y} = \frac{\pi}{8}$

27. $I_x = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx = \int_{-2}^2 \left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx = \frac{2}{3}\int_{-2}^2 (4-x^2)^{3/2}\,dx = 4\pi$; $I_y = 4\pi$, by symmetry;
$I_o = I_x + I_y = 8\pi$

28. $I_y = \int_\pi^{2\pi} \int_0^{(\sin^2 x)/x^2} x^2\,dy\,dx = \int_0^{2\pi} (\sin^2 x - 0)\,dx = \frac{1}{2}\int_\pi^{2\pi} (1 - \cos 2x)\,dx = \frac{\pi}{2}$

29. $M = \int_{-\infty}^{0}\int_{0}^{e^x} dy\,dx = \int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty}\int_{b}^{0} e^x\,dx = 1 - \lim_{b\to-\infty} e^b = 1$; $M_y = \int_{-\infty}^{0}\int_{0}^{e^x} x\,dy\,dx = \int_{-\infty}^{0} xe^x\,dx$

$= \lim_{b\to-\infty}\int_{b}^{0} xe^x\,dx = \lim_{b\to-\infty}\left[xe^x - e^x\right]_b^0 = -1 - \lim_{b\to-\infty}\left(be^b - e^b\right) = -1$; $M_x = \int_{-\infty}^{0}\int_{0}^{e^x} y\,dy\,dx$

$= \frac{1}{2}\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}\lim_{b\to-\infty}\int_{b}^{0} e^{2x}\,dx = \frac{1}{4} \Rightarrow \overline{x} = -1$ and $\overline{y} = \frac{1}{4}$

30. $M_y = \int_{0}^{\infty}\int_{0}^{e^{-x^2/2}} x\,dy\,dx = \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2/2}\,dx = -\lim_{b\to\infty}\left[\frac{1}{e^{x^2/2}} - 1\right]_0^b = 1$

31. $M = \int_{0}^{2}\int_{-y}^{y-y^2} (x+y)\,dx\,dy = \int_{0}^{2}\left[\frac{x^2}{2} + xy\right]_{-y}^{y-y^2} dy = \int_{0}^{2}\left(\frac{y^4}{2} - 2y^3 + 2y^2\right) dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3}\right]_0^2 = \frac{8}{15}$;

$I_x = \int_{0}^{2}\int_{-y}^{y-y^2} y^2(x+y)\,dx\,dy = \int_{0}^{2}\left[\frac{x^2y^2}{2} + xy^3\right]_{-y}^{y-y^2} dy = \int_{0}^{2}\left(\frac{y^6}{2} - 2y^5 + 2y^4\right) dy = \frac{64}{105}$;

$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{8}{7}} = 2\sqrt{\frac{2}{7}}$

32. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy = 5\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(12 - 4y^2 - 16y^4\right) dy = 23\sqrt{3}$

33. $M = \int_{0}^{1}\int_{x}^{2-x} (6x+3y+3)\,dy\,dx = \int_{0}^{1}\left[6xy + \frac{3}{2}y^2 + 3y\right]_x^{2-x} dx = \int_{0}^{1}\left(12 - 12x^2\right) dx = 8$;

$M_y = \int_{0}^{1}\int_{x}^{2-x} x(6x+3y+3)\,dy\,dx = \int_{0}^{1}\left(12x - 12x^3\right) dx = 3$; $M_x = \int_{0}^{1}\int_{x}^{2-x} y(6x+3y+3)\,dy\,dx$

$= \int_{0}^{1}\left(14 - 6x - 6x^2 - 2x^3\right) dx = \frac{17}{2} \Rightarrow \overline{x} = \frac{3}{8}$ and $\overline{y} = \frac{17}{16}$

34. $M = \int_{0}^{1}\int_{y^2}^{2y-y^2} (y+1)\,dx\,dy = \int_{0}^{1}\left(2y - 2y^3\right) dy = \frac{1}{2}$; $M_x = \int_{0}^{1}\int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy = \int_{0}^{1}\left(2y^2 - 2y^4\right) dy = \frac{4}{15}$;

$M_y = \int_{0}^{1}\int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_{0}^{1}\left(2y^2 - 2y^4\right) dy = \frac{4}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{8}{15}$; $I_x = \int_{0}^{1}\int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy$

$= 2\int_0^1 (y^3 - y^5)\, dy = \frac{1}{6}$

35. $M = \int_0^1 \int_0^6 (x+y+1)\, dx\, dy = \int_0^1 (6y+24)\, dy = 27$; $M_x = \int_0^1 \int_0^6 y(x+y+1)\, dx\, dy = \int_0^1 y(6y+24)\, dy = 14$;

$M_y = \int_0^1 \int_0^6 x(x+y+1)\, dx\, dy = \int_0^1 (18y+90)\, dy = 99 \Rightarrow \overline{x} = \frac{11}{3}$ and $\overline{y} = \frac{14}{27}$; $I_y = \int_0^1 \int_0^6 x^2(x+y+1)\, dx\, dy$

$= 216\int_0^1 \left(\frac{y}{3} + \frac{11}{6}\right) dy = 432$; $R_y = \sqrt{\frac{I_y}{M}} = 4$

36. $M = \int_{-1}^{1} \int_{x^2}^{1} (y+1)\, dy\, dx = -\int_{-1}^{1} \left(\frac{x^4}{2} + x^2 - \frac{3}{2}\right) dx = \frac{32}{15}$; $M_x = \int_{-1}^{1} \int_{x^2}^{1} y(y+1)\, dy\, dx = \int_{-1}^{1} \left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2}\right) dx$

$= \frac{48}{35}$; $M_y = \int_{-1}^{1} \int_{x^2}^{1} x(y+1)\, dy\, dx = \int_{-1}^{1} \left(\frac{3x}{2} - \frac{x^5}{2} - x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{9}{14}$; $I_y = \int_{-1}^{1} \int_{x^2}^{1} x^2(y+1)\, dy\, dx$

$= \int_{-1}^{1} \left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4\right) dx = \frac{16}{35}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{3}{14}}$

37. $M = \int_{-1}^{1} \int_{0}^{x^2} (7y+1)\, dy\, dx = \int_{-1}^{1} \left(\frac{7x^4}{2} + x^2\right) dx = \frac{31}{15}$; $M_x = \int_{-1}^{1} \int_{0}^{x^2} y(7y+1)\, dy\, dx = \int_{-1}^{1} \left(\frac{7x^6}{3} + \frac{x^4}{2}\right) dx = \frac{13}{15}$;

$M_y = \int_{-1}^{1} \int_{0}^{x^2} x(7y+1)\, dy\, dx = \int_{-1}^{1} \left(\frac{7x^5}{2} + x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{13}{31}$; $I_y = \int_{-1}^{1} \int_{0}^{x^2} x^2(7y+1)\, dy\, dx$

$= \int_{-1}^{1} \left(\frac{7x^6}{2} + x^4\right) dx = \frac{7}{5}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{21}{31}}$

38. $M = \int_0^{20} \int_{-1}^{1} \left(1 + \frac{x}{20}\right) dy\, dx = \int_0^{20} \left(2 + \frac{x}{10}\right) dx = 60$; $M_x = \int_0^{20} \int_{-1}^{1} y\left(1 + \frac{x}{20}\right) dy\, dx = \int_0^{20} \left[\left(1 + \frac{x}{20}\right)\left(\frac{y^2}{2}\right)\right]_{-1}^{1} dx = 0$;

$M_y = \int_0^{20} \int_{-1}^{1} x\left(1 + \frac{x}{20}\right) dy\, dx = \int_0^{20} \left(2x + \frac{x^2}{10}\right) dx = \frac{2000}{3} \Rightarrow \overline{x} = \frac{100}{9}$ and $\overline{y} = 0$; $I_x = \int_0^{20} \int_{-1}^{1} y^2\left(1 + \frac{x}{20}\right) dy\, dx$

$= \frac{2}{3}\int_0^{20} \left(1 + \frac{x}{20}\right) dx = 20$; $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{1}{3}}$

39. $M = \int_0^1 \int_{-y}^{y} (y+1)\,dx\,dy = \int_0^1 (2y^2+2y)\,dy = \frac{5}{3}$; $M_x = \int_0^1 \int_{-y}^{y} y(y+1)\,dx\,dy = 2\int_0^1 (y^3+y^2)\,dy = \frac{7}{6}$;

$M_y = \int_0^1 \int_{-y}^{y} x(y+1)\,dx\,dy = \int_0^1 0\,dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{7}{10}$; $I_x = \int_0^1 \int_{-y}^{y} y^2(y+1)\,dx\,dy = \int_0^1 (2y^4+2y^3)\,dy$

$= \frac{9}{10} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{3\sqrt{6}}{10}$; $I_y = \int_0^1 \int_{-y}^{y} x^2(y+1)\,dx\,dy = \frac{1}{3}\int_0^1 (2y^4+2y^3)\,dy = \frac{3}{10} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \frac{3\sqrt{2}}{10}$;

$I_o = I_x + I_y = \frac{6}{5} \Rightarrow R_0 = \sqrt{\frac{I_o}{M}} = \frac{3\sqrt{2}}{5}$

40. $M = \int_0^1 \int_{-y}^{y} (3x^2+1)\,dx\,dy = \int_0^1 (2y^3+2y)\,dy = \frac{3}{2}$; $M_x = \int_0^1 \int_{-y}^{y} y(3x^2+1)\,dx\,dy = \int_0^1 (2y^4+2y^2)\,dy = \frac{16}{15}$;

$M_y = \int_0^1 \int_{-y}^{y} x(3x^2+1)\,dx\,dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{32}{45}$; $I_x = \int_0^1 \int_{-y}^{y} y^2(3x^2+1)\,dx\,dy = \int_0^1 (2y^5+2y^3)\,dy = \frac{5}{6}$

$\Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{\sqrt{5}}{3}$; $I_y = \int_0^1 \int_{-y}^{y} x^2(3x^2+1)\,dx\,dy = 2\int_0^1 \left(\frac{3}{5}y^5 + \frac{1}{3}y^3\right)dy = \frac{11}{30} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{11}{45}}$;

$I_o = I_x + I_y = \frac{6}{5} \Rightarrow R_o = \sqrt{\frac{I_o}{M}} = \frac{2}{\sqrt{5}}$

41. $\int_{-5}^{5} \int_{-2}^{0} \frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000(1-e^{-2}) \int_{-5}^{5} \frac{dx}{1+\frac{|x|}{2}} = 10{,}000(1-e^{-2})\left[\int_{-5}^{0} \frac{dx}{1-\frac{x}{2}} + \int_0^5 \frac{dx}{1+\frac{x}{2}}\right]$

$= 10{,}000(1-e^{-2})\left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + 10{,}000(1-e^{-2})\left[2\ln\left(1+\frac{x}{2}\right)\right]_0^5$

$= 10{,}000(1-e^{-2})\left[2\ln\left(1+\frac{5}{2}\right)\right] + 10{,}000(1-e^{-2})\left[2\ln\left(1+\frac{5}{2}\right)\right] = 40{,}000(1-e^{-2})\ln\left(\frac{7}{2}\right) \approx 43{,}329$

42. $\int_0^1 \int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_0^1 \left[100(y+1)x\right]_{y^2}^{2y-y^2} dy = \int_0^1 100(y+1)(2y-2y^2)\,dy = 200\int_0^1 (y-y^3)\,dy$

$= 200\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = (200)\left(\frac{1}{4}\right) = 50$

43. $M = \int_{-1}^{1} \int_0^{a(1-x^2)} dy\,dx = 2a\int_0^1 (1-x^2)\,dx = 2a\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4a}{3}$; $M_x = \int_{-1}^{1} \int_0^{a(1-x^2)} y\,dy\,dx$

49. (a) $\bar{x} = \frac{M_y}{M} = 0 \Rightarrow M_y = \iint\limits_R x\delta(x,y)\,dy\,dx = 0$

(b) $I_L = \iint\limits_R (x-h)^2\delta(x,y)\,dA = \iint\limits_R x^2\delta(x,y)\,dA - \iint\limits_R 2hx\,\delta(x,y)\,dA + \iint\limits_R h^2\delta(x,y)\,dA$

$= I_y - 0 + h^2 \iint\limits_R \delta(x,y)\,dA = I_{c.m.} + mh^2$

50. (a) $I_{c.m.} = I_L - mh^2 \Rightarrow I_{x=5/7} = I_y - mh^2 = \frac{39}{5} - 14\left(\frac{5}{7}\right)^2 = \frac{23}{35}$; $I_{y=11/14} = I_x - mh^2 = 12 - 14\left(\frac{11}{14}\right)^2 = \frac{47}{14}$

(b) $I_{x=1} = I_{x=5/7} + mh^2 = \frac{23}{35} + 14\left(\frac{2}{7}\right)^2 = \frac{9}{5}$; $I_{y=2} = I_{y=11/14} + mh^2 = \frac{47}{14} + 14\left(\frac{17}{14}\right)^2 = 24$

51. $M_{x_{P_1 \cup P_2}} = \iint\limits_{R_1} y\,dA_1 + \iint\limits_{R_2} y\,dA_2 = M_{x_1} + M_{x_2} \Rightarrow \bar{x} = \frac{M_{x_1}+M_{x_2}}{m_1+m_2}$; likewise, $\bar{y} = \frac{M_{y_1}+M_{y_2}}{m_1+m_2}$;

thus $\mathbf{c} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} = \frac{1}{m_1+m_2}\left[\left(M_{x_1}+M_{x_2}\right)\mathbf{i} + \left(M_{y_1}+M_{y_2}\right)\mathbf{j}\right] = \frac{1}{m_1+m_2}\left[(m_1\bar{x}_1 + m_2\bar{x}_2)\mathbf{i} + (m_1\bar{y}_1 + m_2\bar{y}_2)\mathbf{j}\right]$

$= \frac{1}{m_1+m_2}\left[m_1(\bar{x}_1\mathbf{i} + \bar{y}_1\mathbf{j}) + m_2(\bar{x}_2\mathbf{i} + \bar{y}_2\mathbf{j})\right] = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1+m_2}$

52. From Exercise 51 we have that Pappus's formula is true for $n = 2$. Assume that Pappus's formula is true for $n = k-1$, i.e., that $\mathbf{c}(k-1) = \frac{\sum_{i=1}^{k-1} m_i\mathbf{c}_i}{\sum_{i=1}^{k-1} m_i}$. The first moment about x of k nonoverlapping plates is

$\sum_{i=1}^{k-1}\left(\iint\limits_{R_i} y\,dA_i\right) + \iint\limits_{R_k} y\,dA_k = M_{x_{c(k-1)}} + M_{x_k} \Rightarrow \bar{x} = \frac{M_{x_{c(k-1)}} + M_{x_k}}{\left(\sum_{i=1}^{k-1} m_i\right) + m_k}$; similarly, $\bar{y} = \frac{M_{y_{c(k-1)}} + M_{y_k}}{\left(\sum_{i=1}^{k-1} m_i\right) + m_k}$;

thus $\mathbf{c}(k) = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} = \frac{1}{\sum_{i=1}^{k} m_i}\left[\left(M_{x_{c(k-1)}} + M_{x_k}\right)\mathbf{i} + \left(M_{y_{c(k-1)}} + M_{y_k}\right)\mathbf{j}\right]$

$= \frac{1}{\sum_{i=1}^{k} m_i}\left[\left(\left(\sum_{i=1}^{k-1} m_i\right)\bar{x}_c + m_k\bar{x}_k\right)\mathbf{i} + \left(\left(\sum_{i=1}^{k-1} m_i\right)\bar{y}_c + m_k\bar{y}_k\right)\mathbf{j}\right]$

$= \frac{1}{\sum_{i=1}^{k} m_i}\left[\left(\sum_{i=1}^{k-1} m_i\right)(\bar{x}_c\mathbf{i} + \bar{y}_c\mathbf{j}) + m_k(\bar{x}_k\mathbf{i} + \bar{y}_k\mathbf{j})\right] = \frac{\left(\sum_{i=1}^{k-1} m_i\right)\mathbf{c}(k-1) + m_k\mathbf{c}_k}{\sum_{i=1}^{k} m_i}$

$= \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + \ldots + m_{k-1}\mathbf{c}_{k-1} + m_k\mathbf{c}_k}{m_1 + m_2 + \ldots + m_{k-1} + m_k}$, and by mathematical induction the statement follows.

53. (a) $\mathbf{c} = \frac{8(\mathbf{i}+3\mathbf{j}) + 2(3\mathbf{i}+3.5\mathbf{j})}{8+2} = \frac{14\mathbf{j}+31\mathbf{k}}{10} \Rightarrow \bar{x} = \frac{7}{5}$ and $\bar{y} = \frac{31}{10}$

(b) $\mathbf{c} = \frac{8(\mathbf{i}+3\mathbf{j}) + 6(5\mathbf{i}+2\mathbf{j})}{14} = \frac{38\mathbf{i}+36\mathbf{j}}{14} \Rightarrow \bar{x} = \frac{19}{7}$ and $\bar{y} = \frac{18}{7}$

(c) $\mathbf{c} = \frac{2(3\mathbf{i} + 3.5\mathbf{j}) + 6(5\mathbf{i} + 2\mathbf{j})}{8} = \frac{36\mathbf{i} + 19\mathbf{j}}{8} \Rightarrow \bar{x} = \frac{9}{2}$ and $\bar{y} = \frac{19}{8}$

(d) $\mathbf{c} = \frac{8(\mathbf{i} + 3\mathbf{j}) + 2(3\mathbf{i} + 3.5\mathbf{j}) + 6(5\mathbf{i} + 2\mathbf{j})}{16} = \frac{44\mathbf{i} + 43\mathbf{j}}{16} \Rightarrow \bar{x} = \frac{11}{4}$ and $\bar{y} = \frac{43}{16}$

54. $\mathbf{c} = \frac{15\left(\frac{3}{4}\mathbf{i} + 7\mathbf{j}\right) + 48(12\mathbf{i} + \mathbf{j})}{15 + 48} = \frac{15(3\mathbf{i} + 28\mathbf{j}) + 48(48\mathbf{i} + 4\mathbf{j})}{4 \cdot 63} = \frac{2349\,\mathbf{i} + 612\,\mathbf{j}}{4 \cdot 63} = \frac{261\,\mathbf{i} + 68\mathbf{j}}{4 \cdot 7}$

$\Rightarrow \bar{x} = \frac{261}{28}$ and $\bar{y} = \frac{17}{7}$

55. Place the midpoint of the triangle's base at the origin and above the semicircle. Then the center of mass of the triangle is $\left(0, \frac{h}{3}\right)$, and the center of mass of the disk is $\left(0, -\frac{4a}{3\pi}\right)$ from Exercise 25. From Pappus's formula, $\mathbf{c} = \frac{(ah)\left(\frac{h}{3}\mathbf{j}\right) + \left(\frac{\pi a^2}{2}\right)\left(-\frac{4a}{3\pi}\mathbf{j}\right)}{\left(ah + \frac{\pi a^2}{2}\right)} = \frac{\left(\frac{ah^2 - 2a^3}{3}\right)\mathbf{j}}{\left(ah + \frac{\pi a^2}{2}\right)}$, so the centroid is on the boundary if $ah^2 - 2a^3 = 0 \Rightarrow h^2 = 2a^2 \Rightarrow h = a\sqrt{2}$. In order for the center of mass to be inside T we must have $ah^2 - 2a^3 > 0$ or $h > a\sqrt{2}$.

56. Place the midpoint of the triangle's base at the origin and above the square. From Pappus's formula, $\mathbf{c} = \frac{\left(\frac{sh}{2}\right)\left(\frac{h}{3}\mathbf{j}\right) + s^2\left(-\frac{s}{2}\mathbf{j}\right)}{\left(\frac{sh}{2} + s^2\right)}$, so the centroid is on the boundary if $\frac{sh^2}{6} - \frac{s^3}{2} = 0 \Rightarrow h^2 - 3s^2 = 0 \Rightarrow h = s\sqrt{3}$.

12.3 DOUBLE INTEGRALS IN POLAR FORM

1. $\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} dy\,dx = \int_{0}^{\pi} \int_{0}^{1} r\,dr\,d\theta = \frac{1}{2} \int_{0}^{\pi} d\theta = \frac{\pi}{2}$

2. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_{0}^{2\pi} \int_{0}^{1} r\,dr\,d\theta = \frac{1}{2} \int_{0}^{2\pi} d\theta = \pi$

3. $\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy = \int_{0}^{\pi/2} \int_{0}^{1} r^3\,dr\,d\theta = \frac{1}{4} \int_{0}^{\pi/2} d\theta = \frac{\pi}{8}$

4. $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy = \int_{0}^{2\pi} \int_{0}^{1} r^3\,dr\,d\theta = \frac{1}{4} \int_{0}^{2\pi} d\theta = \frac{\pi}{2}$

5. $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_0^{2\pi} \int_0^a r\,dr\,d\theta = \frac{a^2}{2} \int_0^{2\pi} d\theta = \pi a^2$

6. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2} \int_0^2 r^3\,dr\,d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$

7. $\int_0^6 \int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2} \int_0^{6\csc\theta} r^2 \cos\theta\,dr\,d\theta = 72 \int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\,d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

8. $\int_0^2 \int_0^x y\,dy\,dx = \int_0^{\pi/4} \int_0^{2\sec\theta} r^2 \sin\theta\,dr\,d\theta = \frac{8}{3} \int_0^{\pi/4} \tan\theta \sec^2\theta\,d\theta = \frac{4}{3}$

9. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx = \int_\pi^{3\pi/2} \int_0^1 \frac{2r}{1+r}\,dr\,d\theta = 2 \int_\pi^{3\pi/2} \int_0^1 \left(1-\frac{1}{1+r}\right) dr\,d\theta = 2 \int_\pi^{3\pi/2} (1-\ln 2)\,d\theta$
$= (1-\ln 2)\pi$

10. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^0 \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dx\,dy = \int_{\pi/2}^{3\pi/2} \int_0^1 \frac{4r^2}{1+r^2}\,dr\,d\theta = 4 \int_{\pi/2}^{3\pi/2} \int_0^1 \left(1-\frac{1}{1+r^2}\right) dr\,d\theta = 4 \int_{\pi/2}^{3\pi/2} \left(1-\frac{\pi}{4}\right) d\theta$
$= 4\pi - \pi^2$

11. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2} \int_0^{\ln 2} re^r\,dr\,d\theta = \int_0^{\pi/2} (2\ln 2 - 1)\,d\theta = \frac{\pi}{2}(2\ln 2 - 1)$

12. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2} \int_0^1 re^{-r^2}\,dr\,d\theta = -\frac{1}{2} \int_0^{\pi/2} \left(\frac{1}{e}-1\right) d\theta = \frac{\pi(e-1)}{4e}$

13. $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r(\cos\theta+\sin\theta)}{r^2}\,r\,dr\,d\theta = \int_0^{\pi/2} \left(2\cos^2\theta + 2\sin\theta\cos\theta\right) d\theta$
$= \left[\theta + \frac{\sin 2\theta}{2} + \sin^2\theta\right]_0^{\pi/2} = \frac{\pi+2}{2} = \frac{\pi}{2} + 1$

14. $\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2\,dx\,dy = \int_{\pi/2}^{\pi} \int_0^{2\sin\theta} \sin^2\theta \cos\theta\, r^4\,dr\,d\theta = \frac{32}{5} \int_{\pi/2}^{\pi} \sin^7\theta \cos\theta\,d\theta = \frac{4}{5}\left[\sin^8\theta\right]_{\pi/2}^{\pi} = -\frac{4}{5}$

15. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right) dx\,dy = 4\int_{0}^{\pi/2}\int_{0}^{1} \ln\left(r^2+1\right) r\ dr\,d\theta = 2\int_{0}^{\pi/2} (\ln 4-1)\ d\theta = \pi(\ln 4-1)$

16. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\, dy\,dx = 4\int_{0}^{\pi/2}\int_{0}^{1} \frac{2r}{\left(1+r^2\right)^2}\, dr\,d\theta = 4\int_{0}^{\pi/2} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_{0}^{\pi/2} d\theta = \pi$

17. $\int_{0}^{\pi/2}\int_{0}^{2\sqrt{2-\sin 2\theta}} r\ dr\,d\theta = 2\int_{0}^{\pi/2} (2-\sin 2\theta)\ d\theta = 2(\pi-1)$

18. $A = 2\int_{0}^{\pi/2}\int_{1}^{1+\cos\theta} r\ dr\,d\theta = \int_{0}^{\pi/2} \left(2\cos\theta+\cos^2\theta\right) d\theta = \frac{8+\pi}{4}$

19. $A = 2\int_{0}^{\pi/6}\int_{0}^{12\cos 3\theta} r\ dr\,d\theta = 144\int_{0}^{\pi/6} \cos^2 3\theta\ d\theta = 12\pi$

20. $A = \int_{0}^{2\pi}\int_{0}^{4\theta/3} r\ dr\,d\theta = \frac{8}{9}\int_{0}^{2\pi} \theta^2\ d\theta = \frac{64\pi^3}{27}$

21. $A = \int_{0}^{\pi/2}\int_{0}^{1+\sin\theta} r\ dr\,d\theta = \frac{1}{2}\int_{0}^{\pi/2} \left(\frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{8}+1$

22. $A = 4\int_{0}^{\pi/2}\int_{0}^{1-\cos\theta} r\ dr\,d\theta = 2\int_{0}^{\pi/2} \left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{2}-4$

23. $M_x = \int_{0}^{\pi}\int_{0}^{1-\cos\theta} 3r^2\sin\theta\ dr\,d\theta = 2\int_{0}^{\pi} (1-\cos\theta)^3\sin\theta\ d\theta = 4$

24. $I_x = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\left[k\left(x^2+y^2\right)\right] dy\,dx = k\int_{0}^{2\pi}\int_{0}^{a} r^5\sin^2\theta\ dr\,d\theta = \frac{ka^6}{6}\int_{0}^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{ka^6\pi}{6};$

$I_o = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} k\left(x^2+y^2\right)^2 dy\,dx = k\int_{0}^{2\pi}\int_{0}^{a} r^5\ dr\,d\theta = \frac{ka^6}{6}\int_{0}^{2\pi} d\theta = \frac{ka^6\pi}{3}$

25. $M = 2 \int_{\pi/6}^{\pi/2} \int_{3}^{6 \sin \theta} dr\, d\theta = 2 \int_{\pi/6}^{\pi/2} (6 \sin \theta - 3)\, d\theta = 6[-2 \cos \theta - \theta]_{\pi/6}^{\pi/2} = 6\sqrt{3} - 2\pi$

26. $I_o = \int_{\pi/2}^{3\pi/2} \int_{1}^{1-\cos \theta} r\, dr\, d\theta = \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left(\cos^2 \theta - 2 \cos \theta\right) d\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2} - 2 \sin \theta\right]_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}$

27. $M = 2 \int_{0}^{\pi} \int_{0}^{1+\cos \theta} r\, dr\, d\theta = \int_{0}^{\pi} (1 + \cos \theta)^2\, d\theta = \frac{3\pi}{2}$; $M_y = 2 \int_{0}^{2\pi} \int_{0}^{1+\cos \theta} r^2 \cos \theta\, dr\, d\theta$

$= \int_{0}^{2\pi} \left(\frac{4 \cos \theta}{3} + \frac{15}{24} + \cos 2\theta - \sin^2 \theta \cos \theta + \frac{\cos 4\theta}{4}\right) d\theta = \frac{5\pi}{4} \Rightarrow \overline{x} = \frac{5}{6}$ and $\overline{y} = 0$, by symmetry

28. $I_o = \int_{0}^{2\pi} \int_{0}^{1+\cos \theta} r^3\, dr\, d\theta = \frac{1}{4} \int_{0}^{2\pi} (1 + \cos \theta)^4\, d\theta = \frac{35\pi}{16}$

29. average $= \frac{4}{\pi a^2} \int_{0}^{\pi/2} \int_{0}^{a} r\sqrt{a^2 - r^2}\, dr\, d\theta = \frac{4}{3\pi a^2} \int_{0}^{\pi/2} a^3\, d\theta = \frac{2a}{3}$

30. average $= \frac{4}{\pi a^2} \int_{0}^{\pi/2} \int_{0}^{a} r^2\, dr\, d\theta = \frac{4}{3\pi a^2} \int_{0}^{\pi/2} a^3\, d\theta = \frac{2a}{3}$

31. average $= \frac{1}{\pi a^2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \frac{1}{\pi a^2} \int_{0}^{2\pi} \int_{0}^{a} r^2\, dr\, d\theta = \frac{a}{3\pi} \int_{0}^{2\pi} d\theta = \frac{2a}{3}$

32. average $= \frac{1}{\pi} \iint_R \left[(1 - x)^2 + y^2\right] dy\, dx = \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} \left[(1 - r \cos \theta)^2 + r^2 \sin^2 \theta\right] r\, dr\, d\theta$

$= \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} \left(r^3 - 2r^2 \cos \theta + r\right) dr\, d\theta = \frac{1}{\pi} \int_{0}^{2\pi} \left(\frac{3}{4} - \frac{2 \cos \theta}{3}\right) d\theta = \frac{1}{\pi}\left[\frac{3}{4}\theta - \frac{2 \sin \theta}{3}\right]_0^{2\pi} = \frac{3}{2}$

33. $\int_{0}^{2\pi} \int_{1}^{\sqrt{e}} \left(\frac{\ln r^2}{r}\right) r\, dr\, d\theta = \int_{0}^{2\pi} \int_{1}^{\sqrt{e}} 2 \ln r\, dr\, d\theta = 2 \int_{0}^{2\pi} [r \ln r - r]_1^{e^{1/2}}\, d\theta = 2 \int_{0}^{2\pi} \left[\sqrt{e}\left(\frac{1}{2} - 1\right) + 1\right] d\theta = 2\pi(2 - \sqrt{e})$

34. $\int_{0}^{2\pi} \int_{1}^{e} \left(\frac{\ln r^2}{r}\right) dr\, d\theta = \int_{0}^{2\pi} \int_{1}^{e} \left(\frac{2 \ln r}{r}\right) dr\, d\theta = \int_{0}^{2\pi} \left[(\ln r)^2\right]_1^e\, d\theta = \int_{0}^{2\pi} d\theta = 2\pi$

35. $V = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r^2\cos\theta\, dr\, d\theta = \frac{2}{3}\int_0^{\pi/2}\left(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right) d\theta$

$= \frac{2}{3}\left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$

36. $V = 4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\, dr\, d\theta = -\frac{4}{3}\int_0^{\pi/4}\left[(2-2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$

$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\int_0^{\pi/4}\left(1-\cos^2\theta\right)\sin\theta\, d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$

37. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\, dx\, dy = \int_0^{\pi/2}\int_0^\infty \left(e^{-r^2}\right) r\, dr\, d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\, dr\right] d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\lim_{b\to\infty}\left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\lim_{x\to\infty}\int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\, dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a)

38. $\int_0^\infty\int_0^\infty \frac{1}{\left(1+x^2+y^2\right)^2}\, dx\, dy = \int_0^{\pi/2}\int_0^\infty \frac{r}{\left(1+r^2\right)^2}\, dr\, d\theta = \frac{\pi}{2}\lim_{b\to\infty}\int_0^b \frac{r}{\left(1+r^2\right)^2}\, dr = \frac{\pi}{4}\lim_{b\to\infty}\left[-\frac{1}{1+r^2}\right]_0^b$

$= \frac{\pi}{4}\lim_{b\to\infty}\left(1 - \frac{1}{1+b^2}\right) = \frac{\pi}{4}$

39. Over the disk $x^2+y^2 \le \frac{3}{4}$: $\iint_R \frac{1}{1-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}/2} \frac{r}{1-r^2}\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\ln\left(1-r^2\right)\right]_0^{\sqrt{3}/2} d\theta$

$= \int_0^{2\pi}\left(-\frac{1}{2}\ln\frac{1}{4}\right) d\theta = (\ln 2)\int_0^{2\pi} d\theta = \pi\ln 4$

Over the disk $x^2+y^2 \le 1$: $\iint_R \frac{1}{1-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^1 \frac{r}{1-r^2}\, dr\, d\theta = \int_0^{2\pi}\left[\lim_{a\to 1^-}\int_0^a \frac{r}{1-r^2}\, dr\right] d\theta$

$= \int_0^{2\pi}\lim_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right] d\theta = 2\pi\cdot\lim_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right] = 2\pi\cdot\infty$, so the integral does not exist over $x^2+y^2 \le 1$

40. The area in polar coordinates is given by $A = \int_\alpha^\beta\int_0^{f(\theta)} r\, dr\, d\theta = \int_\alpha^\beta\left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\, d\theta = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta$, where $r = f(\theta)$

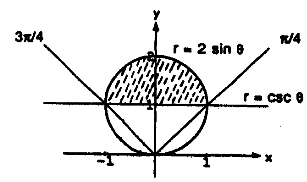
41. $\text{average} = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \left[(r \cos\theta - h)^2 + r^2 \sin^2\theta\right] r\, dr\, d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \left(r^3 - 2r^2 h \cos\theta + rh^2\right) dr\, d\theta$

$= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{a^4}{4} - \frac{2a^3 h \cos\theta}{3} + \frac{a^2 h^2}{2}\right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{a^2}{4} - \frac{2ah\cos\theta}{3} + \frac{h^2}{2}\right) d\theta = \frac{1}{\pi}\left[\frac{a^2\theta}{4} - \frac{2ah\sin\theta}{3} + \frac{h^2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{2}\left(a^2 + 2h^2\right)$

42. $A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\, dr\, d\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} \left(4\sin^2\theta - \csc^2\theta\right) d\theta$

$= \frac{1}{2}\left[2\theta - \sin 2\theta + \cot\theta\right]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$



43-46. Example CAS commands:

Maple:

```
with(plots): y:='y'; x:='x';
bdy1:= y = 0; bdy2:= y = 2 - x; bdy3:= y = x;
implicitplot({bdy1, bdy2, bdy3}, x=0..2,y=0..1, scaling=CONSTRAINED,title=`ORIGINAL PLOT`);
X:= r*cos(theta): Y:= r*sin(theta):
r1:= solve(Y=0,r); theta1:= evalf(solve(Y=0,theta));
r2:=solve(Y=2-X,r);theta2:=solve(Y=2-X,theta);
r3:=solve(Y=X,r); theta3:=solve(Y=X,theta);
trbdy1:= theta=theta1; trbyd2:= r = r2; trbdy3:= theta=theta3;
implicitplot({trbdy1,trbdy2,trbdy3}, theta=0..1, r=0..2, title=`TRANSFORMED PLOT`);
f:= (x,y) -> sqrt(x+y);
subs(x=X, y=Y, f(x,y));
g:= unapply(%,(r,theta));
int(int(g(r,theta), r=0..r2),theta=0..theta3);
evalf(%);
```

Mathematica:

```
Clear[x,y,r,t]
topolar = {x -> r Cos[t], y -> r Sin[t]}
<< Graphics`ImplicitPlot`
f = Sqrt[x+y]
bdy1 = x == y
bdy2 = x == 2-y
ImplicitPlot[{bdy1,bdy2},{x,0,2},{y,0,1}]
bdy3 = y == 0
bdy1 /. topolar
```

Note: Mathematica cannot solve this directly, so we need to help by dividing the equation by the right-hand side:

```
%[[1]]/%[[2]] == 1
Solve[ %, t ]
```

```
t1 = t /. First[%]
bdy2 /. topolar
Solve[ %, r ]
r2 = r /. First[%]
bdy3 /. topolar
Solve[ %, t ]
t2 = t /. First[%]
r1 = 0
ImplicitPlot[{r==r1,r==r2,t==t1,t==t2},{t,0,1},{r,0,2}]
f /. topolar
f = Simplify[%]
Integrate[ f r, {t,t2,t1}, {r,r1,r2} ]
N[%]
```

12.4 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

1. $\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\,dz\,dx = \int_0^1 \int_0^{1-x} (1-x+z)\,dz\,dx$

$= \int_0^1 \left[\frac{(1-x)-x(1-x)-(1-x)^2}{2}\right] dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$

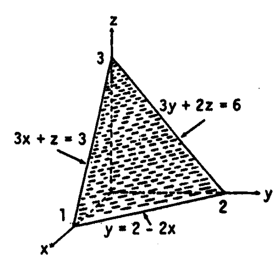
2. $\int_0^1 \int_0^2 \int_0^3 dz\,dy\,dx = \int_0^1 \int_0^2 3\,dy\,dx = \int_0^1 6\,dx = 6,$ $\int_0^2 \int_0^1 \int_0^3 dz\,dx\,dy,$ $\int_0^3 \int_0^2 \int_0^1 dx\,dy\,dz,$ $\int_0^2 \int_0^3 \int_0^1 dx\,dz\,dy,$

$\int_0^3 \int_0^1 \int_0^2 dy\,dx\,dz,$ $\int_0^1 \int_0^3 \int_0^2 dy\,dz\,dx$

3. $\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\,dy\,dx = \int_0^1 \int_0^{2-2x} \left(3-3x-\frac{3}{2}y\right) dy\,dx$

$= \int_0^1 \left[3(1-x)\cdot 2(1-x) - \frac{3}{4}\cdot 4(1-x)^2\right] dx$

$= 3\int_0^1 (1-x)^2\,dx = \left[-(1-x)^3\right]_0^1 = 1,$



$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz\,dx\,dy,$ $\int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy\,dz\,dx,$

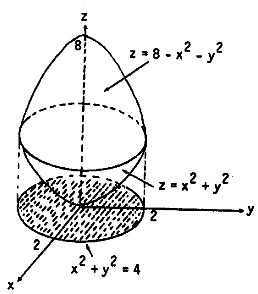
$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy\,dx\,dz,$ $\int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx\,dz\,dy,$ $\int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx\,dy\,dz$

4. $\int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz\,dy\,dx = \int_0^2 \int_0^3 \sqrt{4-x^2}\,dy\,dx = \int_0^2 3\sqrt{4-x^2}\,dx = \frac{3}{2}\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_0^2 = 6\sin^{-1}1 = 3\pi,$

$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz\,dx\,dy, \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy\,dz\,dx, \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy\,dx\,dz, \int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx\,dy\,dz,$

$\int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx\,dz\,dy$

5. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$



$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left[8 - 2\left(x^2+y^2\right)\right] dy\,dx$

$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\,dx = 8\int_0^{\pi/2} \int_0^2 \left(4 - r^2\right) r\,dr\,d\theta$

$= 8\int_0^{\pi/2} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = 32\int_0^{\pi/2} d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi,$

$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy, \int_{-2}^{2} \int_{y^2}^{4} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^{2} \int_{4}^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy,$

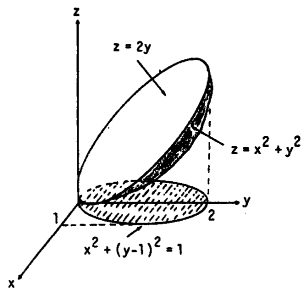
$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz,$

$\int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^{2} \int_{4}^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx,$

$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$

6. The projection of D onto the xy-plane has the boundary $x^2+y^2=2y \Rightarrow x^2+(y-1)^2=1$, which is a circle. Therefore the two integrals are:

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy \text{ and } \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz\,dy\,dx$$



7. $\int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}} \int_0^{3y} (8-2x^2-4y^2)\,dx\,dy = \int_0^{\sqrt{2}} \left[8x-\frac{2}{3}x^3-4xy^2\right]_0^{3y} dy$

$= \int_0^{\sqrt{2}} (24y-18y^3-12y^3)\,dy = \left[12y^2-\frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24-30=-6$

9. $\int_1^e \int_1^e \int_1^e \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e \int_1^e \left[\frac{\ln x}{yz}\right]_1^e dy\,dz = \int_1^e \int_1^e \frac{1}{yz}\,dy\,dz = \int_1^e \left[\frac{\ln y}{z}\right]_1^e dz = \int_1^e \frac{1}{z}\,dz = 1$

10. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\,dy\,dx = \int_0^1 \int_0^{3-3x} (3-3x-y)\,dy\,dx = \int_0^1 \left[(3-3x)^2-\frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1 (1-x)^2\,dx$

$= -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$

11. $\int_0^1 \int_0^{\pi} \int_0^{\pi} y \sin z\,dx\,dy\,dz = \int_0^1 \int_0^{\pi} \pi y \sin z\,dy\,dz = \frac{\pi^3}{2}\int_0^1 \sin z\,dz = \frac{\pi^3}{2}(1-\cos 1)$

12. $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x+y+z)\,dy\,dx\,dz = \int_{-1}^1 \int_{-1}^1 \left[xy+\frac{1}{2}y^2+zy\right]_{-1}^1 dx\,dz = \int_{-1}^1 \int_{-1}^1 (2x+2z)\,dx\,dz = \int_{-1}^1 \left[x^2+2zx\right]_{-1}^1 dz$

$= \int_{-1}^1 4z\,dz = 0$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dy\,dx = \int_0^3 (9-x^2)\,dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 18$

14. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\,dx\,dy = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y)\,dx\,dy = \int_0^2 \left[x^2+xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 \left(4-y^2\right)^{1/2}(2y)\,dy$
$= \left[-\frac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$

15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\,dy\,dx = \int_0^1 \int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^1 \left[(2-x)^2 - \frac{1}{2}(2-x)^2\right] dx = \frac{1}{2}\int_0^1 (2-x)^2\,dx$
$= \left[-\frac{1}{6}(2-x)^3\right]_0^1 = -\frac{1}{6}+\frac{8}{6} = \frac{7}{6}$

16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\,dz\,dy\,dx = \int_0^1 \int_0^{1-x^2} x\left(1-x^2-y\right)dy\,dx = \int_0^1 x\left[\left(1-x^2\right)^2 - \frac{1}{2}\left(1-x^2\right)\right]dx = \int_0^1 \frac{1}{2}x\left(1-x^2\right)^2 dx$
$= \left[-\frac{1}{12}\left(1-x^2\right)^3\right]_0^1 = \frac{1}{12}$

17. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)\,du\,dv\,dw = \int_0^\pi \int_0^\pi [\sin(w+v+\pi) - \sin(w+v)]\,dv\,dw$
$= \int_0^\pi [(-\cos(w+2\pi)+\cos(w+\pi)) + (\cos(w+\pi) - \cos w)]\,dw$
$= [-\sin(w+2\pi) + \sin(w+\pi) - \sin w + \sin(w+\pi)]_0^\pi = 0$

18. $\int_1^e \int_1^e \int_1^e \ln r \ln s \ln t\,dt\,dr\,ds = \int_1^e \int_1^e (\ln r \ln s)[t \ln t - t]_1^e\,dr\,ds = \int_1^e (\ln s)[r \ln r - r]_1^e\,ds = [s \ln s - s]_1^e = 1$

19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x\,dx\,dt\,dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b\to-\infty}\left(e^{2t}-e^b\right)dt\,dv = \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t}\,dt\,dv = \int_0^{\pi/4} \left[\frac{1}{2}e^{2\ln \sec v} - \frac{1}{2}\right]dv$
$= \int_0^{\pi/4} \left[\frac{\sec^2 v}{2} - \frac{1}{2}\right]dv = \left[\frac{\tan v}{2} - \frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$

20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\,dq\,dr = \int_0^7 \frac{1}{3(r+1)}\left[-\left(4-q^2\right)^{3/2}\right]_0^2 dr = \frac{8}{3}\int_0^7 \frac{1}{r+1}\,dr$
$= \frac{8\ln 8}{3} = 8\ln 2$

21. (a) $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$ (b) $\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$ (c) $\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$

(d) $\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$ (e) $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\,dx\,dy$

22. (a) $\int_{0}^{1}\int_{0}^{1}\int_{-1}^{-\sqrt{z}} dy\,dz\,dx$ (b) $\int_{0}^{1}\int_{0}^{1}\int_{-1}^{-\sqrt{z}} dy\,dx\,dz$ (c) $\int_{0}^{1}\int_{-1}^{-\sqrt{z}}\int_{0}^{1} dx\,dy\,dz$

(d) $\int_{-1}^{0}\int_{0}^{y^2}\int_{0}^{1} dx\,dz\,dy$ (e) $\int_{-1}^{0}\int_{0}^{1}\int_{0}^{y^2} dz\,dx\,dy$

23. $V = \int_{0}^{1}\int_{-1}^{1}\int_{0}^{y^2} dz\,dy\,dx = \int_{0}^{1}\int_{-1}^{1} y^2\,dy\,dx = \frac{2}{3}\int_{0}^{1} dx = \frac{2}{3}$

24. $V = \int_{0}^{1}\int_{0}^{1-x}\int_{0}^{2-2z} dy\,dz\,dx = \int_{0}^{1}\int_{0}^{1-x} (2-2z)\,dz\,dx = \int_{0}^{1} \left[2z - z^2\right]_{0}^{1-x} dx = \int_{0}^{1} \left(1-x^2\right) dx = \left[x - \frac{x^3}{3}\right]_{0}^{1} = \frac{2}{3}$

25. $V = \int_{0}^{4}\int_{0}^{\sqrt{4-x}}\int_{0}^{2-y} dz\,dy\,dx = \int_{0}^{4}\int_{0}^{\sqrt{4-x}} (2-y)\,dy\,dx = \int_{0}^{4} \left[2\sqrt{4-x} - \left(\frac{4-x}{2}\right)\right] dx$

$= \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2\right]_{0}^{4} = \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) = \frac{32}{3} - 4 = \frac{20}{3}$

26. $V = 2\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx = -2\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{0} y\,dy\,dx = \int_{0}^{1} \left(1-x^2\right) dx = \frac{2}{3}$

27. $V = \int_{0}^{1}\int_{0}^{2-2x}\int_{0}^{3-3x-3y/2} dz\,dy\,dx = \int_{0}^{1}\int_{0}^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\,dx = \int_{0}^{1} \left[6(1-x)^2 - \frac{3}{4}\cdot 4(1-x)^2\right] dx$

$= \int_{0}^{1} 3(1-x)^2\,dx = \left[-(1-x)^3\right]_{0}^{1} = 1$

28. $V = \int_{0}^{1}\int_{0}^{1-x}\int_{0}^{\cos(\pi x/2)} dz\,dy\,dx = \int_{0}^{1}\int_{0}^{1-x} \cos\left(\frac{\pi x}{2}\right) dy\,dx = \int_{0}^{1} \left(\cos\frac{\pi x}{2}\right)(1-x)\,dx$

$= \int_{0}^{1} \cos\left(\frac{\pi x}{2}\right) dx - \int_{0}^{1} x\cos\left(\frac{\pi x}{2}\right) dx = \left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{0}^{1} - \frac{4}{\pi^2}\int_{0}^{\pi/2} u\cos u\,du = \frac{2}{\pi} - \frac{4}{\pi^2}\left[\cos u + u\sin u\right]_{0}^{\pi/2}$

$$= \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2} - 1\right) = \frac{4}{\pi^2}$$

29. $V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx = 8 \int_0^1 (1-x^2)\,dx = \frac{16}{3}$

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx = \int_0^2 \int_0^{4-x^2} (4-x^2-y)\,dy\,dx = \int_0^2 \left[(4-x^2)^2 - \frac{1}{2}(4-x^2)^2\right] dx$

$$= \frac{1}{2} \int_0^2 (4-x^2)^2\,dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \frac{128}{15}$$

31. $V = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} \int_0^{4-y} dx\,dz\,dy = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} (4-y)\,dz\,dy = \int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\,dy$

$$= \int_0^4 2\sqrt{16-y^2}\,dy - \frac{1}{2} \int_0^4 y\sqrt{16-y^2}\,dy = \left[y\sqrt{16-y^2} + 16\sin^{-1}\frac{y}{4}\right]_0^4 + \left[\frac{1}{6}(16-y^2)^{3/2}\right]_0^4$$

$$= 16\left(\frac{\pi}{2}\right) - \frac{1}{6}(16)^{3/2} = 8\pi - \frac{32}{3}$$

32. $V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz\,dy\,dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x)\,dy\,dx = 2 \int_{-2}^2 (3-x)\sqrt{4-x^2}\,dx$

$$= 3 \int_{-2}^2 2\sqrt{4-x^2}\,dx - 2 \int_{-2}^2 x\sqrt{4-x^2}\,dx = 3\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[\frac{2}{3}(4-x^2)^{3/2}\right]_{-2}^2$$

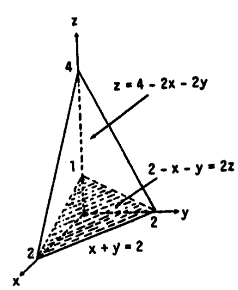
$$= 12\sin^{-1}1 - 12\sin^{-1}(-1) = 12\left(\frac{\pi}{2}\right) - 12\left(-\frac{\pi}{2}\right) = 12\pi$$

33. $\int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx = \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2}\right) dy\,dx$

$$= \int_0^2 \left[3\left(1 - \frac{x}{2}\right)(2-x) - \frac{3}{4}(2-x)^2\right] dx$$

$$= \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4}\right] dx$$

$$= \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4}\right]_0^2 = (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2$$



34. $V = \int_0^4 \int_z^8 \int_z^{8-z} dx\,dy\,dz = \int_0^4 \int_z^8 (8-2z)\,dy\,dz = \int_0^4 (8-2z)(8-z)\,dz = \int_0^4 \left(64-24z+2z^2\right)dz$

$= \left[64z - 12z^2 + \frac{2}{3}z^3\right]_0^4 = \frac{320}{3}$

35. $V = 2\int_{-2}^{2} \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz\,dy\,dx = 2\int_{-2}^{2} \int_0^{\sqrt{4-x^2}/2} (x+2)\,dy\,dx = \int_{-2}^{2} (x+2)\sqrt{4-x^2}\,dx$

$= \int_{-2}^{2} 2\sqrt{4-x^2}\,dx + \int_{-2}^{2} x\sqrt{4-x^2}\,dx = \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2} + \left[-\frac{1}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^{2}$

$= 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi$

36. $V = 2\int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz\,dx\,dy = 2\int_0^1 \int_0^{1-y^2} \left(x^2+y^2\right)dx\,dy = 2\int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y^2} dy$

$= 2\int_0^1 \left(1-y^2\right)\left[\frac{1}{3}\left(1-y^2\right)^2 + y^2\right]dy = 2\int_0^1 \left(1-y^2\right)\left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4\right)dy = \frac{2}{3}\int_0^1 \left(1-y^6\right)dy$

$= \frac{2}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = \frac{4}{7}$

37. $\text{average} = \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 \left(x^2+9\right)dz\,dy\,dx = \frac{1}{8}\int_0^2 \int_0^2 \left(2x^2+18\right)dy\,dx = \frac{1}{8}\int_0^2 \left(4x^2+36\right)dx = \frac{31}{3}$

38. $\text{average} = \frac{1}{2}\int_0^1 \int_0^1 \int_0^2 (x+y-z)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2x-1)\,dx = 0$

39. $\text{average} = \int_0^1 \int_0^1 \int_0^1 \left(x^2+y^2+z^2\right)dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right)dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right)dx = 1$

40. $\text{average} = \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 xyz\,dz\,dy\,dx = \frac{1}{4}\int_0^2 \int_0^2 xy\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = 1$

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dy\,dx\,dz = \int_0^4 \int_0^2 \frac{x\cos\left(x^2\right)}{2\sqrt{z}}\,dx\,dz = \int_0^4 \left(\frac{\sin 4}{2}\right)z^{-1/2}\,dz$

$= \left[(\sin 4)z^{1/2}\right]_0^4 = 2\sin 4$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz\,e^{zy^2}\,dy\,dx\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\,e^{zy^2}\,dx\,dy\,dz = \int_0^1 \int_0^1 6yz\,e^{zy^2}\,dy\,dz = \int_0^1 \left[3e^{zy^2}\right]_0^1 dz$

$= 3\int_0^1 \left(e^z - 1\right) dz = 3\left[e^z - 1\right]_0^1 = 3e - 6$

43. $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin\left(\pi y^2\right)}{y^2}\,dx\,dy\,dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi \sin\left(\pi y^2\right)}{y^2}\,dy\,dz = \int_0^1 \int_0^{y^3} \frac{4\pi \sin\left(\pi y^2\right)}{y^2}\,dz\,dy$

$= \int_0^1 4\pi y \sin\left(\pi y^2\right) dy = \left[-2\cos\left(\pi y^2\right)\right]_0^1 = -2(-1) + 2(1) = 4$

44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2 \int_0^{4-x^2} \frac{x \sin 2z}{4-z}\,dz\,dx = \int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z}\right) x\,dx\,dz = \int_0^4 \left(\frac{\sin 2z}{4-z}\right)\frac{1}{2}(4-z)\,dz$

$= \left[-\frac{1}{4}\cos 2z\right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$

45. $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} \left(4 - x^2 - y - a\right) dy\,dx = \frac{4}{15}$

$\Rightarrow \int_0^1 \left[\left(4-a-x^2\right)^2 - \frac{1}{2}\left(4-a-x^2\right)^2\right] dx = \frac{4}{15} \Rightarrow \frac{1}{2}\int_0^1 \left(4-a-x^2\right)^2 dx = \frac{4}{15} \Rightarrow \int_0^1 \left[(4-a)^2 - 2x^2(4-a) + x^4\right] dx$

$= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3}x^3(4-a) + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$

$\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4 - a = -\frac{1}{3}$ or $4 - a = 1 \Rightarrow a = \frac{13}{3}$ or $a = 3$

46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3$.

47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that $4x^2 + 4y^2 + z^2 - 4 \le 0$ or $4x^2 + 4y^2 + z^2 \le 4$, which is a solid ellipsoid centered at the origin.

48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that $1 - x^2 - y^2 - z^2 \ge 0$ or $x^2 + y^2 + z^2 \le 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
int(int(int(z/(x^2+y^2+z^2)^(3/2), z=sqrt(x^2+y^2)..1), y=-sqrt(1-x^2)..sqrt(1-x^2)), x =-1..1);
evalf(%);
```

Mathematica:

```
Integrate[ z/(x^2+y^2+z^2)^(3/2),
  {x,-1,1}, {y,-Sqrt[1-x^2], Sqrt[1-x^2]},
  {z,Sqrt [x^2+y^2],1} ]
N[%]
```

12.5 MASSES AND MOMENTS IN THREE DIMENSIONS

1. $I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2+z^2)\,dx\,dy\,dz = a\int_{-c/2}^{c/2}\int_{-b/2}^{b/2} (y^2+z^2)\,dy\,dz = a\int_{-c/2}^{c/2}\left[\frac{y^3}{3}+yz^2\right]_{-b/2}^{b/2} dz$

$= a\int_{-c/2}^{c/2}\left(\frac{b^3}{12}+bz^2\right)dz = ab\left[\frac{b^2}{12}z+\frac{z^3}{3}\right]_{-c/2}^{c/2} = ab\left(\frac{b^2c}{12}+\frac{c^3}{12}\right) = \frac{abc}{12}(b^2+c^2) = \frac{M}{12}(b^2+c^2);$

$R_x = \sqrt{\frac{b^2+c^2}{12}}$; likewise $R_y = \sqrt{\frac{a^2+c^2}{12}}$ and $R_z = \sqrt{\frac{a^2+b^2}{12}}$, by symmetry

2. The plane $z = \frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} (y^2+z^2)\,dz\,dy\,dx$

$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{8y^2}{3}-\frac{2y^3}{3}+\frac{8(2-y)^3}{81}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\frac{104}{3}\,dx = 208;\ I_y = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} (x^2+z^2)\,dz\,dy\,dx$

$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{(4-2y)^3}{81}+\frac{x^2(4-2y)}{3}+\frac{4x^2}{3}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\left(12x^2+\frac{32}{3}\right)dx = 280;$

$I_z = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} (x^2+y^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}(x^2+y^2)\left(\frac{8}{3}-\frac{2y}{3}\right)dy\,dx = 12\int_{-3}^{3}(x^2+2)\,dx = 360$

3. $I_x = \int_0^a\int_0^b\int_0^c (y^2+z^2)\,dz\,dy\,dx = \int_0^a\int_0^b\left(cy^2+\frac{c^3}{3}\right)dy\,dx = \int_0^a\left(\frac{cb^3}{3}+\frac{c^3b}{3}\right)dx = \frac{abc\left(b^2+c^2\right)}{3}$

$= \frac{M}{3}\left(b^2+c^2\right)$ where $M = abc$; $I_y = \frac{M}{3}\left(a^2+c^2\right)$ and $I_z = \frac{M}{3}\left(a^2+b^2\right)$, by symmetry

4. (a) $M = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx = \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx = \int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx = \frac{1}{6};$

$$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz\, dy\, dx = \int_0^1 \int_0^{1-x} x(1-x-y)\, dy\, dx = \frac{1}{2}\int_0^1 (x^3 - 2x^2 + x)\, dx = \frac{1}{24}$$

$$\Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{1}{4}, \text{ by symmetry; } I_z = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2+z^2)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[y^2 - xy^2 - y^3 + \frac{(1-x-y)^3}{3}\right] dy\, dx = \frac{1}{6}\int_0^1 (1-x)^4\, dx = \frac{1}{30} \Rightarrow I_y = I_x = \frac{1}{30}, \text{ by symmetry}$$

(b) $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \approx 0.4472$; the distance from the centroid to the x-axis is $\sqrt{0^2 + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{1}{8}} = \frac{\sqrt{2}}{4}$ ≈ 0.3536

5. $$M = 4\int_0^1 \int_0^1 \int_{4y^2}^4 dz\, dy\, dx = 4\int_0^1 \int_0^1 (4 - 4y^2)\, dy\, dx = 16\int_0^1 \frac{2}{3}\, dx = \frac{32}{3};\ M_{xy} = 4\int_0^1 \int_0^1 \int_{4y^2}^4 z\, dz\, dy\, dx$$

$$= 2\int_0^1 \int_0^1 (16 - 16y^4)\, dy\, dx = \frac{128}{5}\int_0^1 dx = \frac{128}{5} \Rightarrow \bar{z} = \frac{12}{5}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry;}$$

$$I_x = 4\int_0^1 \int_0^1 \int_{4y^2}^4 (y^2+z^2)\, dz\, dy\, dx = 4\int_0^1 \int_0^1 \left[\left(4y^2 + \frac{64}{3}\right) - \left(4y^4 + \frac{64y^6}{3}\right)\right] dy\, dx = 4\int_0^1 \frac{1976}{105}\, dx = \frac{7904}{105};$$

$$I_y = 4\int_0^1 \int_0^1 \int_{4y^2}^4 (x^2+z^2)\, dz\, dy\, dx = 4\int_0^1 \int_0^1 \left[\left(4x^2 + \frac{64}{3}\right) - \left(4x^2y^2 + \frac{64y^6}{3}\right)\right] dy\, dx = 4\int_0^1 \left(\frac{8}{3}x^2 + \frac{128}{7}\right) dx$$

$$= \frac{4832}{63};\ I_z = 4\int_0^1 \int_0^1 \int_{4y^2}^4 (x^2+y^2)\, dz\, dy\, dx = 16\int_0^1 \int_0^1 (x^2 - x^2y^2 + y^2 - y^4)\, dy\, dx$$

$$= 16\int_0^1 \left(\frac{2x^2}{3} + \frac{2}{15}\right) dx = \frac{256}{45}$$

6. (a) $$M = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} \int_0^{2-x} dz\, dy\, dx = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} (2-x)\, dy\, dx = \int_{-2}^{2} (2-x)\left(\sqrt{4-x^2}\right) dx = 4\pi;$$

$$M_{yz} = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} \int_0^{2-x} x\, dz\, dy\, dx = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} x(2-x)\, dy\, dx = \int_{-2}^{2} x(2-x)\left(\sqrt{4-x^2}\right) dx = -2\pi;$$

$$M_{xz} = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} \int_0^{2-x} y\, dz\, dy\, dx = \int_{-2}^{2} \int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} y(2-x)\, dy\, dx$$